Prob. 7.8 $\quad x(t) = \sum_{k=0}^{5} \left(\frac{1}{2}\right)^k \sin(k\pi t)$

$\forall t$

The frequencies embeded in $x(t)$:

$$\left.\begin{matrix} \pi, & 2\pi, & 3\pi, & 4\pi, & 5\pi \\ k=1 & k=2 & k=3 & k=4 & k=5 \end{matrix}\right\}$$

no $k=0$ term since $\sin(0 \cdot \pi t) = 0 \ \forall t$

$$\boxed{\omega_M = 5\pi}$$

Sampling rate $= \omega_s = \frac{2\pi}{T} = \frac{2\pi}{.2} = 10\pi$

No aliasing for the frequencies: $\pi, 2\pi, 3\pi, 4\pi$

But 5π is an issue. Let's focus on the $k=5$ term

$x_5(t) = \left(\frac{1}{2}\right)^5 \sin(5\pi t)$

Sample by replacing t with $nT_s = n.2 = \frac{n}{5}$

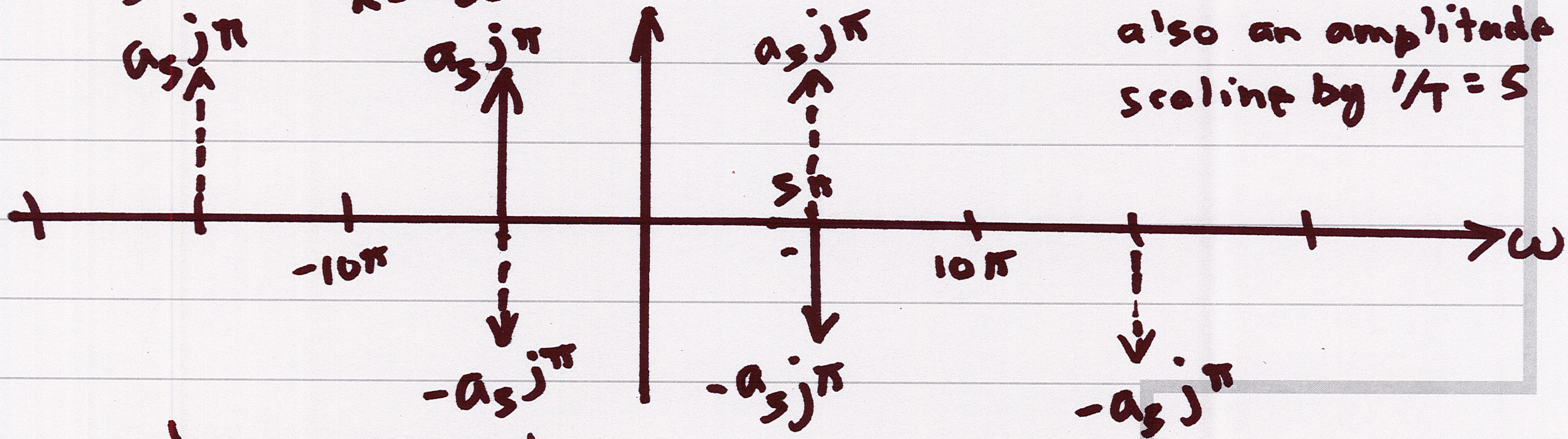
$$x_5[n] = \frac{1}{32} \sin\left(5\pi \frac{n}{5}\right)$$

$x_5[n] = \frac{1}{32}\sin(\pi n) = 0 \;\; \forall n \;\; \Rightarrow$ aliasing!

Examine in the frequency domain:

$$X_{s_5}(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_5(\omega - k10\pi) \qquad \text{Define: } a_5 = \frac{1}{32}$$

also an amplitude scaling by $1/T = 5$



dashed = replica centered at $-\omega_s = -10\pi$

dashed = replica centered at $\omega_s = 10\pi$

$\Rightarrow$ all Dirac Delta functions cancel $\Rightarrow X_{s_5}(\omega) = 0 \;\; \forall \omega$

$\Rightarrow$ aliasing! $\Rightarrow x_5[n] = 0$

Prob. 7.8 (b) no aliasing for the other 4 sinewaves ⇒ reconstructed perfectly! $k=5$ term gone

$$g(t) = x_r(t) = \sum_{k=0}^{4} \left(\frac{1}{2}\right)^k \sin(k\pi t)$$

Theoretically, one can sample at the Nyquist Rate and reconstruct perfectly IF $X(\omega)$ rolls down to at $\omega = \omega_M$ ⇒ $X(\omega_M) = 0$

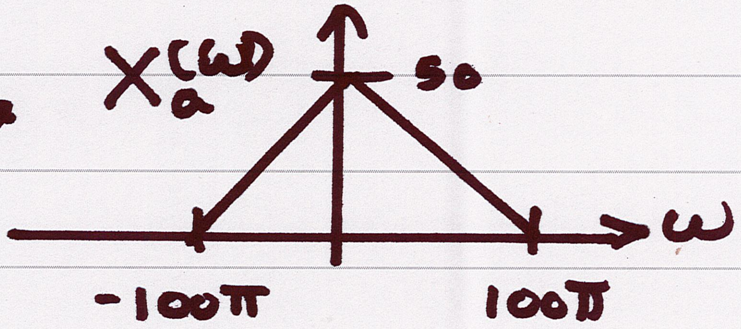
At the Nyquist Rate, the replicas just touch and one would need an Ideal LPF = perfect rectangle passing $-\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2}$

Prob. 7.9:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\,\delta(t-nT) \xleftrightarrow{\mathcal{F}} X_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

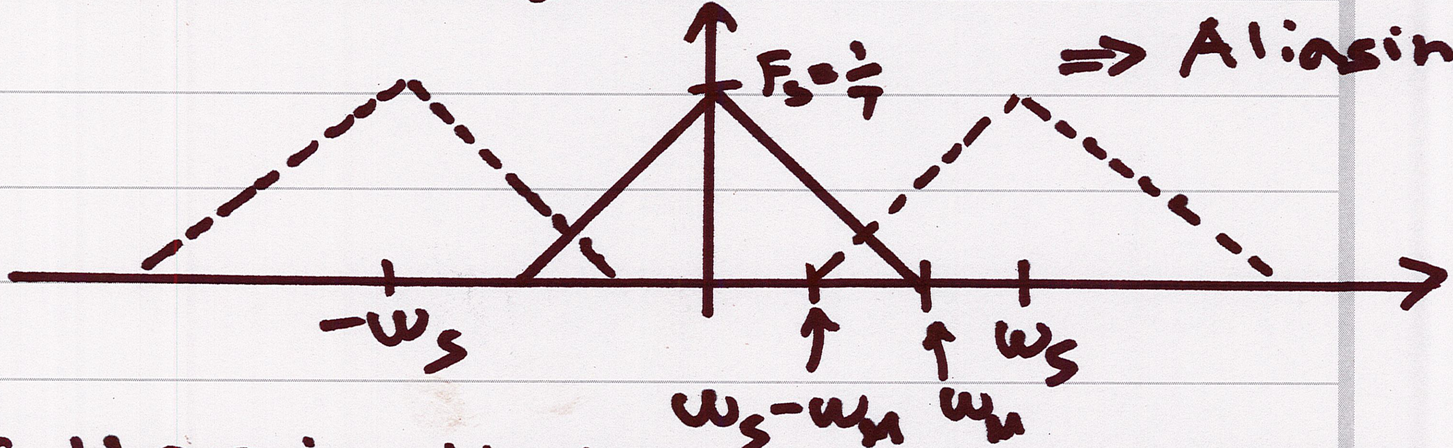
$$F_s = \frac{1}{T} \qquad \omega_s = 2\pi F_s = \frac{2\pi}{T}$$

$$x_a(t) = \left\{\frac{\sin(50\pi t)}{\pi t}\right\}^2 \xleftrightarrow{\mathcal{F}}$$



- Nyquist Rate $= 2(100\pi) = 200\pi$
- BUT sampling rate was: $\omega_s = 150\pi < 200\pi$

$\Rightarrow$ Aliasing!



If there is aliasing, it starts at $\omega_s - \omega_M$

• In this problem, $\omega_s - \omega_M = 150\pi - 100\pi$

$= 50\pi$

So, for $-50\pi < \omega < 50\pi \Rightarrow G(\omega)$

$= X_s(\omega) = 75\, X(\omega)$

since: $\frac{1}{T} = \frac{\omega_s}{2\pi} = \frac{150\pi}{2\pi} = 75$

7.12 In Chap. 5 we will introduce the DTFT (Discrete Time Fourier Transform)

• **If:** $x_d[n] = x_a(nT_s)$ and $x_a(t) \xleftrightarrow{F} X_a(\omega)$

and $X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$

• **Then:** $X_d(\omega) = X_s(F_s\,\omega)$

(Recall: Chap. 1: $y(t) = x(at)$
$a > 1$: compression)

• So: the DTFT $X_d(\omega)$ is $X_s(\omega)$ compressed by the sampling rate in Hz

• Can also view it as a normalization by $F_s = \frac{1}{T}$

• effectively: all analog frequencies divided by $F_s = \frac{1}{T}$

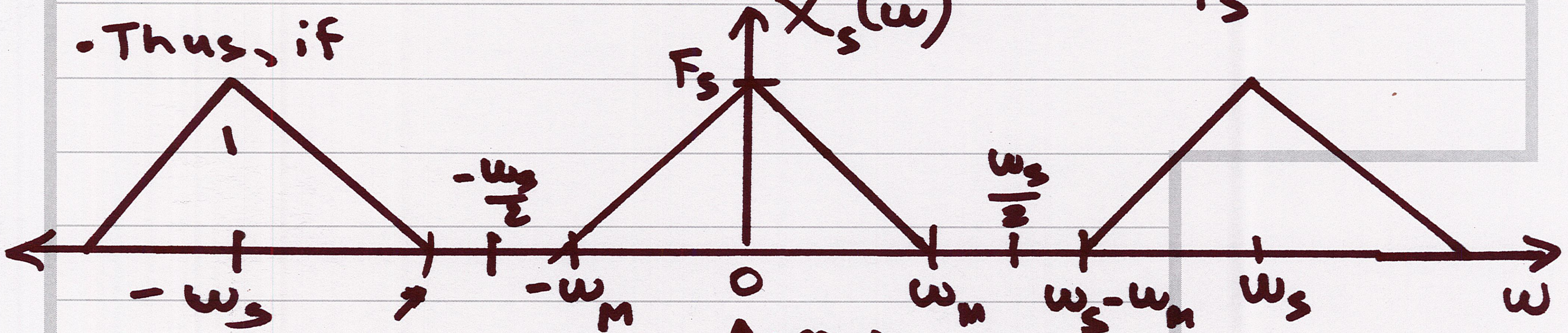
- you can see this when you sample a sinewave

$$x_a(t) = e^{j\omega_a t} \quad \Rightarrow x_d[n] = x_a(nT_s) = x_a\left(\frac{n}{F_s}\right)$$

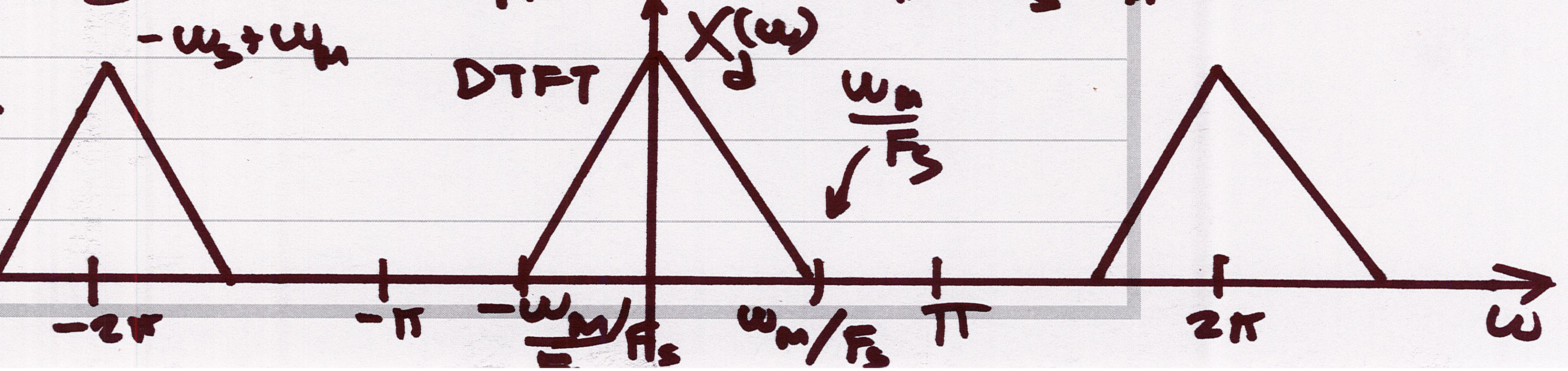
$$x_d[n] = e^{j\omega_a \frac{n}{F_s}} = e^{j\left(\frac{\omega_a}{F_s}\right)n} = e^{j\omega_d n}$$

$\Rightarrow$ The digital frequency is $\omega_d = \frac{\omega_a}{F_s}$

- Thus, if



Then:



- So, for this problem: $T = 10^{-3} \Rightarrow F_s = \frac{1}{T} = 1\text{ KHz}$
  $= 10^3\text{ Hz}$

- $\frac{\omega_M}{F_s} = \frac{\omega_M}{10^3} = \frac{3\pi}{4} \Rightarrow \omega_M = \frac{3\pi}{4} \times 10^3 \frac{\text{rads}}{\text{s}}$

Answer $\Rightarrow$ $X_a(\omega) = 0$ for $|\omega| > \omega_M = \frac{3\pi}{4} \times 10^3 \frac{\text{rads}}{\text{s}}$

- More observations: recall: $\omega_s = 2\pi F_s \Rightarrow F_s = \frac{\omega_s}{2\pi}$

ω_s is mapped to: $\frac{\omega_s}{F_s} = \frac{\omega_s}{\omega_s/2\pi} = 2\pi$

<u>Similarly</u>:

$\frac{\omega_s}{2}$ is mapped to: $\frac{\omega_s/2}{F_s} = \pi$

- If $\frac{\omega_M}{F_s} < \pi \Rightarrow$ no aliasing $\Rightarrow$ above Nyquist
- π is the highest (unambiguous) DT frequency
  (Digital)