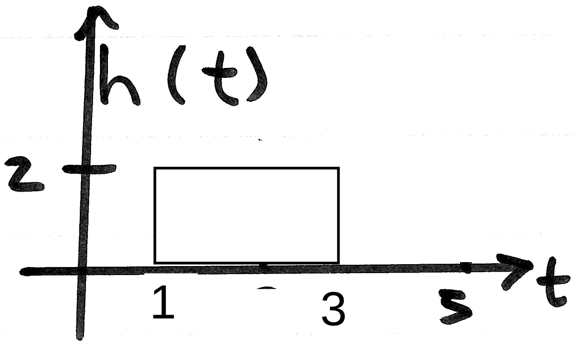
Prob. 2.22 (c)

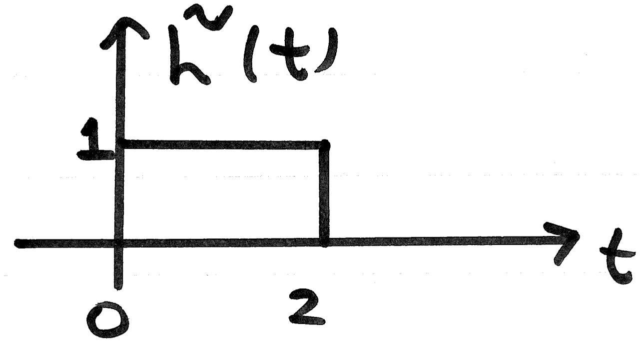
- **Convolve:** $x(t)$ with



$$y(t) = x(t) * h(t)$$

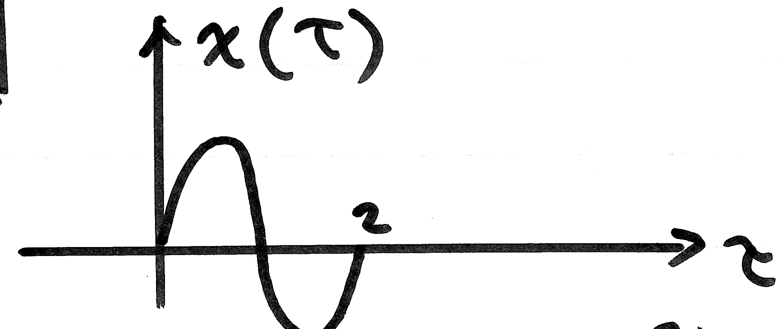
- **Instead:**

$$\tilde{y}(t) = x(t) * \tilde{h}(t)$$



- **Then:**

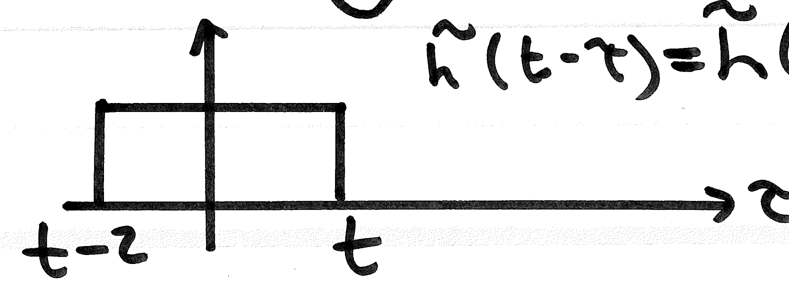
$$\boxed{y(t) = 2\,\tilde{y}(t-1)}$$



Method 1 :

For $0 < t < 2$:

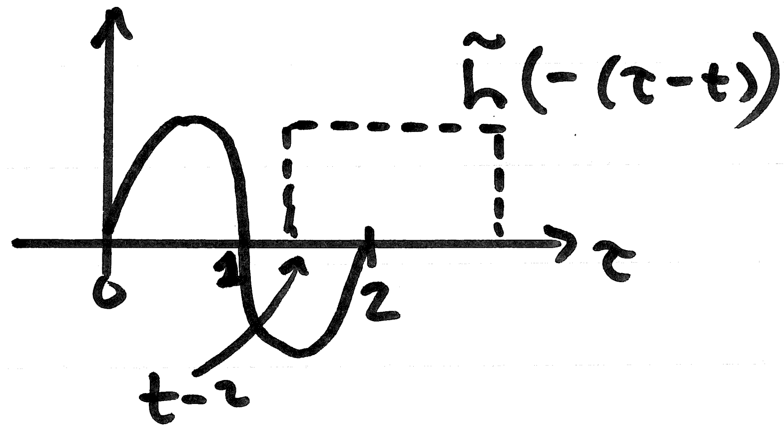
$$\tilde{h}(t-\tau) = \tilde{h}(-(\tau - t))$$



$$\tilde{y}(t) = \int_0^t \sin(\pi\tau)\,d\tau = \frac{-1}{\pi}\cos(\pi\tau)\Big]_0^t$$

$$= -\frac{1}{\pi}\left\{\cos(\pi t) - 1\right\} = \frac{1}{\pi}\left\{1 - \cos(\pi t)\right\}$$

<u>For $2 < t < 4$:</u>



$$\tilde{y}(t) = \int_{t-2}^{2} \sin(\pi\tau)\,d\tau = -\frac{1}{\pi}\cos(\pi\tau)\Big]_{t-2}^{2}$$

$$= -\frac{1}{\pi}\left\{\cos(2\pi) - \cos(\pi(t-2))\right\}$$

$$= \frac{-1}{\pi}\left\{1 - \cos(\pi t)\right\}$$

Summarizing:

$$\tilde{y}(t) = \frac{1}{\pi}\left\{1-\cos(\pi t)\right\}\left(u(t)-u(t-2)\right)$$
$$-\frac{1}{\pi}\left\{1-\cos(\pi t)\right\}\left(u(t-2)-u(t-4)\right)$$

• Thus:

$$y(t) = 2\,\tilde{y}(t-1)$$
$$= \frac{2}{\pi}\left\{1-\cos(\pi(t-1))\right\}\left\{u(t-1)-u(t-3)\right\}$$
$$- \frac{2}{\pi}\left\{1-\cos(\pi(t-1))\right\}\left\{u(t-3)-u(t-5)\right\}$$
$$= \frac{2}{\pi}\left\{1+\cos(\pi t)\right\}\left\{u(t-1)-u(t-3)\right\}$$
$$- \frac{2}{\pi}\left\{1+\cos(\pi t)\right\}\left\{u(t-3)-u(t-5)\right\}$$

- **Alternatively**

$$x(t) = \sin(\pi t)\,\{u(t) - u(t-2)\}$$

$$= \Big(\underbrace{\tfrac{1}{2j}e^{j\pi t}}_{s(t)} - \underbrace{\tfrac{1}{2j}e^{-j\pi t}}_{s^*(t)}\Big)\{u(t) - u(t-2)\}$$

$$s(t) = \tfrac{1}{2j}e^{j\pi t} \times$$

Since $\tilde{h}(t)$ is real-valued:

<u>If:</u> $s(t) * \tilde{h}(t) = v(t)$

<u>Then:</u> $s^*(t) * \tilde{h}(t) = v^*(t)$

<u>Thus:</u> $\tilde{y}(t) = \big(s(t) + s^*(t)\big) * \tilde{h}(t)$

$$= v(t) + v^*(t)$$

$$= 2\,\mathrm{Re}\{v(t)\}$$

- First, find:

$$v(t) = \underbrace{\frac{1}{2j} e^{j\pi t}\{u(t)-u(t-2)\}}_{s(t)} * \overbrace{\{u(t)-u(t-2)\}}^{\tilde{h}(t)}$$

- FOIL:

$$v(t) = \frac{1}{2j}\begin{cases} e^{j\pi t}u(t) * u(t) - e^{j\pi t}u(t) * u(t-2) \\ -e^{j\pi t}u(t-2) * u(t) + e^{j\pi t}u(t-2) * u(t-2) \end{cases}$$

$$= \frac{1}{2j}\begin{cases} e^{j\pi t}u(t) * u(t) - e^{j\pi t}u(t) * u(t-2) \\ -e^{j\pi(t-2)}u(t-2) * u(t) + e^{j\pi(t-2)}u(t-2) * u(t-2) \end{cases}$$

since: $e^{-j2\pi} = 1$

• <u>If</u> $z(t) = e^{j\pi t} u(t) * u(t)$

<u>Then:</u> $$v(t) = \frac{1}{2j}\left\{ \begin{array}{l} z(t) - z(t-2) \\ - z(t-2) + z(t-4) \end{array} \right.$$

and, since:

$$e^{at} u(t) * e^{bt} u(t) = \frac{1}{a-b}\left\{ e^{at} - e^{bt} \right\} u(t)$$

with $a = j\pi$ and $b = 0$

$$z(t) = \frac{1}{j\pi}\left\{ e^{j\pi t} - 1 \right\} u(t)$$

since $e^{-j2\pi} = e^{-j4\pi} = 1$

$$z(t-2) = \frac{1}{j\pi}\left\{ e^{j\pi t} - 1 \right\} u(t-2)$$

$$z(t-4) = \frac{1}{j\pi}\left\{ e^{j\pi t} - 1 \right\} u(t-4)$$

$$v(t) = \frac{1}{2j}\left\{\left(z(t) - z(t-2)\right) - \left(z(t-2) - z(t-4)\right)\right\}$$

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$$= \quad w(t) - w(t-2)$$

where: $w(t) = \frac{1}{2j}\left(z(t) - z(t-2)\right)$

$$= \frac{1}{2j}\frac{1}{j\pi}\left\{e^{j\pi t} - 1\right\}u(t) - \frac{1}{2j}\frac{1}{j\pi}\left\{e^{j\pi t} - 1\right\}u(t-2)$$

$$= \frac{-1}{2\pi}\left\{e^{j\pi t} - 1\right\}\left(u(t) - u(t-2)\right)$$

$$v(t) = \frac{1}{2\pi}\left\{1 - e^{j\pi t}\right\}\left(u(t) - u(t-2)\right) \quad \text{for } 0 < t < 2$$

$$\mathrm{Re}\left\{e^{j\pi t}\right\} = \cos(\pi t)$$

since: $e^{j\pi t} = \cos(\pi t) + j\sin(\pi t)$

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• **Thus:** for $0<t<2$:

$$\tilde{y}(t) = 2\,\mathrm{Re}\{v(t)\}$$

$$= \frac{1}{\pi}\{1-\cos(\pi t)\}(u(t)-u(t-2))$$

• **For $0<t<4$:**

$$\tilde{y}(t) = \begin{cases} \frac{1}{\pi}(1-\cos(\pi t))(u(t)-u(t-2)) \\ -\frac{1}{\pi}(1-\underbrace{\cos(\pi(t-2))}_{=\cos(\pi t)})(u(t-2)-u(t-4)) \end{cases}$$

Then: $y(t) = 2\,\tilde{y}(t-1)$

$$y(t) = \begin{cases} \frac{2}{\pi}(1-\cos(\pi(t-1)))(u(t-1)-u(t-3)) \\ -\frac{2}{\pi}(1-\cos(\pi(t-1)))(u(t-3)-u(t-5)) \end{cases}$$

Since $\cos(\theta - \pi) = -\cos(\theta)$

and $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$$1 - \cos(\pi(t-1)) = 1 + \cos(\pi t)$$
$$= 2\cos^2\left(\frac{\pi}{2}t\right)$$

Thus:

$$y(t) = \begin{cases} \frac{4}{\pi}\cos^2\left(\frac{\pi}{2}t\right) & 1<t<3 \\ -\frac{4}{\pi}\cos^2\left(\frac{\pi}{2}t\right) & 3<t<5 \end{cases}$$

$= 0$ otherwise

$t<1$ $t>5$