PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

In the preceding two sections, we developed the extremely important representations of continuous-time and discrete-time LTI systems in terms of their unit impulse responses. In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral, both of which we repeat here for convenience:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n] \tag{2.39}$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \tag{2.40}$$

As we have pointed out, one consequence of these representations is that the characteristics of an LTI system are completely determined by its impulse response. It is important to emphasize that this property holds in general *only* for LTI systems. In particular, as illustrated in the following example, the unit impulse response of a nonlinear system does *not* completely characterize the behavior of the system.

Example 2.9

Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}. \tag{2.41}$$

If the system is LTI, then eq. (2.41) completely determines its input-output behavior. In particular, by substituting eq. (2.41) into the convolution sum, eq. (2.39), we find the following explicit equation describing how the input and output of this LTI system are related:

$$y[n] = x[n] + x[n-1]. \tag{2.42}$$

On the other hand, there are *many* nonlinear systems with the same response—i.e., that given in eq. (2.41)—to the input $\delta[n]$. For example, both of the following systems have this property:

$$y[n] = (x[n] + x[n-1])^2,$$
$$y[n] = \max(x[n], x[n-1]).$$

Consequently, if the system is nonlinear it is not completely characterized by the impulse response in eq. (2.41).

The preceding example illustrates the fact that LTI systems have a number of properties not possessed by other systems, beginning with the very special representations that they have in terms of convolution sums and integrals. In the remainder of this section, we explore some of the most basic and important of these properties.

2.3.1 The Commutative Property

A basic property of convolution in both continuous and discrete time is that it is a *commutative* operation. That is, in discrete time

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k], \tag{2.43}$$

and in continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau. \tag{2.44}$$

These expressions can be verified in a straightforward manner by means of a substitution of variables in eqs. (2.39) and (2.40). For example, in the discrete-time case, if we let $r = n - k$ or, equivalently, $k = n - r$, eq. (2.39) becomes

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r] = h[n] * x[n]. \tag{2.45}$$

With this substitution of variables, the roles of $x[n]$ and $h[n]$ are interchanged. According to eq. (2.45), the output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$. For example, we could have calculated the convolution in Example 2.4 by first reflecting and shifting $x[k]$, then multiplying the signals $x[n-k]$ and $h[k]$, and finally summing the products for all values of k.

Similarly, eq. (2.44) can be verified by a change of variables, and the implications of this result in continuous time are the same: The output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI system with input $h(t)$ and unit impulse response $x(t)$. Thus, we could have calculated the convolution in Example 2.7 by reflecting and shifting $x(t)$, multiplying the signals $x(t-\tau)$ and $h(\tau)$, and integrating over $-\infty < \tau < +\infty$. In specific cases, one of the two forms for computing convolutions [i.e., eq. (2.39) or (2.43) in discrete time and eq. (2.40) or (2.44) in continuous time] may be easier to visualize, but both forms always result in the same answer.

2.3.2 The Distributive Property

Another basic property of convolution is the *distributive* property. Specifically, convolution distributes over addition, so that in discrete time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n], \tag{2.46}$$

and in continuous time

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \tag{2.47}$$

This property can be verified in a straightforward manner.

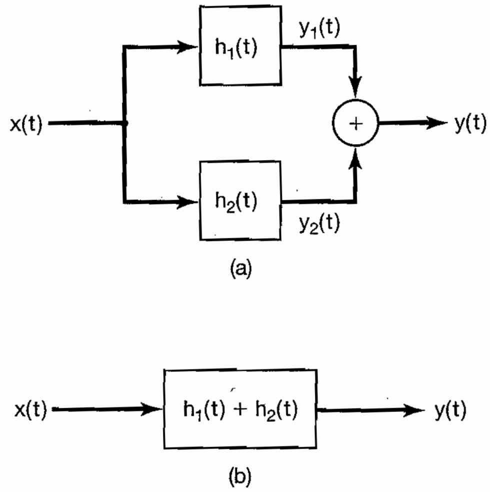

Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

The distributive property has a useful interpretation in terms of system interconnections. Consider two continuous-time LTI systems in parallel, as indicated in Figure 2.23(a). The systems shown in the block diagram are LTI systems with the indicated unit impulse responses. This pictorial representation is a particularly convenient way in which to denote LTI systems in block diagrams, and it also reemphasizes the fact that the impulse response of an LTI system completely characterizes its behavior.

The two systems, with impulse responses $h_1(t)$ and $h_2(t)$, have identical inputs, and their outputs are added. Since

$$y_1(t) = x(t) * h_1(t)$$

and

$$y_2(t) = x(t) * h_2(t),$$

the system of Figure 2.23(a) has output

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t), \tag{2.48}$$

corresponding to the right-hand side of eq. (2.47). The system of Figure 2.23(b) has output

$$y(t) = x(t) * [h_1(t) + h_2(t)], \tag{2.49}$$

corresponding to the left-hand side of eq. (2.47). Applying eq. (2.47) to eq. (2.49) and comparing the result with eq. (2.48), we see that the systems in Figures 2.23(a) and (b) are identical.

There is an identical interpretation in discrete time, in which each of the signals in Figure 2.23 is replaced by a discrete-time counterpart (i.e., $x(t)$, $h_1(t)$, $h_2(t)$, $y_1(t)$, $y_2(t)$, and $y(t)$ are replaced by $x[n]$, $h_1[n]$, $h_2[n]$, $y_1[n]$, $y_2[n]$, and $y[n]$, respectively). In summary, then, by virtue of the distributive property of convolution, a parallel combination of LTI systems can be replaced by a single LTI system whose unit impulse response is the sum of the individual unit impulse responses in the parallel combination.

Also, as a consequence of both the commutative and distributive properties, we have

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \tag{2.50}$$

and

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t), \tag{2.51}$$

which simply state that the response of an LTI system to the sum of two inputs must equal the sum of the responses to these signals individually.

As illustrated in the next example, the distributive property of convolution can also be exploited to break a complicated convolution into several simpler ones.

Example 2.10

Let $y[n]$ denote the convolution of the following two sequences:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n], \tag{2.52}$$

$$h[n] = u[n]. \tag{2.53}$$

Note that the sequence $x[n]$ is nonzero along the entire time axis. Direct evaluation of such a convolution is somewhat tedious. Instead, we may use the distributive property to express $y[n]$ as the sum of the results of two simpler convolution problems. In particular, if we let $x_1[n] = (1/2)^n u[n]$ and $x_2[n] = 2^n u[-n]$, it follows that

$$y[n] = (x_1[n] + x_2[n]) * h[n]. \tag{2.54}$$

Using the distributive property of convolution, we may rewrite eq. (2.54) as

$$y[n] = y_1[n] + y_2[n], \tag{2.55}$$

where

$$y_1[n] = x_1[n] * h[n] \tag{2.56}$$

and

$$y_2[n] = x_2[n] * h[n]. \tag{2.57}$$

The convolution in eq. (2.56) for $y_1[n]$ can be obtained from Example 2.3 (with $\alpha = 1/2$), while $y_2[n]$ was evaluated in Example 2.5. Their sum is $y[n]$, which is shown in Figure 2.24.

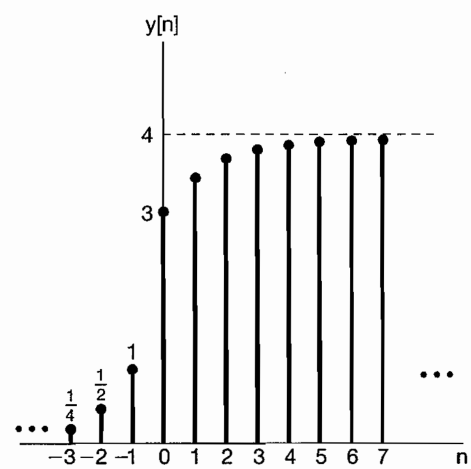

Figure 2.24 The signal $y[n] = x[n] * h[n]$ for Example 2.10.

2.3.3 The Associative Property

Another important and useful property of convolution is that it is *associative*. That is, in discrete time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n], \tag{2.58}$$

and in continuous time

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t). \tag{2.59}$$

This property is proven by straightforward manipulations of the summations and integrals involved. Examples verifying it are given in Problem 2.43.

As a consequence of the associative property, the expressions

$$y[n] = x[n] * h_1[n] * h_2[n] \tag{2.60}$$

and

$$y(t) = x(t) * h_1(t) * h_2(t) \tag{2.61}$$

are unambiguous. That is, according to eqs. (2.58) and (2.59), it does not matter in which order we convolve these signals.

An interpretation of the associative property is illustrated for discrete-time systems in Figures 2.25(a) and (b). In Figure 2.25(a),

$$\begin{aligned} y[n] &= w[n] * h_2[n] \\ &= (x[n] * h_1[n]) * h_2[n]. \end{aligned}$$

In Figure 2.25(b),

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * (h_1[n] * h_2[n]). \end{aligned}$$

According to the associative property, the series interconnection of the two systems in Figure 2.25(a) is equivalent to the single system in Figure 2.25(b). This can be generalized to an arbitrary number of LTI systems in cascade, and the analogous interpretation and conclusion also hold in continuous time.

By using the commutative property together with the associative property, we find another very important property of LTI systems. Specifically, from Figures 2.25(a) and (b), we can conclude that the impulse response of the cascade of two LTI systems is the convolution of their individual impulse responses. Since convolution is commutative, we can compute this convolution of $h_1[n]$ and $h_2[n]$ in either order. Thus, Figures 2.25(b) and (c) are equivalent, and from the associative property, these are in turn equivalent to the system of Figure 2.25(d), which we note is a cascade combination of two systems as in Figure 2.25(a), but with the order of the cascade reversed. Consequently, the unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded. In fact, this holds for an arbitrary number of LTI systems in cascade: The order in which they are cascaded does not matter as far as the overall system impulse response is concerned. The same conclusions hold in continuous time as well.

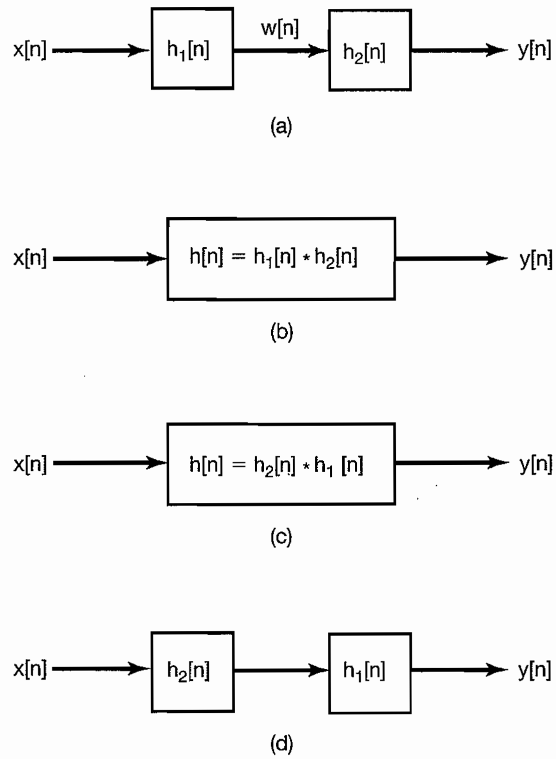

Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

It is important to emphasize that the behavior of LTI systems in cascade—and, in particular, the fact that the overall system response does not depend upon the order of the systems in the cascade—is very special to such systems. In contrast, the order in which nonlinear systems are cascaded cannot be changed, in general, without changing the overall response. For instance, if we have two memoryless systems, one being multiplication by 2 and the other squaring the input, then if we multiply first and square second, we obtain

$$y[n] = 4x^2[n].$$

However, if we multiply by 2 after squaring, we have

$$y[n] = 2x^2[n].$$

Thus, being able to interchange the order of systems in a cascade is a characteristic particular to LTI systems. In fact, as shown in Problem 2.51, we need both linearity *and* time invariance in order for this property to be true in general.

2.3.4 LTI Systems with and without Memory

As specified in Section 1.6.1, a system is memoryless if its output at any time depends only on the value of the input at that same time. From eq. (2.39), we see that the only way that this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$. In this case the impulse response has the form

$$h[n] = K\delta[n], \tag{2.62}$$

where $K = h[0]$ is a constant, and the convolution sum reduces to the relation

$$y[n] = Kx[n]. \tag{2.63}$$

If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory. An example of an LTI system with memory is the system given by eq. (2.42). The impulse response for this system, given in eq. (2.41), is nonzero for $n = 1$.

From eq. (2.40), we can deduce similar properties for continuous-time LTI systems with and without memory. In particular, a continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$, and such a memoryless LTI system has the form

$$y(t) = Kx(t) \tag{2.64}$$

for some constant K and has the impulse response

$$h(t) = K\delta(t). \tag{2.65}$$

Note that if $K = 1$ in eqs. (2.62) and (2.65), then these systems become identity systems, with output equal to the input and with unit impulse response equal to the unit impulse. In this case, the convolution sum and integral formulas imply that

$$x[n] = x[n] * \delta[n]$$

and

$$x(t) = x(t) * \delta(t),$$

which reduce to the sifting properties of the discrete-time and continuous-time unit impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

2.3.5 Invertibility of LTI Systems

Consider a continuous-time LTI system with impulse response $h(t)$. Based on the discussion in Section 1.6.2, this system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system. Furthermore, if an LTI system is invertible, then it has an LTI inverse. (See Problem 2.50.) Therefore, we have the picture shown in Figure 2.26. We are given a system with impulse response $h(t)$. The inverse system, with impulse response $h_1(t)$, results in $w(t) = x(t)$—such that the series interconnection in Figure 2.26(a) is identical to the

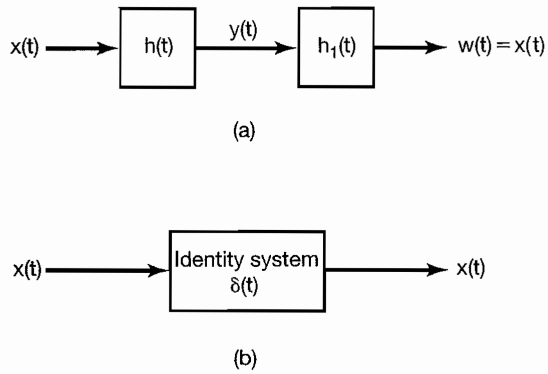

Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

identity system in Figure 2.26(b). Since the overall impulse response in Figure 2.26(a) is $h(t) * h_1(t)$, we have the condition that $h_1(t)$ must satisfy for it to be the impulse response of the inverse system, namely,

$$h(t) * h_1(t) = \delta(t). \tag{2.66}$$

Similarly, in discrete time, the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]. \tag{2.67}$$

The following two examples illustrate invertibility and the construction of an inverse system.

Example 2.11

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0). \tag{2.68}$$

Such a system is a *delay* if $t_0 > 0$ and an *advance* if $t_0 < 0$. For example, if $t_0 > 0$, then the output at time t equals the value of the input at the earlier time $t - t_0$. If $t_0 = 0$, the system in eq. (2.68) is the identity system and thus is memoryless. For any other value of t_0, this system has memory, as it responds to the value of the input at a time other than the current time.

The impulse response for the system can be obtained from eq. (2.68) by taking the input equal to $\delta(t)$, i.e.,

$$h(t) = \delta(t - t_0). \tag{2.69}$$

Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0). \tag{2.70}$$

That is, the convolution of a signal with a shifted impulse simply shifts the signal.

To recover the input from the output, i.e., to invert the system, all that is required is to shift the output back. The system with this compensating time shift is then the inverse system. That is, if we take

$$h_1(t) = \delta(t + t_0),$$

then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

Similarly, a pure time shift in discrete time has the unit impulse response $\delta[n - n_0]$, so that convolving a signal with a shifted impulse is the same as shifting the signal. Furthermore, the inverse of the LTI system with impulse response $\delta[n - n_0]$ is the LTI system that shifts the signal in the opposite direction by the same amount—i.e., the LTI system with impulse response $\delta[n + n_0]$.

Example 2.12

Consider an LTI system with impulse response

$$h[n] = u[n]. \tag{2.71}$$

Using the convolution sum, we can calculate the response of this system to an arbitrary input:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]u[n - k]. \tag{2.72}$$

Since $u[n - k]$ is 0 for $n - k < 0$ and 1 for $n - k \geq 0$, eq. (2.72) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{2.73}$$

That is, this system, which we first encountered in Section 1.6.1 [see eq. (1.92)], is a summer or accumulator that computes the running sum of all the values of the input up to the present time. As we saw in Section 1.6.2, such a system is invertible, and its inverse, as given by eq. (1.99), is

$$y[n] = x[n] - x[n - 1], \tag{2.74}$$

which is simply a *first difference* operation. Choosing $x[n] = \delta[n]$, we find that the impulse response of the inverse system is

$$h_1[n] = \delta[n] - \delta[n - 1]. \tag{2.75}$$

As a check that $h[n]$ in eq. (2.71) and $h_1[n]$ in eq. (2.75) are indeed the impulse responses of LTI systems that are inverses of each other, we can verify eq. (2.67) by direct calculation:

$$\begin{aligned} h[n] * h_1[n] &= u[n] * \{\delta[n] - \delta[n - 1]\} \\ &= u[n] * \delta[n] - u[n] * \delta[n - 1] \\ &= u[n] - u[n - 1] \\ &= \delta[n]. \end{aligned} \tag{2.76}$$

2.3.6 Causality for LTI Systems

In Section 1.6.3, we introduced the property of causality: The output of a causal system depends only on the present and past values of the input to the system. By using the convolution sum and integral, we can relate this property to a corresponding property of the impulse response of an LTI system. Specifically, in order for a discrete-time LTI system to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$. From eq. (2.39), we see that for this to be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero. This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0. \tag{2.77}$$

According to eq. (2.77), the impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality. More generally, as shown in Problem 1.44, causality for a linear system is equivalent to the condition of *initial rest*; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time. It is important to emphasize that the equivalence of causality and the condition of initial rest applies only to linear systems. For example, as discussed in Section 1.6.6, the system $y[n] = 2x[n] + 3$ is not linear. However, it is causal and, in fact, memoryless. On the other hand, if $x[n] = 0$, $y[n] = 3 \neq 0$, so it does not satisfy the condition of initial rest.

For a causal discrete-time LTI system, the condition in eq. (2.77) implies that the convolution sum representation in eq. (2.39) becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k], \tag{2.78}$$

and the alternative equivalent form, eq. (2.43), becomes

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]. \tag{2.79}$$

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0, \tag{2.80}$$

and in this case the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{2.81}$$

Both the accumulator ($h[n] = u[n]$) and its inverse ($h[n] = \delta[n] - \delta[n-1]$), described in Example 2.12, satisfy eq. (2.77) and therefore are causal. The pure time shift with impulse response $h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$ (when the time shift is a delay), but is noncausal for $t_0 < 0$ (in which case the time shift is an advance, so that the output anticipates future values of the input).

Finally, while causality is a property of systems, it is common terminology to refer to a signal as being causal if it is zero for $n < 0$ or $t < 0$. The motivation for this terminology comes from eqs. (2.77) and (2.80): Causality of an LTI system is equivalent to its impulse response being a causal signal.

2.3.7 Stability for LTI Systems

Recall from Section 1.6.4 that a system is *stable* if every bounded input produces a bounded output. In order to determine conditions under which LTI systems are stable, consider an input $x[n]$ that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n. \tag{2.82}$$

Suppose that we apply this input to an LTI system with unit impulse response $h[n]$. Then, using the convolution sum, we obtain an expression for the magnitude of the output:

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|. \tag{2.83}$$

Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers, it follows from eq. (2.83) that

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|. \tag{2.84}$$

From eq. (2.82), $|x[n-k]| < B$ for all values of k and n. Together with eq. (2.84), this implies that

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n. \tag{2.85}$$

From eq. (2.85), we can conclude that if the impulse response is *absolutely summable*, that is, if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty, \tag{2.86}$$

then $y[n]$ is bounded in magnitude, and hence, the system is stable. Therefore, eq. (2.86) is a sufficient condition to guarantee the stability of a discrete-time LTI system. In fact, this condition is also a necessary condition, since, as shown in Problem 2.49, if eq. (2.86) is not satisfied, there are bounded inputs that result in unbounded outputs. Thus, the stability of a discrete-time LTI system is completely equivalent to eq. (2.86).

In continuous time, we obtain an analogous characterization of stability in terms of the impulse response of an LTI system. Specifically, if $|x(t)| < B$ for all t, then, in analogy with eqs. (2.83)–(2.85), it follows that

$$\begin{aligned}|y(t)| &= \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right| \\ &\leq \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau \\ &\leq B\int_{-\infty}^{+\infty} |h(\tau)|d\tau.\end{aligned}$$

Therefore, the system is stable if the impulse response is *absolutely integrable*, i.e., if

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty. \tag{2.87}$$

As in discrete time, if eq. (2.87) is not satisfied, there are bounded inputs that produce unbounded outputs; therefore, the stability of a continuous-time LTI system is equivalent to eq. (2.87). The use of eqs (2.86) and (2.87) to test for stability is illustrated in the next two examples.

Example 2.13

Consider a system that is a pure time shift in either continuous time or discrete time. Then, in discrete time

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n-n_0]| = 1, \tag{2.88}$$

while in continuous time

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \int_{-\infty}^{+\infty} |\delta(\tau-t_0)|d\tau = 1, \tag{2.89}$$

and we conclude that both of these systems are stable. This should not be surprising, since if a signal is bounded in magnitude, so is any time-shifted version of that signal.

Now consider the accumulator described in Example 2.12. As we discussed in Section 1.6.4, this is an unstable system, since, if we apply a constant input to an accumulator, the output grows without bound. That this system is unstable can also be seen from the fact that its impulse response $u[n]$ is not absolutely summable:

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} u[n] = \infty.$$

Similarly, consider the integrator, the continuous-time counterpart of the accumulator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau. \tag{2.90}$$

This is an unstable system for precisely the same reason as that given for the accumulator; i.e., a constant input gives rise to an output that grows without bound. The impulse response for the integrator can be found by letting $x(t) = \delta(t)$, in which case

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

and

$$\int_{-\infty}^{+\infty} |u(\tau)|d\tau = \int_{0}^{+\infty} d\tau = \infty.$$

Since the impulse response is not absolutely integrable, the system is not stable.

2.3.8 The Unit Step Response of an LTI System

Up to now, we have seen that the representation of an LTI system in terms of its unit impulse response allows us to obtain very explicit characterizations of system properties. Specifically, since $h[n]$ or $h(t)$ completely determines the behavior of an LTI system, we have been able to relate system properties such as stability and causality to properties of the impulse response.

There is another signal that is also used quite often in describing the behavior of LTI systems: the *unit step response*, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$. We will find it useful on occasion to refer to the step response, and therefore, it is worthwhile relating it to the impulse response. From the convolution-sum representation, the step response of a discrete-time LTI system is the convolution of the unit step with the impulse response; that is,

$$s[n] = u[n] * h[n].$$

However, by the commutative property of convolution, $s[n] = h[n] * u[n]$, and therefore, $s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$. As we have seen in Example 2.12, $u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{2.91}$$

From this equation and from Example 2.12, it is clear that $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1]. \tag{2.92}$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response [eq. (2.91)]. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response [eq. (2.92)].

Similarly, in continuous time, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \tag{2.93}$$

and from eq. (2.93), the unit impulse response is the first derivative of the unit step response,[1] or

$$h(t) = \frac{ds(t)}{dt} = s'(t). \tag{2.94}$$

Therefore, in both continuous and discrete time, the unit step response can also be used to characterize an LTI system, since we can calculate the unit impulse response from it. In Problem 2.45, expressions analogous to the convolution sum and convolution integral are derived for the representations of an LTI system in terms of its unit step response.

2.4 CAUSAL LTI SYSTEMS DESCRIBED BY DIFFERENTIAL AND DIFFERENCE EQUATIONS

An extremely important class of continuous-time systems is that for which the input and output are related through a *linear constant-coefficient differential equation*. Equations of this type arise in the description of a wide variety of systems and physical phenomena. For example, as we illustrated in Chapter 1, the response of the *RC* circuit in Figure 1.1 and the motion of a vehicle subject to acceleration inputs and frictional forces, as depicted in Figure 1.2, can both be described through linear constant-coefficient differential equations. Similar differential equations arise in the description of mechanical systems containing restoring and damping forces, in the kinetics of chemical reactions, and in many other contexts as well.

Correspondingly, an important class of discrete-time systems is that for which the input and output are related through a *linear constant-coefficient difference equation*. Equations of this type are used to describe the sequential behavior of many different processes. For instance, in Example 1.10 we saw how difference equations arise in describing the accumulation of savings in a bank account, and in Example 1.11 we saw how they can be used to describe a digital simulation of a continuous-time system described by a differential equation. Difference equations also arise quite frequently in the specification of discrete-time systems designed to perform particular operations on the input signal. For example, the system that calculates the difference between successive input values, as in eq. (1.99), and the system described by eq. (1.104) that computes the average value of the input over an interval are described by difference equations.

Throughout this book, there will be many occasions in which we will consider and examine systems described by linear constant-coefficient differential and difference equations. In this section we take a first look at these systems to introduce some of the basic ideas involved in solving differential and difference equations and to uncover and explore some of the properties of systems described by such equations. In subsequent chapters, we develop additional tools for the analysis of signals and systems that will add considerably both to our ability to analyze systems described by such equations and to our understanding of their characteristics and behavior.

[1]Throughout this book, we will use both the notations indicated in eq. (2.94) to denote first derivatives. Analogous notation will also be used for higher derivatives.

2.4.1 Linear Constant-Coefficient Differential Equations

To introduce some of the important ideas concerning systems specified by linear constant-coefficient differential equations, let us consider a first-order differential equation as in eq. (1.85), viz.,

$$\frac{dy(t)}{dt} + 2y(t) = x(t), \tag{2.95}$$

where $y(t)$ denotes the output of the system and $x(t)$ is the input. For example, comparing eq. (2.95) to the differential equation (1.84) for the velocity of a vehicle subject to applied and frictional forces, we see that eq. (2.95) would correspond exactly to this system if $y(t)$ were identified with the vehicle's velocity $v(t)$, if $x(t)$ were taken as the applied force $f(t)$, and if the parameters in eq. (1.84) were normalized in units such that $b/m = 2$ and $1/m = 1$.

A very important point about differential equations such as eq. (2.95) is that they provide an *implicit* specification of the system. That is, they describe a relationship between the input and the output, rather than an explicit expression for the system output as a function of the input. In order to obtain an explicit expression, we must solve the differential equation. To find a solution, we need more information than that provided by the differential equation alone. For example, to determine the speed of an automobile at the end of a 10-second interval when it has been subjected to a constant acceleration of 1 m/sec^2 for 10 seconds, we would also need to know how fast the vehicle was moving at the *start* of the interval. Similarly, if we are told that a constant source voltage of 1 volt is applied to the *RC* circuit in Figure 1.1 for 10 seconds, we cannot determine what the capacitor voltage is at the end of that interval without also knowing what the initial capacitor voltage is.

More generally, to solve a differential equation, we must specify one or more auxiliary conditions, and once these are specified, we can then, in principle, obtain an explicit expression for the output in terms of the input. In other words, a differential equation such as eq. (2.95) describes a constraint between the input and the output of a system, but to characterize the system completely, we must also specify auxiliary conditions. Different choices for these auxiliary conditions then lead to different relationships between the input and the output. For the most part, in this book we will focus on the use of differential equations to describe causal LTI systems, and for such systems the auxiliary conditions take a particular, simple form. To illustrate this and to uncover some of the basic properties of the solutions to differential equations, let us take a look at the solution of eq. (2.95) for a specific input signal $x(t)$.[2]

[2]Our discussion of the solution of linear constant-coefficient differential equations is brief, since we assume that the reader has some familiarity with this material. For review, we recommend a text on the solution of ordinary differential equations, such as *Ordinary Differential Equations* (3rd ed.), by G. Birkhoff and G.-C. Rota (New York: John Wiley and Sons, 1978), or *Elementary Differential Equations* (3rd ed.), by W.E. Boyce and R.C. DiPrima (New York: John Wiley and Sons, 1977). There are also numerous texts that discuss differential equations in the context of circuit theory. See, for example, *Basic Circuit Theory*, by L.O. Chua, C.A. Desoer, and E.S. Kuh (New York: McGraw-Hill Book Company, 1987). As mentioned in the text, in the following chapters we present other very useful methods for solving linear differential equations that will be sufficient for our purposes. In addition, a number of exercises involving the solution of differential equations are included in the problems at the end of the chapter.

Example 2.14

Consider the solution of eq. (2.95) when the input signal is

$$x(t) = Ke^{3t}u(t), \tag{2.96}$$

where K is a real number.

The complete solution to eq. (2.96) consists of the sum of a *particular solution*, $y_p(t)$, and a *homogeneous solution*, $y_h(t)$, i.e.,

$$y(t) = y_p(t) + y_h(t), \tag{2.97}$$

where the particular solution satisfies eq. (2.95) and $y_h(t)$ is a solution of the homogeneous differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0. \tag{2.98}$$

A common method for finding the particular solution for an exponential input signal as in eq. (2.96) is to look for a so-called *forced response*—i.e., a signal of the same form as the input. With regard to eq. (2.95), since $x(t) = Ke^{3t}$ for $t > 0$, we hypothesize a solution for $t > 0$ of the form

$$y_p(t) = Ye^{3t}, \tag{2.99}$$

where Y is a number that we must determine. Substituting eqs. (2.96) and (2.99) into eq. (2.95) for $t > 0$ yields

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}. \tag{2.100}$$

Canceling the factor e^{3t} from both sides of eq. (2.100), we obtain

$$3Y + 2Y = K, \tag{2.101}$$

or

$$Y = \frac{K}{5}, \tag{2.102}$$

so that

$$y_p(t) = \frac{K}{5}e^{3t}, \quad t > 0. \tag{2.103}$$

In order to determine $y_h(t)$, we hypothesize a solution of the form

$$y_h(t) = Ae^{st}. \tag{2.104}$$

Substituting this into eq. (2.98) gives

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s + 2) = 0. \tag{2.105}$$

From this equation, we see that we must take $s = -2$ and that Ae^{-2t} is a solution to eq. (2.98) for *any* choice of A. Utilizing this fact and eq. (2.103) in eq. (2.97), we find that the solution of the differential equation for $t > 0$ is

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \quad t > 0. \tag{2.106}$$

As noted earlier, the differential equation (2.95) by itself does not specify uniquely the response $y(t)$ to the input $x(t)$ in eq. (2.96). In particular, the constant A in eq. (2.106) has not yet been determined. In order for the value of A to be determined, we need to specify an auxiliary condition in addition to the differential equation (2.95). As explored in Problem 2.34, different choices for this auxiliary condition lead to different solutions $y(t)$ and, consequently, to different relationships between the input and the output. As we have indicated, for the most part in this book we focus on differential and difference equations used to describe systems that are LTI and causal, and in this case the auxiliary condition takes the form of the condition of initial rest. That is, as shown in Problem 1.44, for a causal LTI system, if $x(t) = 0$ for $t < t_0$, then $y(t)$ must also equal 0 for $t < t_0$. From eq. (2.96), we see that for our example $x(t) = 0$ for $t < 0$, and thus, the condition of initial rest implies that $y(t) = 0$ for $t < 0$. Evaluating eq. (2.106) at $t = 0$ and setting $y(0) = 0$ yields

$$0 = A + \frac{K}{5},$$

or

$$A = -\frac{K}{5}.$$

Thus, for $t > 0$,

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right], \tag{2.107}$$

while for $t < 0$, $y(t) = 0$, because of the condition of initial rest. Combining these two cases, we obtain the full solution

$$y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right]u(t). \tag{2.108}$$

Example 2.14 illustrates several very important points concerning linear constant-coefficient differential equations and the systems they represent. First, the response to an input $x(t)$ will generally consist of the sum of a particular solution to the differential equation and a homogeneous solution—i.e., a solution to the differential equation with the input set to zero. The homogeneous solution is often referred to as the *natural response* of the system. The natural responses of simple electrical circuits and mechanical systems are explored in Problems 2.61 and 2.62.

In Example 2.14 we also saw that, in order to determine completely the relationship between the input and the output of a system described by a differential equation such as eq. (2.95), we must specify auxiliary conditions. An implication of this fact, which is illustrated in Problem 2.34, is that different choices of auxiliary conditions lead to different relationships between the input and the output. As we illustrated in the example, for the most part we will use the condition of initial rest for systems described by differential equations. In the example, since the input was 0 for $t < 0$, the condition of initial rest implied the initial condition $y(0) = 0$. As we have stated, and as illustrated in

Problem 2.33, under the condition of initial rest the system described by eq. (2.95) is LTI and causal.[3] For example, if we multiply the input in eq. (2.96) by 2, the resulting output would be twice the output in eq. (2.108).

It is important to emphasize that the condition of initial rest does not specify a zero initial condition at a fixed point in time, but rather adjusts this point in time so that the response is zero *until* the input becomes nonzero. Thus, if $x(t) = 0$ for $t \leq t_0$ for the causal LTI system described by eq. (2.95), then $y(t) = 0$ for $t \leq t_0$, and we would use the initial condition $y(t_0) = 0$ to solve for the output for $t > t_0$. As a physical example, consider again the circuit in Figure 1.1, also discussed in Example 1.8. Initial rest for this example corresponds to the statement that, until we connect a nonzero voltage source to the circuit, the capacitor voltage is zero. Thus, if we begin to use the circuit at noon today, the initial capacitor voltage as we connect the voltage source at noon today is zero. Similarly, if we begin to use the circuit at noon tomorrow instead, the initial capacitor voltage as we connect the voltage source at noon tomorrow is zero.

This example also provides us with some intuition as to why the condition of initial rest makes a system described by a linear constant-coefficient differential equation time invariant. For example, if we perform an experiment on the circuit, starting from initial rest, then, assuming that the coefficients R and C don't change over time, we would expect to get the same results whether we ran the experiment today or tomorrow. That is, if we perform identical experiments on the two days, where the circuit starts from initial rest at noon on each day, then we would expect to see identical responses—i.e., responses that are simply time-shifted by one day with respect to each other.

While we have used the first-order differential equation (2.95) as the vehicle for the discussion of these issues, the same ideas extend directly to systems described by higher order differential equations. A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{2.109}$$

The order refers to the highest derivative of the output $y(t)$ appearing in the equation. In the case when $N = 0$, eq. (2.109) reduces to

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{2.110}$$

In this case, $y(t)$ is an explicit function of the input $x(t)$ and its derivatives. For $N \geq 1$, eq. (2.109) specifies the output implicitly in terms of the input. In this case, the analysis of the equation proceeds just as in our discussion of the first-order differential equation in Example 2.14. The solution $y(t)$ consists of two parts—a particular solution to eq. (2.109)

[3]In fact, as is also shown in Problem 2.34, if the initial condition for eq. (2.95) is nonzero, the resulting system is incrementally linear. That is, the overall response can be viewed, much as in Figure 1.48, as the superposition of the response to the initial conditions alone (with input set to 0) and the response to the input with an initial condition of 0 (i.e., the response of the causal LTI system described by eq. (2.95)).

plus a solution to the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \tag{2.111}$$

The solutions to this equation are referred to as the *natural responses* of the system.

As in the first-order case, the differential equation (2.109) does not completely specify the output in terms of the input, and we need to identify auxiliary conditions to determine completely the input-output relationship for the system. Once again, different choices for these auxiliary conditions result in different input-output relationships, but for the most part, in this book we will use the condition of initial rest when dealing with systems described by differential equations. That is, if $x(t) = 0$ for $t \leq t_0$, we assume that $y(t) = 0$ for $t \leq t_0$, and therefore, the response for $t > t_0$ can be calculated from the differential equation (2.109) with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0. \tag{2.112}$$

Under the condition of initial rest, the system described by eq. (2.109) is causal and LTI. Given the initial conditions in eq. (2.112), the output $y(t)$ can, in principle, be determined by solving the differential equation in the manner used in Example 2.14 and further illustrated in several problems at the end of the chapter. However, in Chapters 4 and 9 we will develop some tools for the analysis of continuous-time LTI systems that greatly facilitate the solution of differential equations and, in particular, provide us with powerful methods for analyzing and characterizing the properties of systems described by such equations.

2.4.2 Linear Constant-Coefficient Difference Equations

The discrete-time counterpart of eq. (2.109) is the Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{2.113}$$

An equation of this type can be solved in a manner exactly analogous to that for differential equations. (See Problem 2.32.)[4] Specifically, the solution $y[n]$ can be written as the sum of a particular solution to eq. (2.113) and a solution to the homogeneous equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0. \tag{2.114}$$

[4]For a detailed treatment of the methods for solving linear constant-coefficient difference equations, see *Finite Difference Equations*, by H. Levy and F. Lessman (New York: Macmillan, Inc., 1961), or *Finite Difference Equations and Simulations* (Englewood Cliffs, NJ: Prentice-Hall, 1968) by F. B. Hildebrand. In Chapter 6, we present another method for solving difference equations that greatly facilitates the analysis of linear time-invariant systems that are so described. In addition, we refer the reader to the problems at the end of this chapter that deal with the solution of difference equations.

The solutions to this homogeneous equation are often referred to as the natural responses of the system described by eq. (2.113).

As in the continuous-time case, eq. (2.113) does not completely specify the output in terms of the input. To do this, we must also specify some auxiliary conditions. While there are many possible choices for auxiliary conditions, leading to different input-output relationships, we will focus for the most part on the condition of initial rest—i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$ as well. With initial rest, the system described by eq. (2.113) is LTI and causal.

Although all of these properties can be developed following an approach that directly parallels our discussion for differential equations, the discrete-time case offers an alternative path. This stems from the observation that eq. (2.113) can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}. \tag{2.115}$$

Equation (2.115) directly expresses the output at time n in terms of previous values of the input and output. From this, we can immediately see the need for auxiliary conditions. In order to calculate $y[n]$, we need to know $y[n-1], \ldots, y[n-N]$. Therefore, if we are given the input for all n and a set of auxiliary conditions such as $y[-N], y[-N+1], \ldots, y[-1]$, eq. (2.115) can be solved for successive values of $y[n]$.

An equation of the form of eq. (2.113) or eq. (2.115) is called a *recursive equation*, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case when $N = 0$, eq. (2.115) reduces to

$$y[n] = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-k]. \tag{2.116}$$

This is the discrete-time counterpart of the continuous-time system given in eq. (2.110). Here, $y[n]$ is an explicit function of the present and previous values of the input. For this reason, eq. (2.116) is often called a *nonrecursive equation*, since we do not recursively use previously computed values of the output to compute the present value of the output. Therefore, just as in the case of the system given in eq. (2.110), we do not need auxiliary conditions in order to determine $y[n]$. Furthermore, eq. (2.116) describes an LTI system, and by direct computation, the impulse response of this system is found to be

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}. \tag{2.117}$$

That is, eq. (2.116) is nothing more than the convolution sum. Note that the impulse response for it has finite duration; that is, it is nonzero only over a finite time interval. Because of this property, the system specified by eq. (2.116) is often called a *finite impulse response (FIR) system.*

Although we do not require auxiliary conditions for the case of $N = 0$, such conditions are needed for the recursive case when $N \ge 1$. To illustrate the solution of such an equation, and to gain some insight into the behavior and properties of recursive difference equations, let us examine the following simple example:

Example 2.15

Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]. \tag{2.118}$$

Eq. (2.118) can also be expressed in the form

$$y[n] = x[n] + \frac{1}{2}y[n-1], \tag{2.119}$$

highlighting the fact that we need the previous value of the output, $y[n-1]$, to calculate the current value. Thus, to begin the recursion, we need an initial condition.

For example, suppose that we impose the condition of initial rest and consider the input

$$x[n] = K\delta[n]. \tag{2.120}$$

In this case, since $x[n] = 0$ for $n \leq -1$, the condition of initial rest implies that $y[n] = 0$ for $n \leq -1$, so that we have as an initial condition $y[-1] = 0$. Starting from this initial condition, we can solve for successive values of $y[n]$ for $n \geq 0$ as follows:

$$y[0] = x[0] + \frac{1}{2}y[-1] = K, \tag{2.121}$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K, \tag{2.122}$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K, \tag{2.123}$$

$$\vdots$$

$$y[n] = x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K. \tag{2.124}$$

Since the system specified by eq. (2.118) and the condition of initial rest is LTI, its input-output behavior is completely characterized by its impulse response. Setting $K = 1$, we see that the impulse response for the system considered in this example is

$$h[n] = \left(\frac{1}{2}\right)^n u[n]. \tag{2.125}$$

Note that the causal LTI system in Example 2.15 has an impulse response of infinite duration. In fact, if $N \geq 1$ in eq. (2.113), so that the difference equation is recursive, it is usually the case that the LTI system corresponding to this equation together with the condition of initial rest will have an impulse response of infinite duration. Such systems are commonly referred to as *infinite impulse response (IIR) systems.*

As we have indicated, for the most part we will use recursive difference equations in the context of describing and analyzing systems that are linear, time-invariant, and causal, and consequently, we will usually make the assumption of initial rest. In Chapters 5 and 10 we will develop tools for the analysis of discrete-time systems that will provide us with very useful and efficient methods for solving linear constant-coefficient difference equations and for analyzing the properties of the systems that they describe.

2.4.3 Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

An important property of systems described by linear constant-coefficient difference and differential equations is that they can be represented in very simple and natural ways in terms of block diagram interconnections of elementary operations. This is significant for a number of reasons. One is that it provides a pictorial representation which can add to our understanding of the behavior and properties of these systems. In addition, such representations can be of considerable value for the simulation or implementation of the systems. For example, the block diagram representation to be introduced in this section for continuous-time systems is the basis for early analog computer simulations of systems described by differential equations, and it can also be directly translated into a program for the simulation of such a system on a digital computer. In addition, the corresponding representation for discrete-time difference equations suggests simple and efficient ways in which the systems that the equations describe can be implemented in digital hardware. In this section, we illustrate the basic ideas behind these block diagram representations by constructing them for the causal first-order systems introduced in Examples 1.8–1.11. In Problems 2.57–2.60 and Chapters 9 and 10, we consider block diagrams for systems described by other, more complex differential and difference equations.

We begin with the discrete-time case and, in particular, the causal system described by the first-order difference equation

$$y[n] + ay[n-1] = bx[n]. \tag{2.126}$$

To develop a block diagram representation of this system, note that the evaluation of eq. (2.126) requires three basic operations: addition, multiplication by a coefficient, and delay (to capture the relationship between $y[n]$ and $y[n-1]$). Thus, let us define three basic network elements, as indicated in Figure 2.27. To see how these basic elements can be used to represent the causal system described by eq. (2.126), we rewrite this equation in the form that directly suggests a recursive algorithm for computing successive values of the output $y[n]$:

$$y[n] = -ay[n-1] + bx[n]. \tag{2.127}$$

This algorithm is represented pictorially in Figure 2.28, which is an example of a feedback system, since the output is fed back through a delay and a multiplication by a coefficient and is then added to $bx[n]$. The presence of feedback is a direct consequence of the recursive nature of eq. (2.127).

The block diagram in Figure 2.28 makes clear the required memory in this system and the consequent need for initial conditions. In particular, a delay corresponds to a memory element, as the element must retain the previous value of its input. Thus, the initial value of this memory element serves as a necessary initial condition for the recursive calculation specified pictorially in Figure 2.28 and mathematically in eq. (2.127). Of course, if the system described by eq. (2.126) is initially at rest, the initial value stored in the memory element is zero.

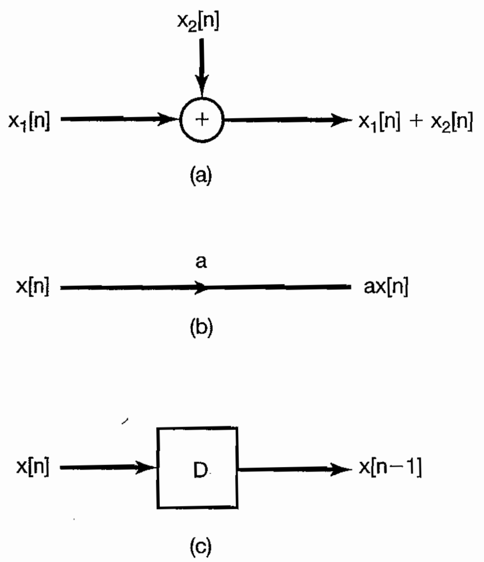

Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

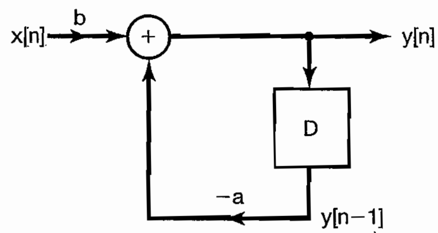

Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t). \tag{2.128}$$

As a first attempt at defining a block diagram representation for this system, let us rewrite it as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t). \tag{2.129}$$

The right-hand side of this equation involves three basic operations: addition, multiplication by a coefficient, and differentiation. Therefore, if we define the three basic network elements indicated in Figure 2.29, we can consider representing eq. (2.129) as an interconnection of these basic elements in a manner analogous to that used for the discrete-time system described previously, resulting in the block diagram of Figure 2.30.

While the latter figure is a valid representation of the causal system described by eq. (2.128), it is not the representation that is most frequently used or the representation that leads directly to practical implementations, since differentiators are both difficult to implement and extremely sensitive to errors and noise. An alternative implementation that

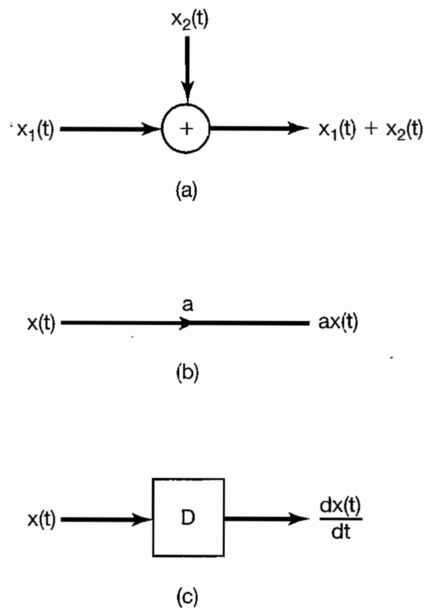

Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

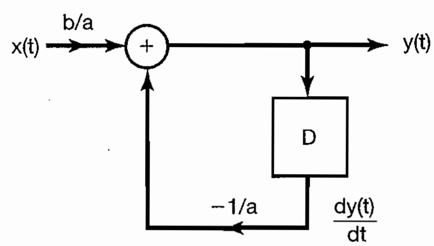

Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

is much more widely used can be obtained by first rewriting eq. (2.128) as

$$\frac{dy(t)}{dt} = bx(t) - ay(t) \tag{2.130}$$

and then integrating from $-\infty$ to t. Specifically, if we assume that in the system described by eq. (2.130) the value of $y(-\infty)$ is zero, then the integral of $dy(t)/dt$ from $-\infty$ to t is precisely $y(t)$. Consequently, we obtain the equation

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]\, d\tau. \tag{2.131}$$

In this form, our system can be implemented using the adder and coefficient multiplier indicated in Figure 2.29, together with an *integrator*, as defined in Figure 2.31. Figure 2.32 is a block diagram representation for this system using these elements.

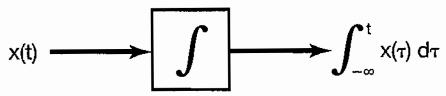

Figure 2.31 Pictorial representation of an integrator.

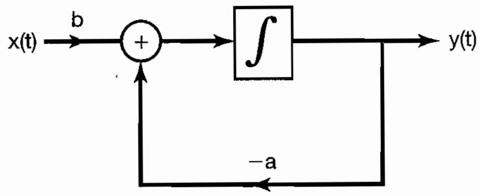

Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

Since integrators can be readily implemented using operational amplifiers, representations such as that in Figure 2.32 lead directly to analog implementations, and indeed, this is the basis for both early analog computers and modern analog computation systems. Note that in the continuous-time case it is the integrator that represents the memory storage element of the system. This is perhaps more readily seen if we consider integrating eq. (2.130) from a finite point in time t_0, resulting in the expression

$$y(t) = y(t_0) + \int_{t_0}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau. \tag{2.132}$$

Equation (2.132) makes clear the fact that the specification of $y(t)$ requires an initial condition, namely, the value of $y(t_0)$. It is precisely this value that the integrator stores at time t_0.

While we have illustrated block diagram constructions only for the simplest first-order differential and difference equations, such block diagrams can also be developed for higher order systems, providing both valuable intuition for and possible implementations of these systems. Examples of block diagrams for higher order systems can be found in Problems 2.58 and 2.60.

5 SINGULARITY FUNCTIONS

In this section, we take another look at the continuous-time unit impulse function in order to gain additional intuitions about this important idealized signal and to introduce a set of related signals known collectively as *singularity functions*. In particular, in Section 1.4.2 we suggested that a continuous-time unit impulse could be viewed as the idealization of a pulse that is "short enough" so that its shape and duration is of no practical consequence—i.e., so that as far as the response of any particular LTI system is concerned, all of the area under the pulse can be thought of as having been applied instantaneously. In this section, we would first like to provide a concrete example of what this means and then use the interpretation embodied within the example to show that the key to the use of unit impulses and other singularity functions is in the specification of how LTI systems respond to these idealized signals; i.e., the signals are in essence defined in terms of how they behave under convolution with other signals.