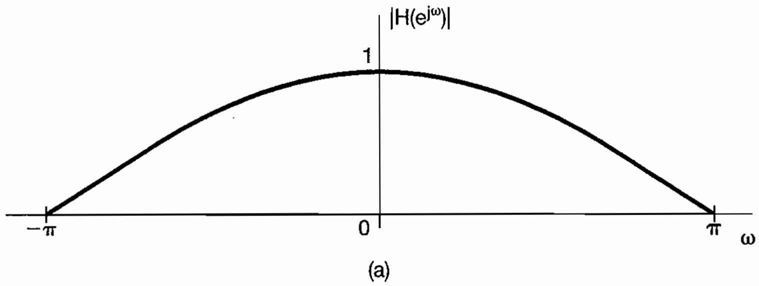


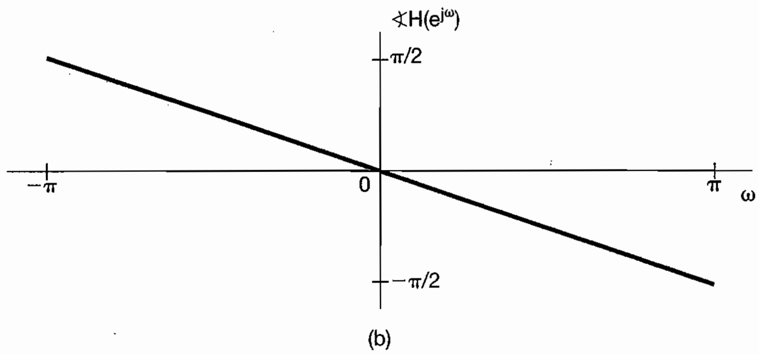


Figure 3.25 (a) Magnitude and (b) phase for the frequency response of the discrete-time LTI system $y[n] = 1/2(x[n] + x[n-1])$.

$x[n] = Ke^{j0\cdot n} = K$—then the output will be

$$y[n] = H(e^{j\cdot 0})Ke^{j\omega 0\cdot n} = K = x[n].$$

On the other hand, if the input is the high-frequency signal $x[n] = Ke^{j\pi n} = K(-1)^n$, then the output will be

$$y[n] = H(e^{j\pi})Ke^{j\pi\cdot n} = 0.$$

Thus, this system separates out the long-term constant value of a signal from its high-frequency fluctuations and, consequently, represents a first example of frequency-selective filtering, a topic we look at more carefully in the next subsection.

3.9.2 Frequency-Selective Filters

Frequency-selective filters are a class of filters specifically intended to accurately or approximately select some bands of frequencies and reject others. The use of frequency-selective filters arises in a variety of situations. For example, if noise in an audio recording is in a higher frequency band than the music or voice on the recording is, it can be removed by frequency-selective filtering. Another important application of frequency-selective filters is in communication systems. As we discuss in detail in Chapter 8, the basis for amplitude modulation (AM) systems is the transmission of information from many different sources simultaneously by putting the information from each channel into a separate frequency band and extracting the individual channels or bands at the receiver using frequency-selective filters. Frequency-selective filters for separating the individual channels and frequency-shaping filters (such as the equalizer illustrated in Figure 3.22) for adjusting the quality of the tone form a major part of any home radio and television receiver.

While frequency selectivity is not the only issue of concern in applications, its broad importance has led to a widely accepted set of terms describing the characteristics of frequency-selective filters. In particular, while the nature of the frequencies to be passed by a frequency-selective filter varies considerably from application to application, several basic types of filter are widely used and have been given names indicative of their function. For example, a *lowpass filter* is a filter that passes low frequencies—i.e., frequencies around $\omega = 0$—and attenuates or rejects higher frequencies. A *highpass filter* is a filter that passes high frequencies and attentuates or rejects low ones, and a *bandpass filter* is a filter that passes a band of frequencies and attenuates frequencies both higher and lower than those in the band that is passed. In each case, the *cutoff frequencies* are the frequencies defining the boundaries between frequencies that are passed and frequencies that are rejected—i.e., the frequencies in the *passband* and *stopband.*

Numerous questions arise in defining and assessing the quality of a frequency-selective filter. How effective is the filter at passing frequencies in the passband? How effective is it at attentuating frequencies in the stopband? How sharp is the transition near the cutoff frequency—i.e., from nearly free of distortion in the passband to highly attenuated in the stopband? Each of these questions involves a comparison of the characteristics of an actual frequency-selective filter with those of a filter with idealized behavior. Specifically, an *ideal frequency-selective filter* is a filter that exactly passes complex exponentials at one set of frequencies without any distortion and completely rejects signals at all other frequencies. For example, a continuous-time *ideal lowpass filter* with cutoff frequency ω_c is an LTI system that passes complex exponentials $e^{j\omega t}$ for values of ω in the range $-\omega_c \leq \omega \leq \omega_c$ and rejects signals at all other frequencies. That is, the frequency response of a continuous-time ideal lowpass filter is

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \tag{3.140}$$

as shown in Figure 3.26.

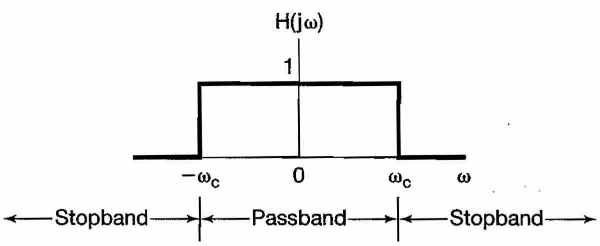


Figure 3.26 Frequency response of an ideal lowpass filter.

Figure 3.27(a) depicts the frequency response of an ideal continuous-time highpass filter with cutoff frequency ω_c, and Figure 3.27(b) illustrates an ideal continuous-time bandpass filter with lower cutoff frequency ω_{c1} and upper cutoff frequency ω_{c2}. Note that each of these filters is symmetric about $\omega = 0$, and thus, there appear to be two passbands for the highpass and bandpass filters. This is a consequence of our having adopted the

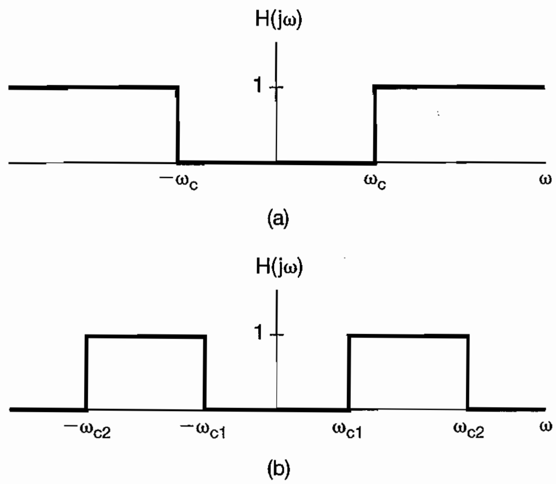


Figure 3.27 (a) Frequency response of an ideal highpass filter; (b) frequency response of an ideal bandpass filter.

use of the complex exponential signal $e^{j\omega t}$, rather than the sinusoidal signals $\sin \omega t$ and $\cos \omega t$, at frequency ω. Since $e^{j\omega t} = \cos \omega t + j \sin \omega t$ and $e^{-j\omega t} = \cos \omega t - j \sin \omega t$, both of these complex exponentials are composed of sinusoidal signals at the same frequency ω. For this reason, we usually define ideal filters so that they have the symmetric frequency response behavior seen in Figures 3.26 and 3.27.

In a similar fashion, we can define the corresponding set of ideal discrete-time frequency-selective filters, the frequency responses for which are depicted in Figure 3.28.

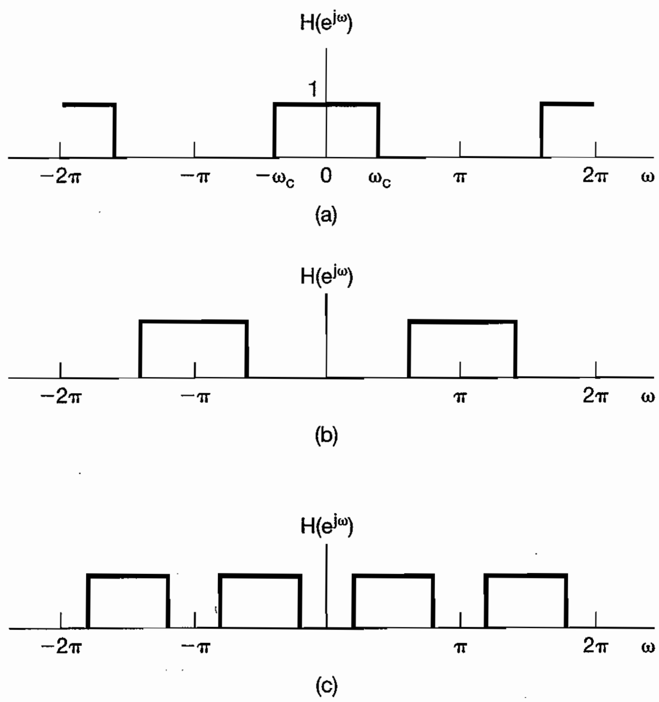


Figure 3.28 Discrete-time ideal frequency-selective filters: (a) lowpass; (b) highpass; (c) bandpass.

In particular, Figure 3.28(a) depicts an ideal discrete-time lowpass filter, Figure 3.28(b) is an ideal highpass filter, and Figure 3.28(c) is an ideal bandpass filter. Note that, as discussed in the preceding section, the characteristics of the continuous-time and discrete-time ideal filters differ by virtue of the fact that, for discrete-time filters, the frequency response $H(e^{j\omega})$ must be periodic with period 2π, with low frequencies near even multiples of π and high frequencies near odd multiples of π.

As we will see on numerous occasions, ideal filters are quite useful in describing idealized system configurations for a variety of applications. However, they are not realizable in practice and must be approximated. Furthermore, even if they could be realized, some of the characteristics of ideal filters might make them undesirable for particular applications, and a nonideal filter might in fact be preferable.

In detail, the topic of filtering encompasses many issues, including design and implementation. While we will not delve deeply into the details of filter design methodologies, in the remainder of this chapter and the following chapters we will see a number of other examples of both continuous-time and discrete-time filters and will develop the concepts and techniques that form the basis of this very important engineering discipline.

3.10 EXAMPLES OF CONTINUOUS-TIME FILTERS DESCRIBED BY DIFFERENTIAL EQUATIONS

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations. The reasons for this are numerous. For example, many physical systems that can be interpreted as performing filtering operations are characterized by differential or difference equations. A good example of this that we will examine in Chapter 6 is an automobile suspension system, which in part is designed to filter out high-frequency bumps and irregularities in road surfaces. A second reason for the use of filters described by differential or difference equations is that they are conveniently implemented using either analog or digital hardware. Furthermore, systems described by differential or difference equations offer an extremely broad and flexible range of designs, allowing one, for example, to produce filters that are close to ideal or that possess other desirable characteristics. In this and the next section, we consider several examples that illustrate the implementation of continuous-time and discrete-time frequency-selective filters through the use of differential and difference equations. In Chapters 4–6, we will see other examples of these classes of filters and will gain additional insights into the properties that make them so useful.

3.10.1 A Simple RC Lowpass Filter

Electrical circuits are widely used to implement continuous-time filtering operations. One of the simplest examples of such a circuit is the first-order *RC* circuit depicted in Figure 3.29, where the source voltage $v_s(t)$ is the system input. This circuit can be used to perform either a lowpass or highpass filtering operation, depending upon what we take as the output signal. In particular, suppose that we take the capacitor voltage $v_c(t)$ as the output. In this case, the output voltage is related to the input voltage through the linear

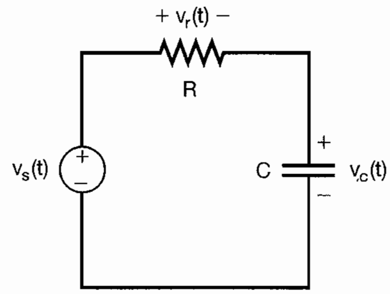


Figure 3.29 First-order *RC* filter.

constant-coefficient differential equation

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t). \tag{3.141}$$

Assuming initial rest, the system described by eq. (3.141) is LTI. In order to determine its frequency response $H(j\omega)$, we note that, by definition, with input voltage $v_s(t) = e^{j\omega t}$, we must have the output voltage $v_c(t) = H(j\omega)e^{j\omega t}$. If we substitute these expressions into eq. (3.141), we obtain

$$RC\frac{d}{dt}[H(j\omega)e^{j\omega t}] + H(j\omega)e^{j\omega t} = e^{j\omega t}, \tag{3.142}$$

or

$$RC\,j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}, \tag{3.143}$$

from which it follows directly that

$$H(j\omega)e^{j\omega t} = \frac{1}{1 + RCj\omega}e^{j\omega t}, \tag{3.144}$$

or

$$H(j\omega) = \frac{1}{1 + RCj\omega}. \tag{3.145}$$

The magnitude and phase of the frequency response $H(j\omega)$ for this example are shown in Figure 3.30. Note that for frequencies near $\omega = 0$, $|H(j\omega)| \approx 1$, while for larger values of ω (positive or negative), $|H(j\omega)|$ is considerably smaller and in fact steadily decreases as $|\omega|$ increases. Thus, this simple *RC* filter (with $v_c(t)$ as output) is a nonideal lowpass filter.

To provide a first glimpse at the trade-offs involved in filter design, let us briefly consider the time-domain behavior of the circuit. In particular, the impulse response of the system described by eq. (3.141) is

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t), \tag{3.146}$$

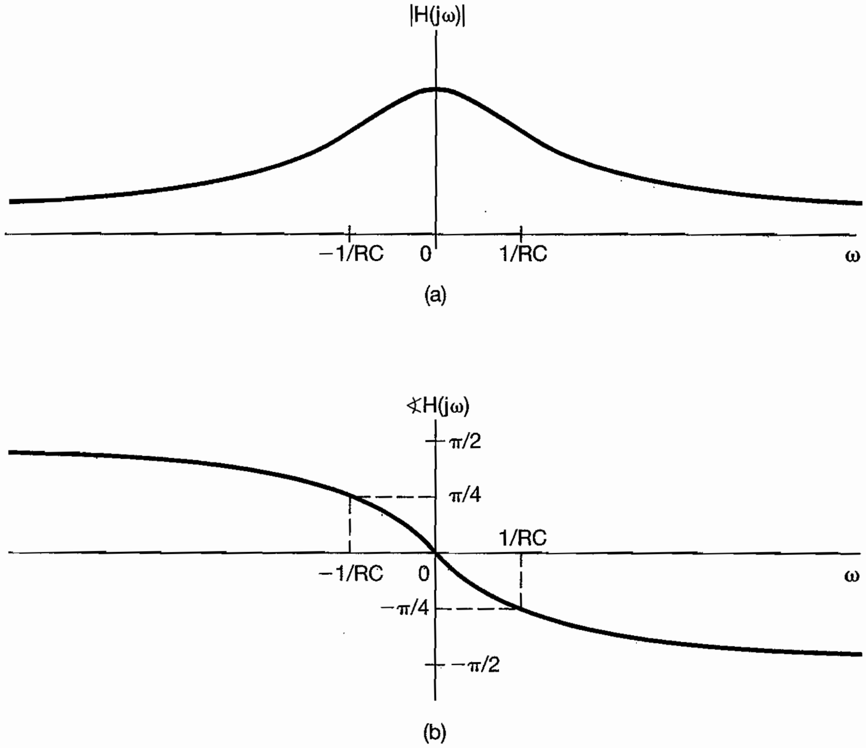


Figure 3.30 (a) Magnitude and (b) phase plots for the frequency response for the *RC* circuit of Figure 3.29 with output $v_c(t)$.

and the step response is

$$s(t) = [1 - e^{-t/RC}]u(t), \tag{3.147}$$

both of which are plotted in Figure 3.31 (where $\tau = RC$). Comparing Figures 3.30 and 3.31, we see a fundamental trade-off. Specifically, suppose that we would like our filter to pass only very low frequencies. From Figure 3.30(a), this implies that $1/RC$ must be small, or equivalently, that RC is large, so that frequencies other than the low ones of interest will be attentuated sufficiently. However, looking at Figure 3.31(b), we see that if RC is large, then the step response will take a considerable amount of time to reach its long-term value of 1. That is, the system responds sluggishly to the step input. Conversely, if we wish to have a faster step response, we need a smaller value of RC, which in turn implies that the filter will pass higher frequencies. This type of trade-off between behavior in the frequency domain and in the time domain is typical of the issues arising in the design and analysis of LTI systems and filters and is a subject we will look at more carefully in Chapter 6.

3.10.2 A Simple RC Highpass Filter

As an alternative to choosing the capacitor voltage as the output in our *RC* circuit, we can choose the voltage across the resistor. In this case, the differential equation relating input

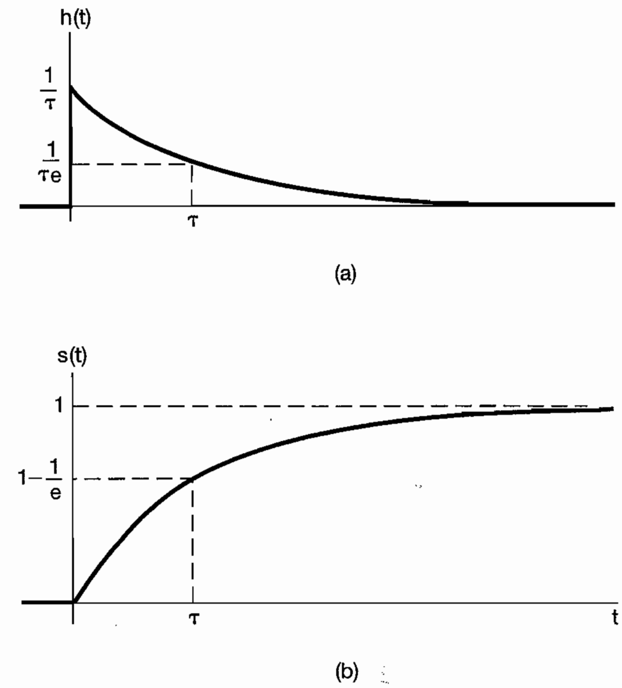


Figure 3.31 (a) Impulse response of the first-order *RC* lowpass filter with $\tau = RC$; (b) step response of *RC* lowpass filter with $\tau = RC$.

and output is

$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}. \tag{3.148}$$

We can find the frequency response $G(j\omega)$ of this system in exactly the same way we did in the previous case: If $v_s(t) = e^{j\omega t}$, then we must have $v_r(t) = G(j\omega)e^{j\omega t}$; substituting these expressions into eq. (3.148) and performing a bit of algebra, we find that

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}. \tag{3.149}$$

The magnitude and phase of this frequency response are shown in Figure 3.32. From the figure, we see that the system attenuates lower frequencies and passes higher frequencies—i.e., those for which $|\omega| \gg 1/RC$—with minimal attenuation. That is, this system acts as a nonideal highpass filter.

As with the lowpass filter, the parameters of the circuit control both the frequency response of the highpass filter and its time response characteristics. For example, consider the step response for the filter. From Figure 3.29, we see that $v_r(t) = v_s(t) - v_c(t)$. Thus, if $v_s(t) = u(t)$, $v_c(t)$ must be given by eq. (3.147). Consequently, the step response of the highpass filter is

$$v_r(t) = e^{-t/RC}u(t), \tag{3.150}$$

which is depicted in Figure 3.33. Consequently, as *RC* is increased, the response becomes more sluggish—i.e., the step response takes a longer time to reach its long-term value

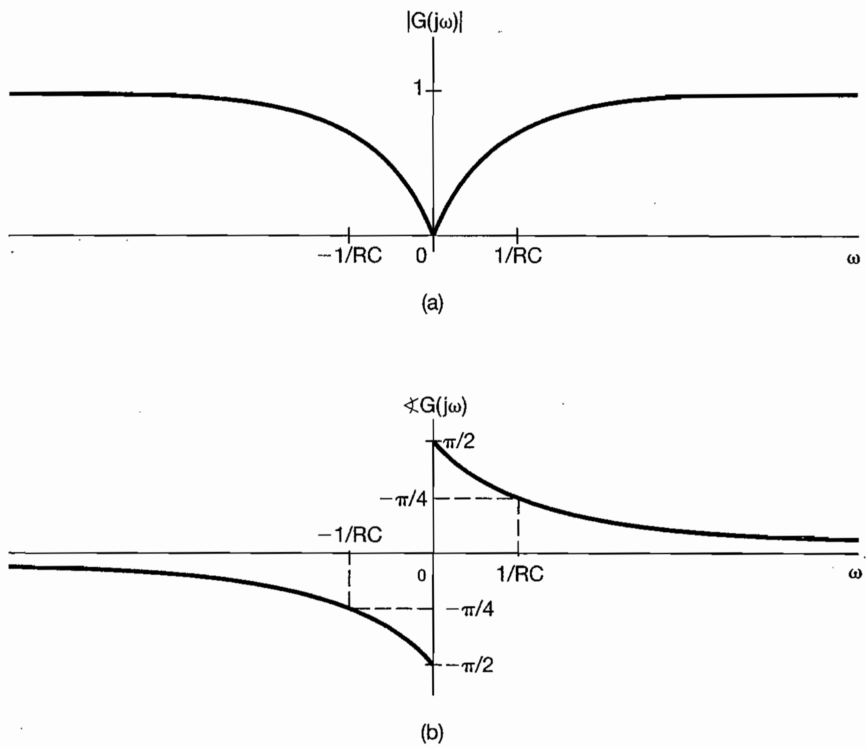


Figure 3.32 (a) Magnitude and (b) phase plots for the frequency response of the *RC* circuit of Figure 3.29 with output $v_r(t)$.

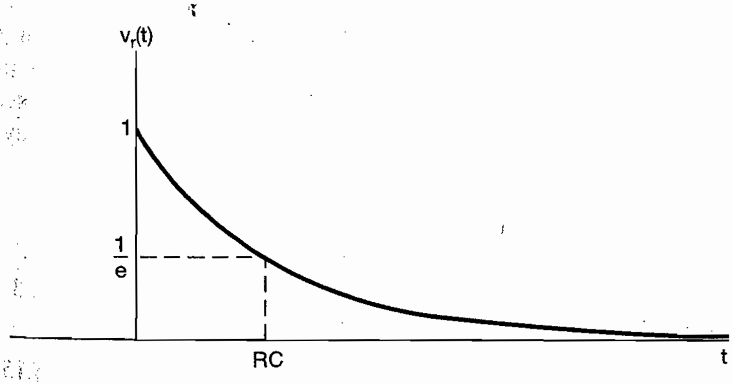


Figure 3.33 Step response of the first-order *RC* highpass filter with $\tau = RC$.

of 0. From Figure 3.32, we see that increasing RC (or equivalently, decreasing $1/RC$) also affects the frequency response, specifically, it extends the passband down to lower frequencies.

We observe from the two examples in this section that a simple RC circuit can serve as a rough approximation to a highpass or a lowpass filter, depending upon the choice of the physical output variable. As illustrated in Problem 3.71, a simple mechanical system using a mass and a mechanical damper can also serve as a lowpass or highpass filter described by analogous first-order differential equations. Because of their simplicity, these examples of electrical and mechanical filters do not have a sharp transition from passband to stopband and, in fact, have only a single parameter (namely, RC in the electrical case) that controls both the frequency response and time response behavior of the system. By designing more complex filters, implemented using more energy storage elements (capacitances and inductances in electrical filters and springs and damping devices in mechanical filters), we obtain filters described by higher order differential equations. Such filters offer considerably more flexibility in terms of their characteristics, allowing, for example, sharper passband-stopband transition or more control over the trade-offs between time response and frequency response.

3.11 EXAMPLES OF DISCRETE-TIME FILTERS DESCRIBED BY DIFFERENCE EQUATIONS

As with their continuous-time counterparts, discrete-time filters described by linear constant-coefficient difference equations are of considerable importance in practice. Indeed, since they can be efficiently implemented in special- or general-purpose digital systems, filters described by difference equations are widely used in practice. As in almost all aspects of signal and system analysis, when we examine discrete-time filters described by difference equations, we find both strong similarities and important differences with the continuous-time case. In particular, discrete-time LTI systems described by difference equations can either be recursive and have impulse responses of infinite length (IIR systems) or be nonrecursive and have finite-length impulse responses (FIR systems). The former are the direct counterparts of continuous-time systems described by differential equations illustrated in the previous section, while the latter are also of considerable practical importance in digital systems. These two classes have distinct sets of advantages and disadvantages in terms of ease of implementation and in terms of the order of filter or the complexity required to achieve particular design objectives. In this section we limit ourselves to a few simple examples of recursive and nonrecursive filters, while in Chapters 5 and 6 we develop additional tools and insights that allow us to analyze and understand the properties of these systems in more detail.

3.11.1 First-Order Recursive Discrete-Time Filters

The discrete-time counterpart of each of the first-order filters considered in Section 3.10 is the LTI system described by the first-order difference equation

$$y[n] - ay[n-1] = x[n]. \tag{3.151}$$

From the eigenfunction property of complex exponential signals, we know that if $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$, where $H(e^{j\omega})$ is the frequency response of the system. Substituting into eq. (3.151), we obtain

$$H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}, \tag{3.152}$$

or

$$[1 - ae^{-j\omega}]H(e^{j\omega})e^{j\omega n} = e^{j\omega n}, \tag{3.153}$$

so that

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{3.154}$$

The magnitude and phase of $H(e^{j\omega})$ are shown in Figure 3.34(a) for $a = 0.6$ and in Figure 3.34(b) for $a = -0.6$. We observe that, for the positive value of a, the difference equation (3.151) behaves like a lowpass filter with minimal attenuation of low frequencies near $\omega = 0$ and increasing attenuation as we increase ω toward $\omega = \pi$. For the negative value of a, the system is a highpass filter, passing frequencies near $\omega = \pi$ and attenuating lower frequencies. In fact, for any positive value of $a < 1$, the system approximates a lowpass filter, and for any negative value of $a > -1$, the system approximates a highpass filter, where $|a|$ controls the size of the filter passband, with broader passbands as $|a|$ is decreased.

As with the continuous-time examples, we again have a trade-off between time domain and frequency domain characteristics. In particular, the impulse response of the system described by eq. (3.151) is

$$h[n] = a^n u[n]. \tag{3.155}$$

The step response $s[n] = u[n] * h[n]$ is

$$s[n] = \frac{1 - a^{n+1}}{1 - a} u[n]. \tag{3.156}$$

From these expressions, we see that $|a|$ also controls the speed with which the impulse and step responses approach their long-term values, with faster responses for smaller values of $|a|$, and hence, for filters with smaller passbands. Just as with differential equations, higher order recursive difference equations can be used to provide sharper filter characteristics and to provide more flexibility in balancing time-domain and frequency-domain constraints.

Finally, note from eq. (3.155) that the system described by eq. (3.151) is unstable if $|a| \geq 1$ and thus does not have a finite response to complex exponential inputs. As we stated previously, Fourier-based methods and frequency domain analysis focus on systems with finite responses to complex exponentials; hence, for examples such as eq. (3.151), we restrict ourselves to stable systems.

3.11.2 Nonrecursive Discrete-Time Filters

The general form of an FIR nonrecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]. \tag{3.157}$$

That is, the output $y[n]$ is a *weighted average* of the $(N + M + 1)$ values of $x[n]$ from $x[n - M]$ through $x[n + N]$, with the weights given by the coefficients b_k. Systems of this form can be used to meet a broad array of filtering objectives, including frequency-selective filtering.

One frequently used example of such a filter is a *moving-average filter,* where the output $y[n]$ for any n—say, n_0—is an average of values of $x[n]$ in the vicinity of n_0. The

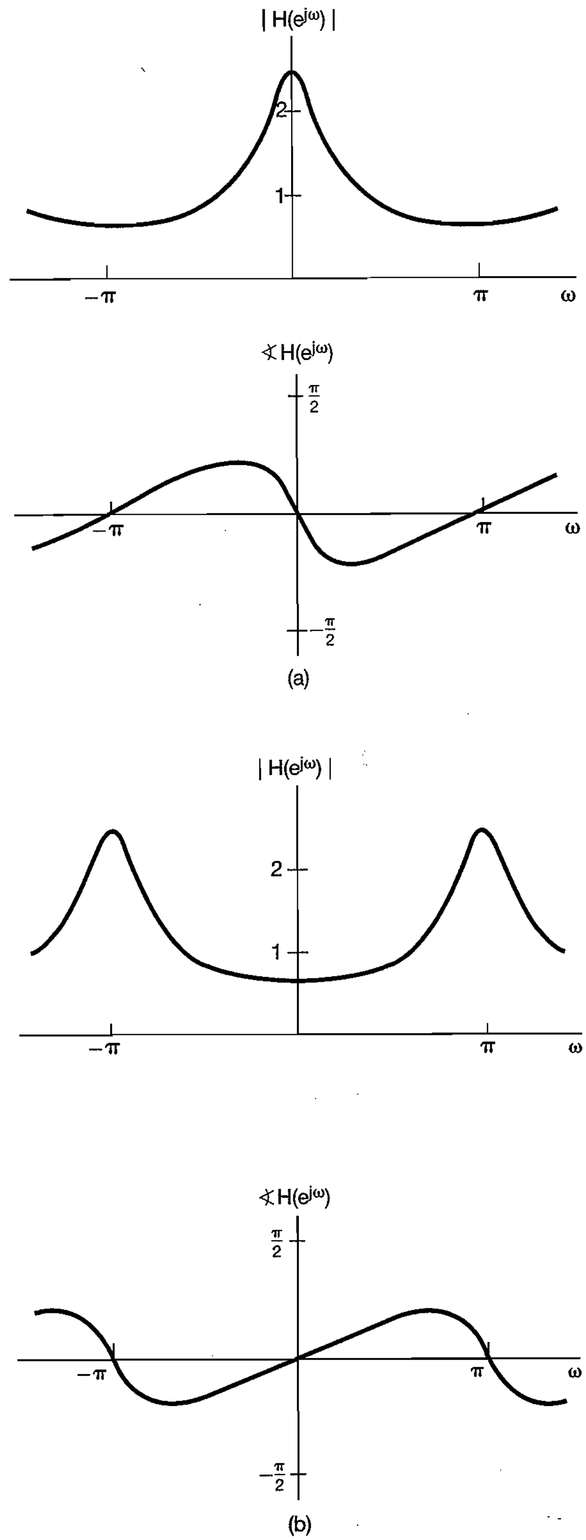


Figure 3.34 Frequency response of the first-order recursive discrete-time filter of eq. (3.151): (a) $a = 0.6$; (b) $a = -0.6$.

basic idea is that by averaging values locally, rapid high-frequency components of the input will be averaged out and lower frequency variations will be retained, corresponding to smoothing or lowpass filtering the original sequence. A simple two-point moving-average filter was briefly introduced in Section 3.9 [eq. (3.138)]. An only slightly more complex example is the three-point moving-average filter, which is of the form

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]), \tag{3.158}$$

so that each output $y[n]$ is the average of three consecutive input values. In this case,

$$h[n] = \frac{1}{3}[\delta[n+1] + \delta[n] + \delta[n-1]],$$

and thus, from eq. (3.122), the corresponding frequency response is

$$H(e^{j\omega}) = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos\omega). \tag{3.159}$$

The magnitude of $H(e^{j\omega})$ is sketched in Figure 3.35. We observe that the filter has the general characteristics of a lowpass filter, although, as with the first-order recursive filter, it does not have a sharp transition from passband to stopband.

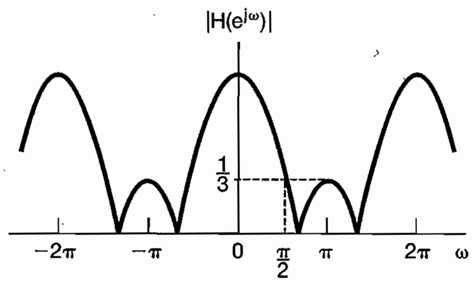


Figure 3.35 Magnitude of the frequency response of a three-point moving-average lowpass filter.

The three-point moving-average filter in eq. (3.158) has no parameters that can be changed to adjust the effective cutoff frequency. As a generalization of this moving-average filter, we can consider averaging over $N + M + 1$ neighboring points—that is, using a difference equation of the form

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]. \tag{3.160}$$

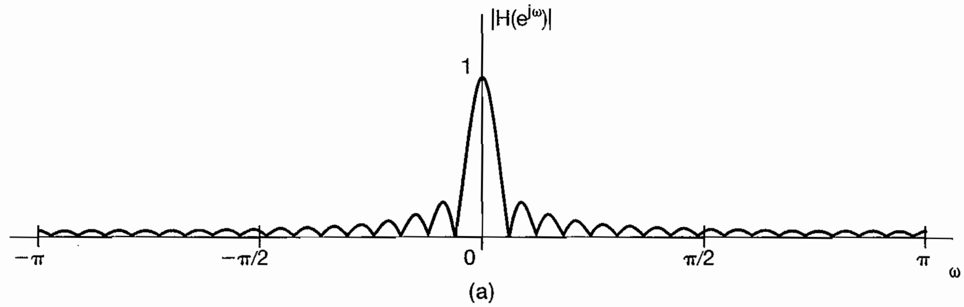
The corresponding impulse response is a rectangular pulse (i.e., $h[n] = 1/(N + M + 1)$ for $-N \leq n \leq M$, and $h[n] = 0$ otherwise). The filter's frequency response is

$$H(e^{j\omega}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-j\omega k}. \tag{3.161}$$

The summation in eq. (3.161) can be evaluated by performing calculations similar to those in Example 3.12, yielding

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}. \tag{3.162}$$

By adjusting the size, $N + M + 1$, of the averaging window we can vary the cutoff frequency. For example, the magnitude of $H(e^{j\omega})$ is shown in Figure 3.36 for $M+N+1 = 33$ and $M + N + 1 = 65$.



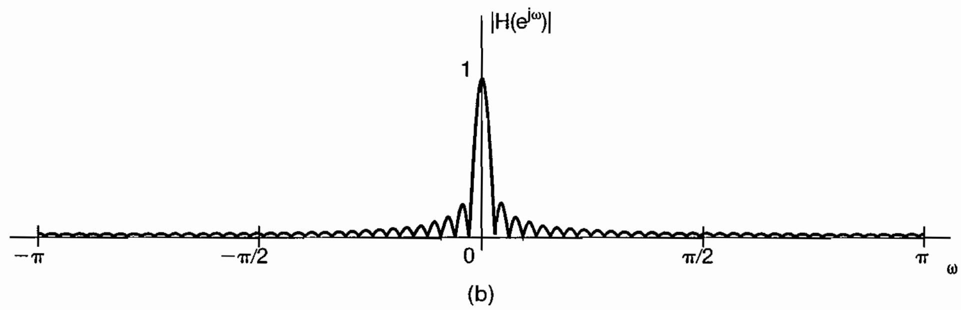


Figure 3.36 Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

Nonrecursive filters can also be used to perform highpass filtering operations. To illustrate this, again with a simple example, consider the difference equation

$$y[n] = \frac{x[n] - x[n-1]}{2}. \tag{3.163}$$

For input signals that are approximately constant, the value of $y[n]$ is close to zero. For input signals that vary greatly from sample to sample, the values of $y[n]$ can be expected to have larger amplitude. Thus, the system described by eq. (3.163) approximates a highpass filtering operation, attenuating slowly varying low-frequency components and passing rapidly varying higher frequency components with little attenuation. To see this more precisely we need to look at the system's frequency response. In this case, $h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]\}$, so that direct application of eq. (3.122) yields

$$H(e^{j\omega}) = \frac{1}{2}[1 - e^{-j\omega}] = je^{j\omega/2}\sin(\omega/2). \tag{3.164}$$

In Figure 3.37 we have plotted the magnitude of $H(e^{j\omega})$, showing that this simple system approximates a highpass filter, albeit one with a very gradual transition from passband to stopband. By considering more general nonrecursive filters, we can achieve far sharper transitions in lowpass, highpass, and other frequency-selective filters.

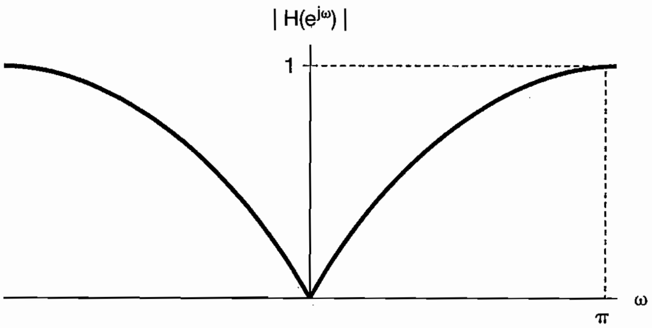


Figure 3.37 Frequency response of a simple highpass filter.

Note that, since the impulse response of any FIR system is of finite length (i.e., from eq. (3.157), $h[n] = b_n$ for $-N \leq n \leq M$ and 0 otherwise), it is always absolutely summable for any choices of the b_n. Hence, all such filters are stable. Also, if $N > 0$ in eq. (3.157), the system is noncausal, since $y[n]$ then depends on future values of the input. In some applications, such as those involving the processing of previously recorded signals, causality is not a necessary constraint, and thus, we are free to use filters with $N > 0$. In others, such as many involving real-time processing, causality is essential, and in such cases we must take $N \leq 0$.

3.12 SUMMARY

In this chapter, we have introduced and developed Fourier series representations for both continuous-time and discrete-time systems and have used these representations to take a first look at one of the very important applications of the methods of signal and system analysis, namely, filtering. In particular, as we discussed in Section 3.2, one of the primary motivations for the use of Fourier series is the fact that complex exponential signals are eigenfunctions of LTI systems. We have also seen, in Sections 3.3–3.7, that any periodic signal of practical interest can be represented in a Fourier series—i.e., as a weighted sum