ECE301: Signals and Systems
Spring 2012
Division 2

HW # 5 Supplementary Solutions

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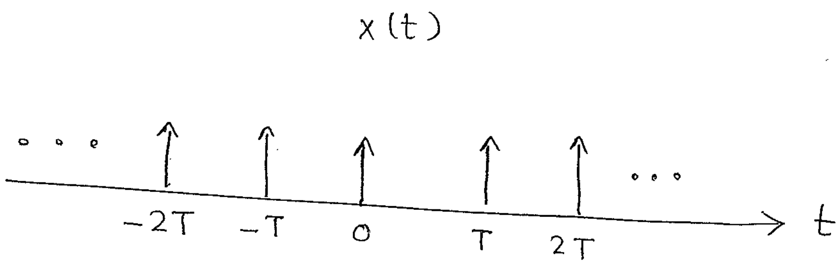
1) Fourier Series Representation for a periodic train of impulses

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-T)$$



$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

For our $x(t)$, $T = T$, take one period of $x(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$ for integration.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

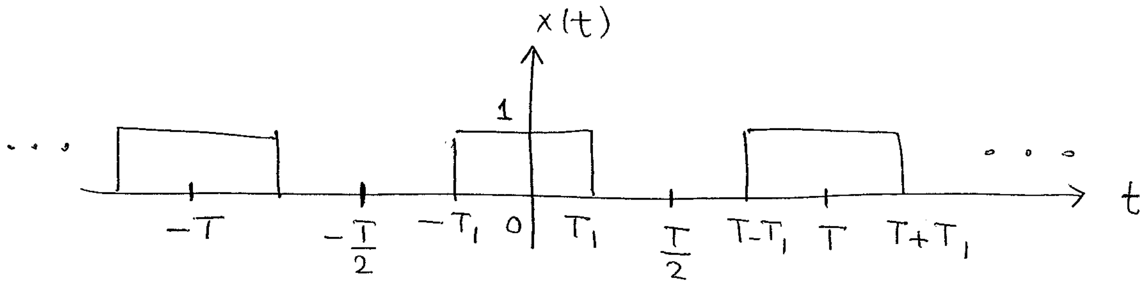
$$= \frac{1}{T}\left[e^{-jk\frac{2\pi}{T}t} \right]_{t=0}$$

$$= \frac{1}{T}\,(1)$$

$$= \frac{1}{T} \leftarrow$$

2) Fourier Series Representation for the periodic square wave

Let $x(t)$ is defined as follows.



$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

For our $x(t)$, $T = T$, take one period of $x(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$ for integration.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} (1)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T} \left. \frac{e^{-jk\frac{2\pi}{T}t}}{-jk\frac{2\pi}{T}} \right|_{-T_1}^{T_1} \qquad k \neq 0$$

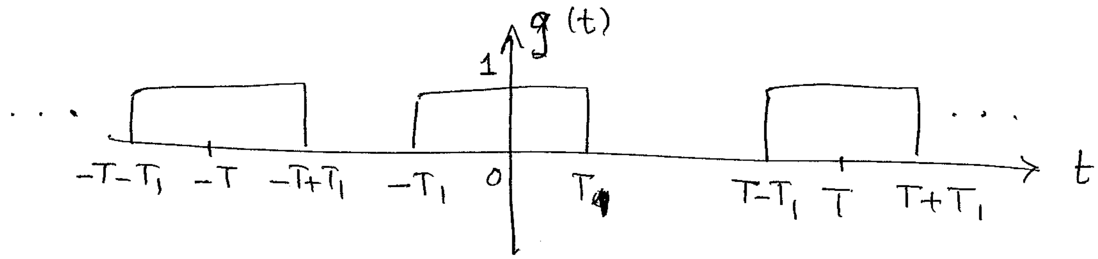
$$= \frac{1}{\cancel{T}} \frac{1}{-jk\frac{2\pi}{\cancel{T}}}\left(e^{-jk\frac{2\pi}{T}T_1} - e^{-jk\frac{2\pi}{T}(-T_1)}\right)$$

$$= \frac{1}{k\pi} \frac{\left(-e^{-jk\frac{2\pi}{T}T_1} + e^{jk\frac{2\pi}{T}T_1}\right)}{2j}$$

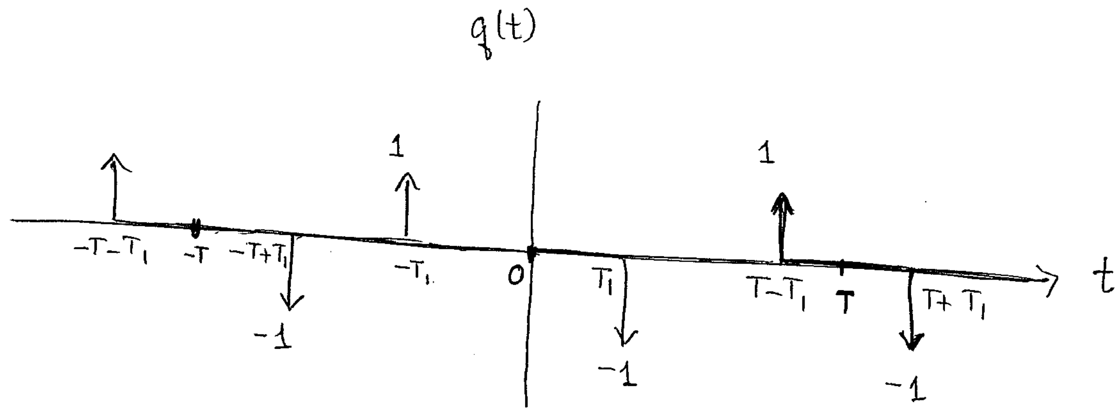
$$= \frac{1}{k\pi}\sin\left(k\frac{2\pi T_1}{T}\right) \leftarrow$$

$$a_k = \frac{1}{T}\int_T x(t)\, dt = \frac{2T_1}{T} \leftarrow k = 0$$

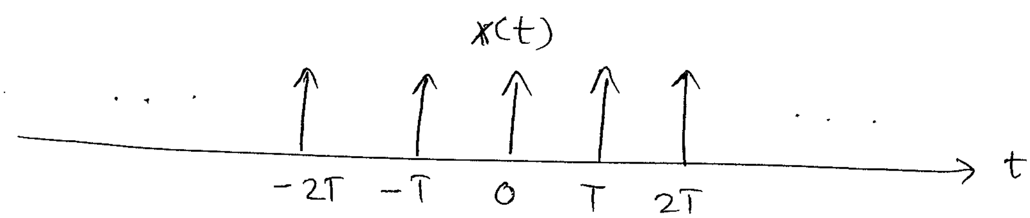
Instead of doing the straight forward way, we can use properties of Fourier Series. to find a_k of $x(t)$
Say our $g(t)$ is the periodic square wave again



Let $q(t) = \dfrac{dg(t)}{dt}$



From the first page, we know that Fourier Series coefficients of a periodic train of impulses $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-T)$



is $a_k = \frac{1}{T}$ ←

We can interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$.

$$q(t) = x(t+T_1) - x(t-T_1)$$

Using Properties of Fourier Series (Table 3-1, page 206) we can compute the Fourier Series coefficients of $q(t)$ and $g(t)$ without any further direct computations.

Let
$$x(t) \overset{F.S}{\longleftrightarrow} a_k$$
$$q(t) \overset{F.S}{\longleftrightarrow} b_k$$
$$g(t) \overset{F.S}{\longleftrightarrow} c_k$$

Then using time shifting & linearity properties,

$$b_k = e^{jk\frac{2\pi}{T}T_1} a_k - e^{-jk\frac{2\pi}{T}T_1} a_k$$

$$= \frac{1}{T}\left(e^{jk\frac{2\pi}{T}T_1} - e^{-jk\frac{2\pi}{T}T_1}\right)$$

$$= \frac{2j}{T} \sin\left(k\frac{2\pi}{T}T_1\right)$$

Next using differentiation property,

$$b_k = jk\omega_0 c_k \qquad \omega_0 = \frac{2\pi}{T}$$

$$c_k = \frac{b_k}{jk\omega_0}$$

$$= \frac{\cancel{2j}\sin\left(k\frac{2\pi}{T}T_1\right)}{\cancel{j}k\frac{\cancel{2}\pi}{\cancel{T}}\cancel{T}}$$

$$= \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi} \qquad k \neq 0$$

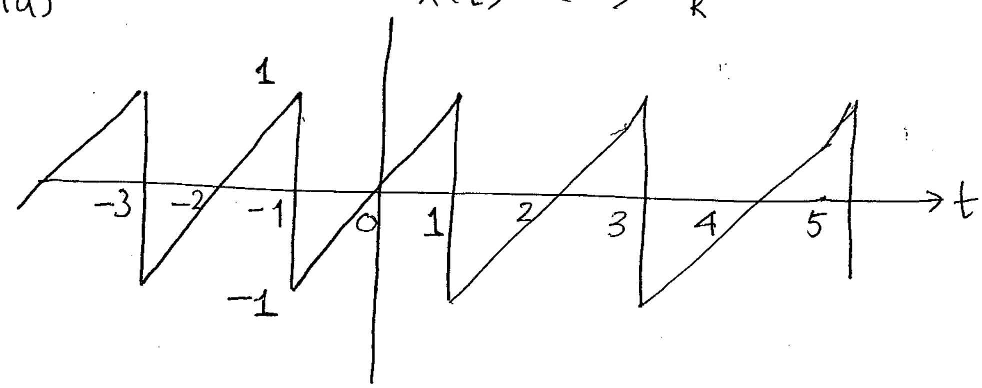
when $k = 0$, $c_k = \frac{2T_1}{T}$ (just average value of $g(t)$ over one period)

We got the same result as before. The use of properties of Fourier Series reduces the calculations required by the direct computations.

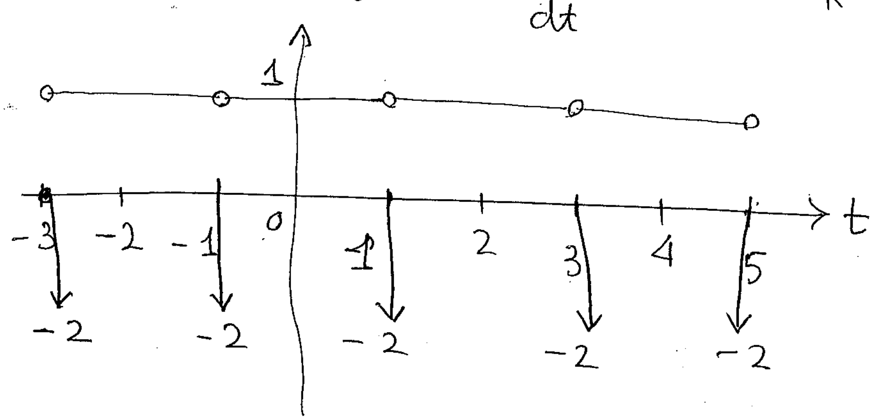
We will use the properties of Fourier Series and the results obtained for two basics Fourier Series (impulse train & periodic square wave) for problem 3.22 (a).

Problem 3.22 (a)

(a) $x(t) \overset{F.S}{\longleftrightarrow} a_k$



$$g(t) = \frac{dx(t)}{dt} \overset{F.S}{\longleftrightarrow} b_k$$



$$b_k = \frac{\sin\left(2\pi k \frac{1}{2}\right)}{k\pi} + (-2)\left(\frac{1}{2}\right) e^{-jk\frac{2\pi}{2}(1)}$$

(Linearity & time-shifting)

$$b_k = 0 - 1\, e^{-jk\pi}$$

$$= -e^{-jk\pi}$$

$$= -(-1)^k \qquad k \neq 0$$

By differentiation property.

$$jk\frac{2\pi}{T} a_k = b_k$$

$$a_k = \frac{\cancel{2}}{\cancel{2}k\pi j} b_k = \frac{b_k}{k\pi j}$$

$$= \frac{-(-1)^k}{k\pi j}$$

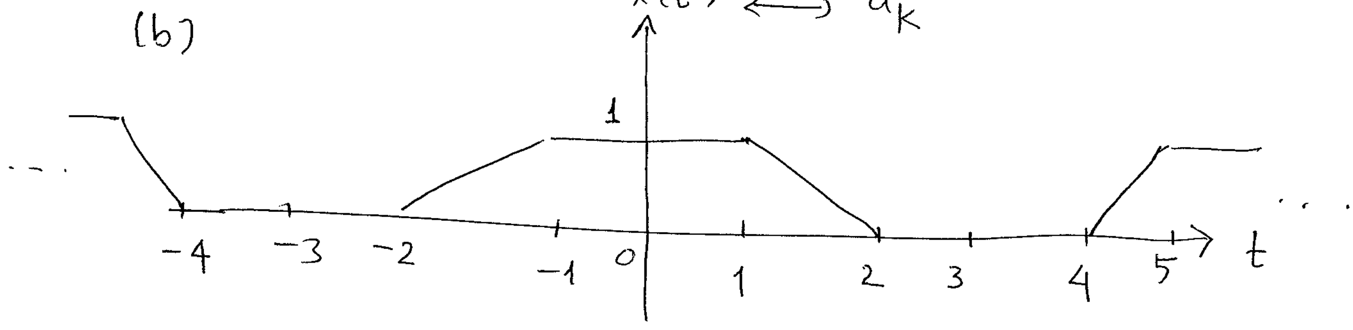
$$= \frac{j(-1)^k}{k\pi} \qquad k \neq 0$$

For $k = 0$,

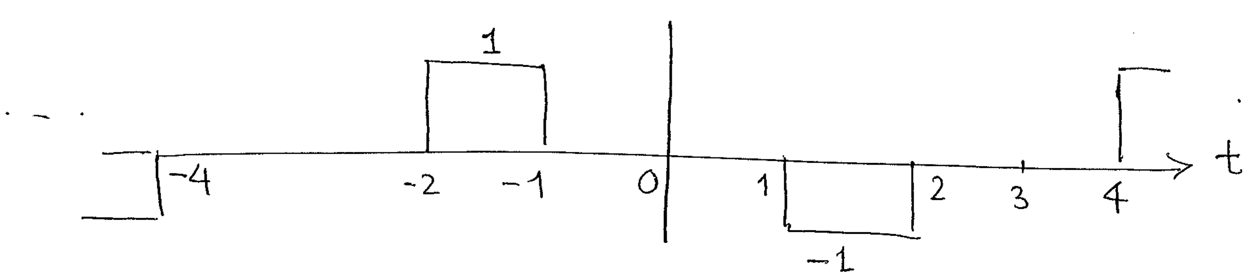
$$a_k = a_0 = 0$$ (average value of $x(t)$ over one period)

(b)

$x(t) \xleftrightarrow{F.S} a_k$



$g(t) \xleftrightarrow{F.S} b_k$



$$b_k = \underbrace{\frac{\sin\left(\frac{k\pi\, 2\, \frac{1}{2}}{6}\right)}{k\pi} e^{-j\frac{2\pi k}{6}\left(\frac{-3}{2}\right)}}_{+\text{ pulse}} + \underbrace{\frac{-\sin\left(\frac{k\pi\, 2\cdot\frac{1}{2}}{6}\right)}{k\pi} e^{-j\frac{2\pi k}{6}\left(\frac{3}{2}\right)}}_{-\text{ pulse}}$$

(By Linearity & time shifting)

$$= \frac{\sin\frac{k\pi}{6}}{k\pi}\left(e^{\frac{j2\pi k}{4}} + (-e^{-j\frac{2\pi k}{4}})\right)$$

$$= \frac{\sin\frac{k\pi}{6}}{k\pi}(j2)\left(\sin\frac{k\pi}{2}\right)$$

$$= \frac{j2}{k\pi}\sin\left(\frac{k\pi}{6}\right)\sin\left(\frac{k\pi}{2}\right) \qquad k \neq 0$$

By differentiation property.

$$a_k = \frac{1}{jk\frac{2\pi}{6}} b_k$$

$$= \frac{1}{\cancel{j}k\frac{\pi}{3}} \frac{\cancel{j}2}{k\pi}\sin\left(\frac{k\pi}{6}\right)\sin\left(\frac{k\pi}{2}\right)$$

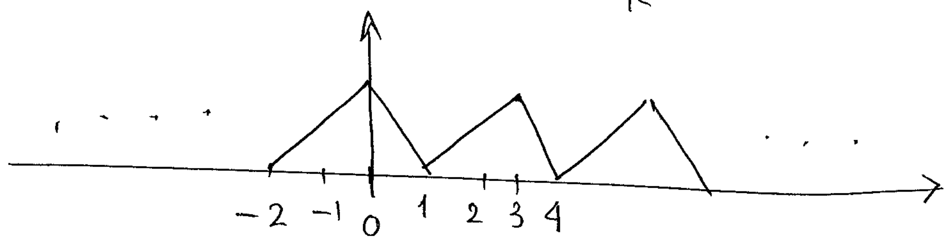
$$= \frac{6}{k^2\pi^2}\sin\left(\frac{k\pi}{6}\right)\sin\left(\frac{k\pi}{2}\right) \qquad k \neq 0$$

For $k = 0$,

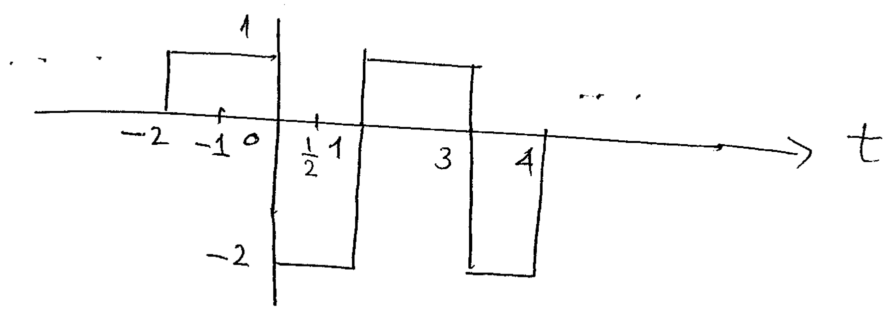
$$a_k = a_0 = \frac{3}{6} = \frac{1}{2} \qquad k = 0$$

(c)

$$x(t) \xleftrightarrow{F.S} a_k$$



$$g(t) = \frac{d\,x(t)}{dt} \xleftrightarrow{F.S} b_k$$



For $k \neq 0$

$$b_k = \underbrace{\frac{1}{k\pi}\sin\left(\frac{k\pi\, 2\,(1)}{3}\right) e^{-jk\frac{2\pi}{3}(-1)}}_{+\text{ pulse}} - \underbrace{\frac{2}{k\pi}\sin\left(\frac{k\pi\, 2\left(\frac{1}{2}\right)}{3}\right) e^{-jk\frac{2\pi}{3}\left(\frac{1}{2}\right)}}_{-\text{ pulse.}}$$

(By linearity & time shift)

By differentiation property,

$$a_k = \frac{1}{jk\frac{2\pi}{3}}\, b_k$$

$$= \frac{1}{k\pi}\,\frac{3}{jk2\pi}\left[\sin\left(\frac{2k\pi}{3}\right) e^{jk\frac{2\pi}{3}} - 2\sin\left(\frac{k\pi}{3}\right) e^{-jk\frac{\pi}{3}}\right]$$

$$\cancel{= \frac{3}{jk^2 2\pi^2}\sin\left(\frac{k\pi}{3}\right)\left[e^{jk\frac{2\pi}{3}} - 2e^{-jk\frac{\pi}{3}}\right]}$$

$$= \frac{3}{2j\,k^2\pi^2}\left[\left(\frac{e^{\frac{j2\pi k}{3}} - e^{\frac{-j2\pi k}{3}}}{2j}\right) e^{jk\frac{2\pi}{3}} - 2\left(\frac{e^{\frac{jk\pi}{3}} - e^{\frac{-jk\pi}{3}}}{2j}\right) e^{-j\frac{k\pi}{3}}\right.$$

$$= \frac{-3}{4\pi^2 k^2}\left[\left(e^{j\frac{4\pi}{3}k} - 1\right) - 2\left(1 - e^{-j\frac{2k\pi}{3}}\right)\right]$$

$$= \frac{-3}{4\pi^2 k^2}\left[e^{j\frac{4\pi}{3}k} - 1 - 2 + 2e^{-j\frac{2k\pi}{3}}\right]$$

$e^{-j\frac{2k\pi}{3}}$

$$= \frac{9}{4\pi^2 k^2} - \frac{3}{4\pi^2 k^2}\left[e^{j\frac{4\pi}{3}k} + 2e^{-j\frac{2k\pi}{3}}\right]$$

$$= \frac{9}{4\pi^2 k^2} - \frac{9}{4\pi^2 k^2} e^{-j\frac{2k\pi}{3}} \leftarrow$$

The solutions given by integration does not agree with this one because there is some mistakes in integration. (Refer to page 13 & 14 of HW#4 solutions)

The integration should be like this.

$$x(t) = \begin{cases} t+2 & -2 \le t < 0 \\ 2-2t & 0 \le t < 1 \end{cases}$$

$$a_k = \frac{1}{3}\int_{-2}^{1} x(t)\, e^{-j\frac{2\pi}{3}kt}\, dt$$

$$= \frac{1}{3}\int_{-2}^{0} (t+2)\, e^{-j\frac{2\pi}{3}kt}\, dt + \frac{1}{3}\int_{0}^{1} (2-2t)\, e^{-j\frac{2\pi}{3}kt}\, dt$$

$$= \frac{1}{3}\left[\frac{t+2}{-j\left(\frac{2\pi}{3}k\right)} e^{-j\left(\frac{2\pi}{3}\right)kt} - \frac{e^{-j\left(\frac{2\pi}{3}\right)kt}}{j^2\left(\frac{2\pi}{3}\right)^2 k^2}\right]_{-2}^{0} +$$

$$\frac{1}{3}\left[\frac{2-2t}{-j\left(\frac{2\pi}{3}k\right)} e^{-j\frac{2\pi}{3}kt} + \frac{2e^{-j\left(\frac{2\pi}{3}\right)kt}}{j^2\left(\frac{2\pi}{3}\right)^2 k^2}\right]_{0}^{1}$$

$$= \frac{1}{3}\left[\frac{2}{-j\frac{2\pi}{3}k} + \frac{1}{\left(\frac{2\pi}{3}\right)^2 k^2} - \frac{e^{+j\frac{4\pi}{3}k}}{\left(\frac{2\pi}{3}\right)^2 k^2}\right] +$$

$= e^{-j\frac{2\pi}{3}k}$

$$\frac{1}{3}\left[0 + \frac{2e^{-j\frac{2\pi}{3}k}}{-\left(\frac{2\pi}{3}\right)^2k^2} + \frac{2}{j\frac{2\pi}{3}k} + \frac{2}{\left(\frac{2\pi}{3}\right)^2k^2}\right]$$

$$= \frac{1}{\cancel{3}}\left[\frac{-\cancel{3}e^{-j\frac{2\pi}{3}k}}{\left(\frac{2\pi}{3}\right)^2k^2} + \frac{\cancel{3}}{\left(\frac{2\pi}{3}\right)^2k^2}\right]$$

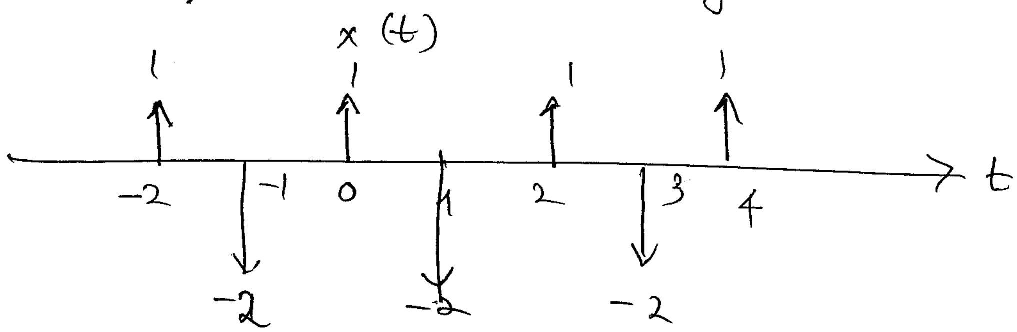
$$= \frac{9}{4\pi^2k^2}\left[1 - e^{-j\frac{2\pi}{3}k}\right]$$

$$= \frac{9}{4\pi^2k^2} - \frac{9}{4\pi^2k^2}e^{-j\frac{2\pi}{3}k}$$

Now they agree. :)

For $k=0$, it is just average value $\Rightarrow \frac{3}{3} = 1 = a_0$

(d)



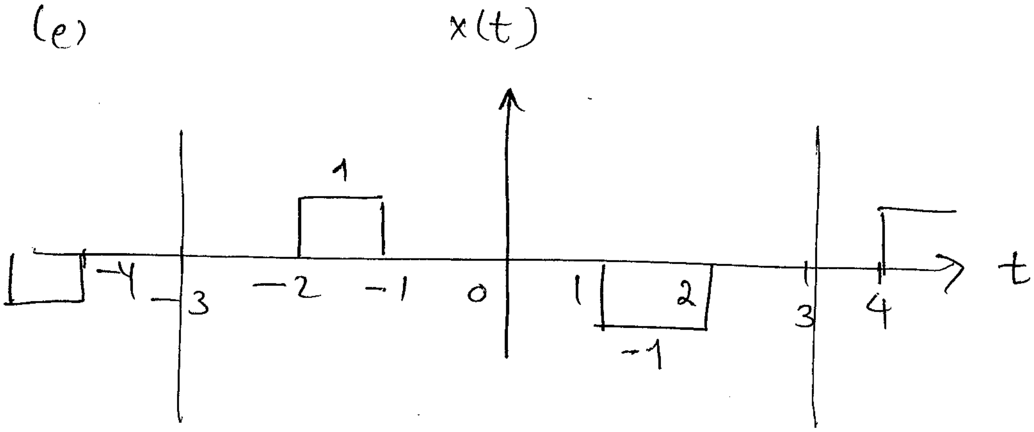
By linearity.

$$a_k = \frac{1}{2} + (\cancel{-2})\frac{1}{\cancel{2}}e^{-jk\frac{2\pi}{2}(1)}$$

$$= \frac{1}{2} - e^{-jk\pi}$$

$$= \frac{1}{2} - (-1)^k \quad \text{for all } k.$$

(e)



from (b)

$$a_k = \frac{j2}{k\pi} \sin\left(\frac{k\pi}{6}\right) \sin\left(\frac{k\pi}{2}\right) \qquad k \neq 0$$

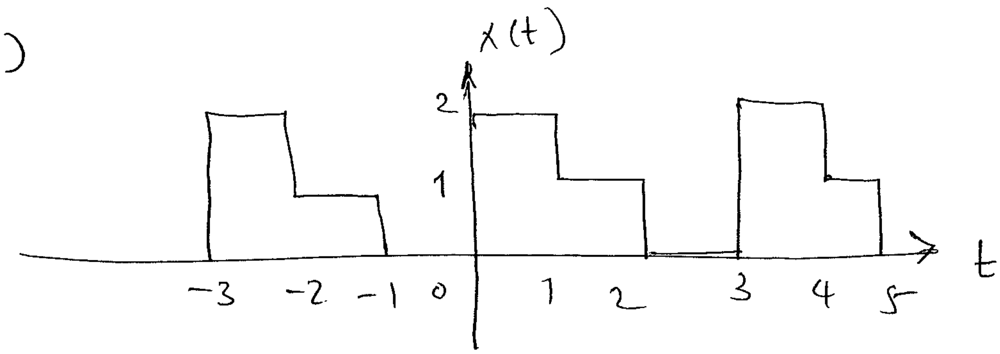
$$= \frac{j}{k\pi}\left[\cos \frac{\pi}{3}k - \cos \frac{2\pi}{3}k\right]$$

$2 \sin A \sin B = \cos(A-B) + \cos(A+B)$

$k=0,$

$$a_k = 0$$

(f)



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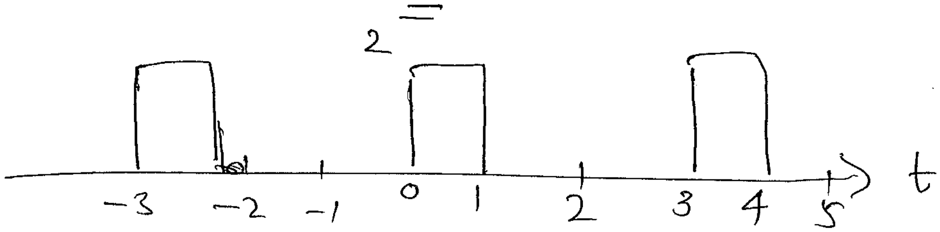

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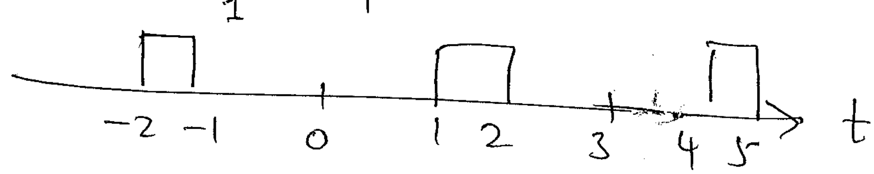

By Linearity

$k \neq 0$

$$a_k = \frac{2}{k\pi} \sin\left(\frac{k\cancel{2}\pi\left(\frac{1}{\cancel{2}}\right)}{3}\right) e^{-jk\frac{2\pi}{3}\left(\frac{1}{2}\right)} +$$

$$\frac{1}{k\pi} \sin\left(\frac{\cancel{2}\pi k\left(\frac{1}{\cancel{2}}\right)}{3}\right) e^{-jk\frac{2\pi}{3}\left(\frac{3}{2}\right)}$$

$$= \frac{\left(\sin \frac{\pi k}{3}\right)}{k\pi} \left[2e^{-j\frac{k\pi}{3}} + e^{-jk\pi}\right]$$

$$= \frac{1}{k\pi} \left(\frac{e^{j\frac{k\pi}{3}} - e^{-j\frac{k\pi}{3}}}{2j}\right) \left[2e^{-j\frac{k\pi}{3}} + e^{-jk\pi}\right]$$

$$= \frac{1}{2jk\pi} \left(2 - e^{-j\frac{2k\pi}{3}} + 2e^{-j\frac{2\pi k}{3}} - e^{\frac{-j\pi 4k}{3}}\right)$$

$$= \frac{1}{2jk\pi} \left(2 - e^{-j\frac{2k\pi}{3}} - e^{\frac{-j\pi 4k}{3}}\right)$$

$$= \frac{j}{2k\pi} \left(e^{-j\frac{2k\pi}{3}} + e^{-j\frac{4k\pi}{3}} - 2\right)$$

$k = 0$

$$a_0 = a_k = \frac{1}{3} \cdot 3 = 1$$