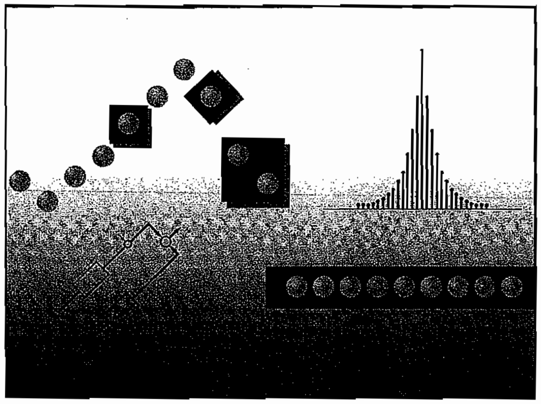
5

THE DISCRETE-TIME FOURIER TRANSFORM



5.0 INTRODUCTION

In Chapter 4, we introduced the continuous-time Fourier transform and developed the many characteristics of that transform which make the methods of Fourier analysis of such great value in analyzing and understanding the properties of continuous-time signals and systems. In the current chapter, we complete our development of the basic tools of Fourier analysis by introducing and examining the discrete-time Fourier transform.

In our discussion of Fourier series in Chapter 3, we saw that there are many similarities and strong parallels in analyzing continuous-time and discrete-time signals. However, there are also important differences. For example, as we saw in Section 3.6, the Fourier series representation of a discrete-time periodic signal is a *finite* series, as opposed to the infinite series representation required for continuous-time periodic signals. As we will see in this chapter, there are corresponding differences between continuous-time and discrete-time Fourier transforms.

In the remainder of the chapter, we take advantage of the similarities between continuous-time and discrete-time Fourier analysis by following a strategy essentially identical to that used in Chapter 4. In particular, we begin by extending the Fourier series description of periodic signals in order to develop a Fourier transform representation for discrete-time aperiodic signals, and we follow with an analysis of the properties and characteristics of the discrete-time Fourier transform that parallels that given in Chapter 4. By doing this, we not only will enhance our understanding of the basic concepts of Fourier analysis that are common to both continuous and discrete time, but also will contrast their differences in order to deepen our understanding of the distinct characteristics of each.

PRESENTATION OF APERIODIC SIGNALS: THE DISCRETE-TIME FOURIER TRANSFORM

5.1.1 Development of the Discrete-Time Fourier Transform

In Section 4.1 [eq. (4.2) and Figure 4.2], we saw that the Fourier series coefficients for a continuous-time periodic square wave can be viewed as samples of an envelope function and that, as the period of the square wave increases, these samples become more and more finely spaced. This property suggested representing an aperiodic signal $x(t)$ by first constructing a periodic signal $\tilde{x}(t)$ that equaled $x(t)$ over one period. Then, as this period approached infinity, $\tilde{x}(t)$ was equal to $x(t)$ over larger and larger intervals of time, and the Fourier series representation for $\tilde{x}(t)$ converged to the Fourier transform representation for $x(t)$. In this section, we apply an analogous procedure to discrete-time signals in order to develop the Fourier transform representation for discrete-time aperiodic sequences.

Consider a general sequence $x[n]$ that is of finite duration. That is, for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. A signal of this type is illustrated in Figure 5.1(a). From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period, as illustrated in Figure 5.1(b). As we choose the period N to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, and as $N \rightarrow \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n.

Let us now examine the Fourier series representation of $\tilde{x}[n]$. Specifically, from eqs. (3.94) and (3.95), we have

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \tag{5.1}$$

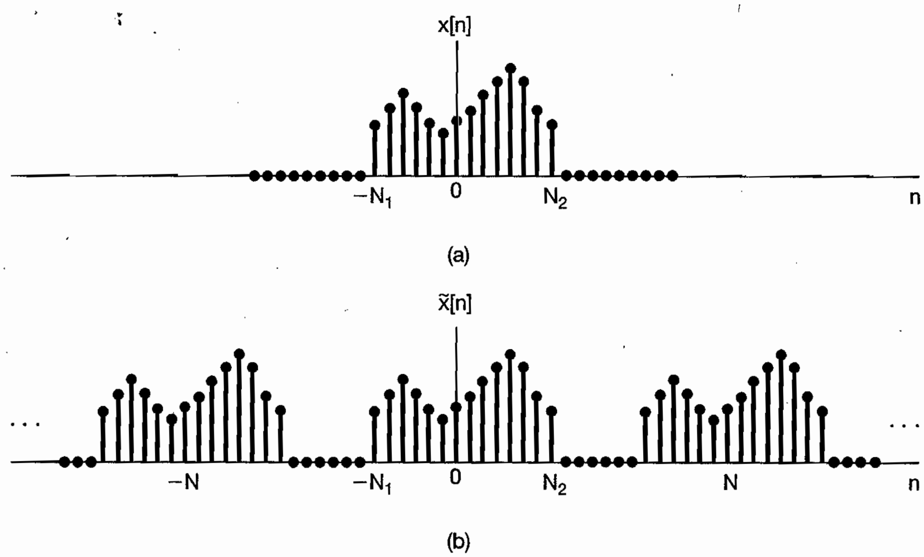


Figure 5.1 (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}. \tag{5.2}$$

Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \le n \le N_2$, it is convenient to choose the interval of summation in eq. (5.2) to include this interval, so that $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n}, \tag{5.3}$$

where in the second equality in eq. (5.3) we have used the fact that $x[n]$ is zero outside the interval $-N_1 \le n \le N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}, \tag{5.4}$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N} X(e^{jk\omega_0}), \tag{5.5}$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining eqs. (5.1) and (5.5) yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}. \tag{5.6}$$

Since $\omega_0 = 2\pi/N$, or equivalently, $1/N = \omega_0/2\pi$, eq. (5.6) can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0. \tag{5.7}$$

As with eq. (4.7), as N increases ω_0 decreases, and as $N \to \infty$ eq. (5.7) passes to an integral. To see this more clearly, consider $X(e^{j\omega})e^{j\omega n}$ as sketched in Figure 5.2. From

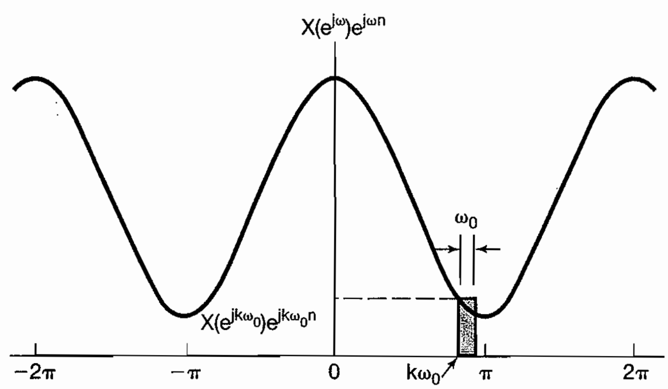


Figure 5.2 Graphical interpretation of eq. (5.7).

eq. (5.4), $X(e^{j\omega})$ is seen to be periodic in ω with period 2π, and so is $e^{j\omega n}$. Thus, the product $X(e^{j\omega})e^{j\omega n}$ will also be periodic. As depicted in the figure, each term in the summation in eq. (5.7) represents the area of a rectangle of height $X(e^{jk\omega_0})e^{j\omega_0 n}$ and width ω_0. As $\omega_0 \to 0$, the summation becomes an integral. Furthermore, since the summation is carried out over N consecutive intervals of width $\omega_0 = 2\pi/N$, the total interval of integration will always have a width of 2π. Therefore, as $N \to \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

where, since $X(e^{j\omega})e^{j\omega n}$ is periodic with period 2π, the interval of integration can be taken as *any* interval of length 2π. Thus, we have the following pair of equations:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{5.8}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \tag{5.9}$$

Equations (5.8) and (5.9) are the discrete-time counterparts of eqs. (4.8) and (4.9). The function $X(e^{j\omega})$ is referred to as the *discrete-time Fourier transform* and the pair of equations as the *discrete-time Fourier transform pair*. Equation (5.8) is the *synthesis equation*, eq. (5.9) the *analysis equation*. Our derivation of these equations indicates how an aperiodic sequence can be thought of as a linear combination of complex exponentials. In particular, the synthesis equation is in effect a representation of $x[n]$ as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X(e^{j\omega})(d\omega/2\pi)$. For this reason, as in continuous time, the Fourier transform $X(e^{j\omega})$ will often be referred to as the *spectrum* of $x[n]$, because it provides us with the information on how $x[n]$ is composed of complex exponentials at different frequencies.

Note also that, as in continuous time, our derivation of the discrete-time Fourier transform provides us with an important relationship between discrete-time Fourier series and transforms. In particular, the Fourier coefficients a_k of a periodic signal $\tilde{x}[n]$ can be expressed in terms of equally spaced *samples* of the Fourier transform of a finite-duration, aperiodic signal $x[n]$ that is equal to $\tilde{x}[n]$ over one period and is zero otherwise. This fact is of considerable importance in practical signal processing and Fourier analysis, and we look at it further in Problem 5.41.

As our derivation indicates, the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the discrete-time transform $X(e^{j\omega})$ and the finite interval of integration in the synthesis equation. Both of these stem from a fact that we have noted several times before: Discrete-time complex exponentials that differ in frequency by a multiple of 2π are identical. In Section 3.6 we saw that, for periodic discrete-time signals, the implications of this statement are that the Fourier series coefficients are periodic and that the Fourier series representation is a finite sum. For aperiodic signals, the analogous implications are that $X(e^{j\omega})$ is periodic (with period 2π) and that the synthesis equation involves an integration only over a frequency interval that produces distinct complex exponentials (i.e., any interval of length 2π). In Section 1.3.3, we noted one further consequence of the periodicity of $e^{j\omega n}$ as a function of ω: $\omega = 0$ and $\omega = 2\pi$ yield the same signal. Signals at frequencies near these values or any other even multiple of π are slowly varying and therefore are all appropriately thought of as low-frequency signals. Similarly, the high frequencies in discrete time are the values of ω near odd multiples of π. Thus, the signal $x_1[n]$ shown in Figure 5.3(a) with Fourier transform depicted in Figure 5.3(b) varies more slowly than the signal $x_2[n]$ in Figure 5.3(c) whose transform is shown in Figure 5.3(d).

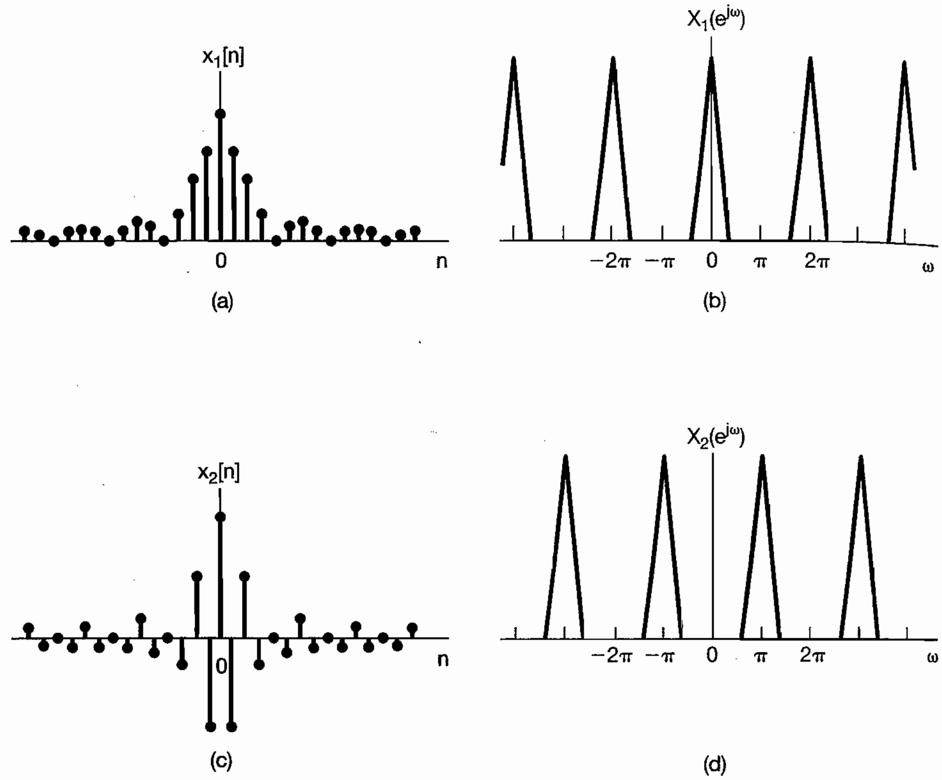


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

5.1.2 Examples of Discrete-Time Fourier Transforms

To illustrate the discrete-time Fourier transform, let us consider several examples.

Example 5.1

Consider the signal

$$x[n] = a^n u[n], \qquad |a| < 1.$$

In this case,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

The magnitude and phase of $X(e^{j\omega})$ are shown in Figure 5.4(a) for $a > 0$ and in Figure 5.4(b) for $a < 0$. Note that all of these functions are periodic in ω with period 2π.

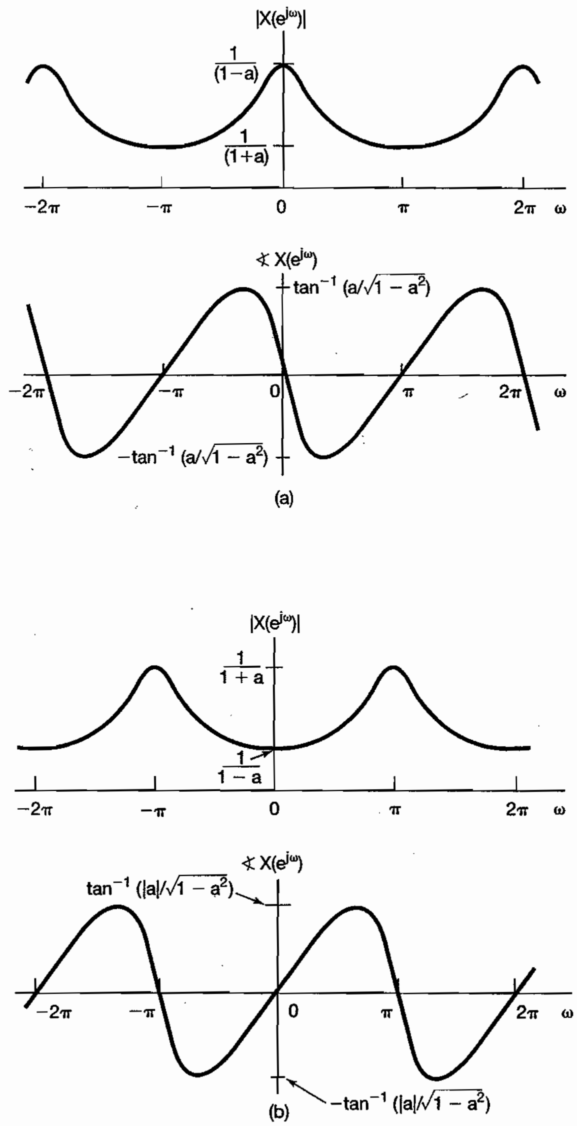


Figure 5.4 Magnitude and phase of the Fourier transform of Example 5.1 for (a) $a > 0$ and (b) $a < 0$.

Example 5.2

Let

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

This signal is sketched for $0 < a < 1$ in Figure 5.5(a). Its Fourier transform is obtained from eq. (5.9):

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}. \end{aligned}$$

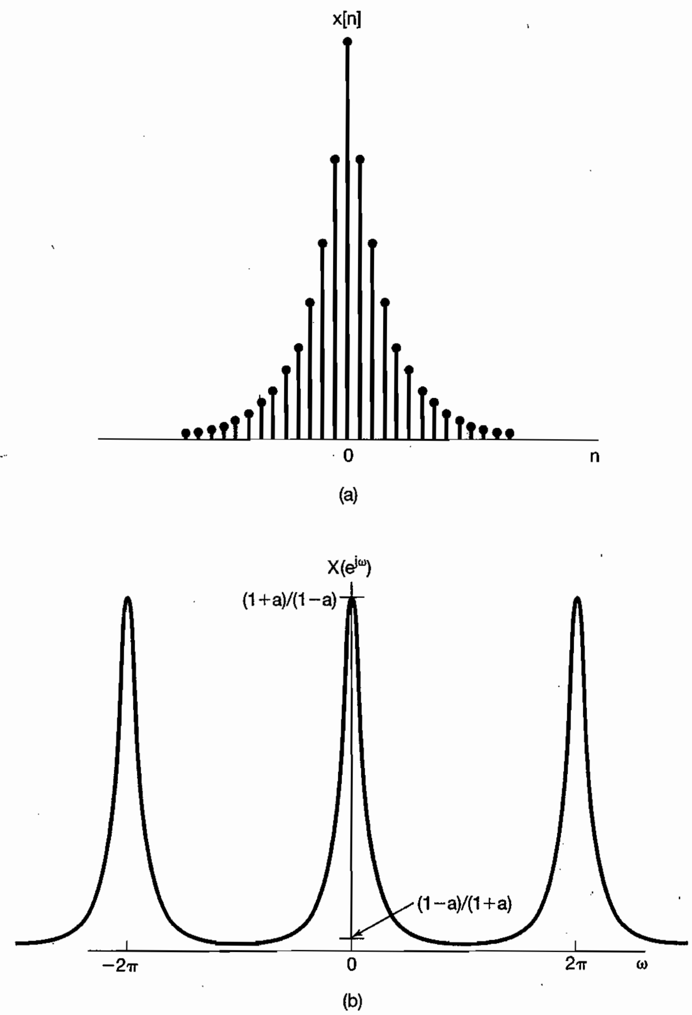


Figure 5.5 (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform ($0 < a < 1$).

Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty}(ae^{-j\omega})^n + \sum_{m=1}^{\infty}(ae^{j\omega})^m.$$

Both of these summations are infinite geometric series that we can evaluate in closed form, yielding

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

In this case, $X(e^{j\omega})$ is real and is illustrated in Figure 5.5(b), again for $0 < a < 1$.

Example 5.3

Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}, \tag{5.10}$$

which is illustrated in Figure 5.6(a) for $N_1 = 2$. In this case,

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}. \tag{5.11}$$

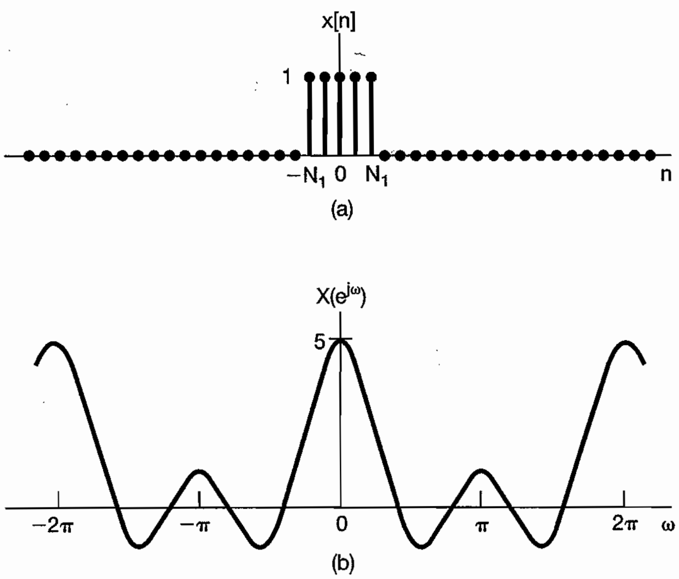


Figure 5.6 (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

Using calculations similar to those employed in obtaining eq. (3.104) in Example 3.12, we can write

$$X(e^{j\omega}) = \frac{\sin \omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}. \tag{5.12}$$

This Fourier transform is sketched in Figure 5.6(b) for $N_1 = 2$. The function in eq. (5.12) is the discrete-time counterpart of the sinc function, which appears in the Fourier transform of the continuous-time rectangular pulse (see Example 4.4). An important difference between these two functions is that the function in eq. (5.12) is periodic with period 2π, whereas the sinc function is aperiodic.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Although the argument we used to derive the discrete-time Fourier transform in Section 5.1.1 was constructed assuming that $x[n]$ was of arbitrary but finite duration, eqs. (5.8) and (5.9) remain valid for an extremely broad class of signals with infinite duration (such as the signals in Examples 5.1 and 5.2). In this case, however, we again must consider the question of convergence of the infinite summation in the analysis equation (5.9). The conditions on $x[n]$ that guarantee the convergence of this sum are direct counterparts of the convergence conditions for the continuous-time Fourier transform.[1] Specifically, eq. (5.9) will converge either if $x[n]$ is absolutely summable, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty, \tag{5.13}$$

or if the sequence has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty. \tag{5.14}$$

In contrast to the situation for the analysis equation (5.9), there are generally no convergence issues associated with the synthesis equation (5.8), since the integral in this equation is over a finite interval of integration. This is very much the same situation as for the discrete-time Fourier series synthesis equation (3.94), which involves a finite sum and consequently has no issues of convergence associated with it either. In particular, if we approximate an aperiodic signal $x[n]$ by an integral of complex exponentials with frequencies taken over the interval $|\omega| \le W$, i.e.,

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}\, d\omega, \tag{5.15}$$

[1]For discussions of the convergence issues associated with the discrete-time Fourier transform, see A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1989), and L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).

then $\hat{x}[n] = x[n]$ for $W = \pi$. Thus, much as in Figure 3.18, we would expect not to see any behavior like the Gibbs phenomenon in evaluating the discrete-time Fourier transform synthesis equation. This is illustrated in the following example.

Example 5.4

Let $x[n]$ be the unit impulse; that is,

$$x[n] = \delta[n].$$

In this case the analysis equation (5.9) is easily evaluated, yielding

$$X(e^{j\omega}) = 1.$$

In other words, just as in continuous time, the unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies. If we then apply eq. (5.15) to this example, we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\, d\omega = \frac{\sin Wn}{\pi n}. \tag{5.16}$$

This is plotted in Figure 5.7 for several values of W. As can be seen, the frequency of the oscillations in the approximation increases as W is increased, which is similar to what we observed in the continuous-time case. On the other hand, in contrast to the continuous-time case, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$ as W is increased, and the oscillations disappear entirely for $W = \pi$.

FOURIER TRANSFORM FOR PERIODIC SIGNALS

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain. To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n}. \tag{5.17}$$

In continuous time, we saw that the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. Therefore, we might expect the same type of transform to result for the discrete-time signal of eq. (5.17). However, the discrete-time Fourier transform must be periodic in ω with period 2π. This then suggests that the Fourier transform of $x[n]$ in eq. (5.17) should have impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In fact, the Fourier transform of $x[n]$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l), \tag{5.18}$$

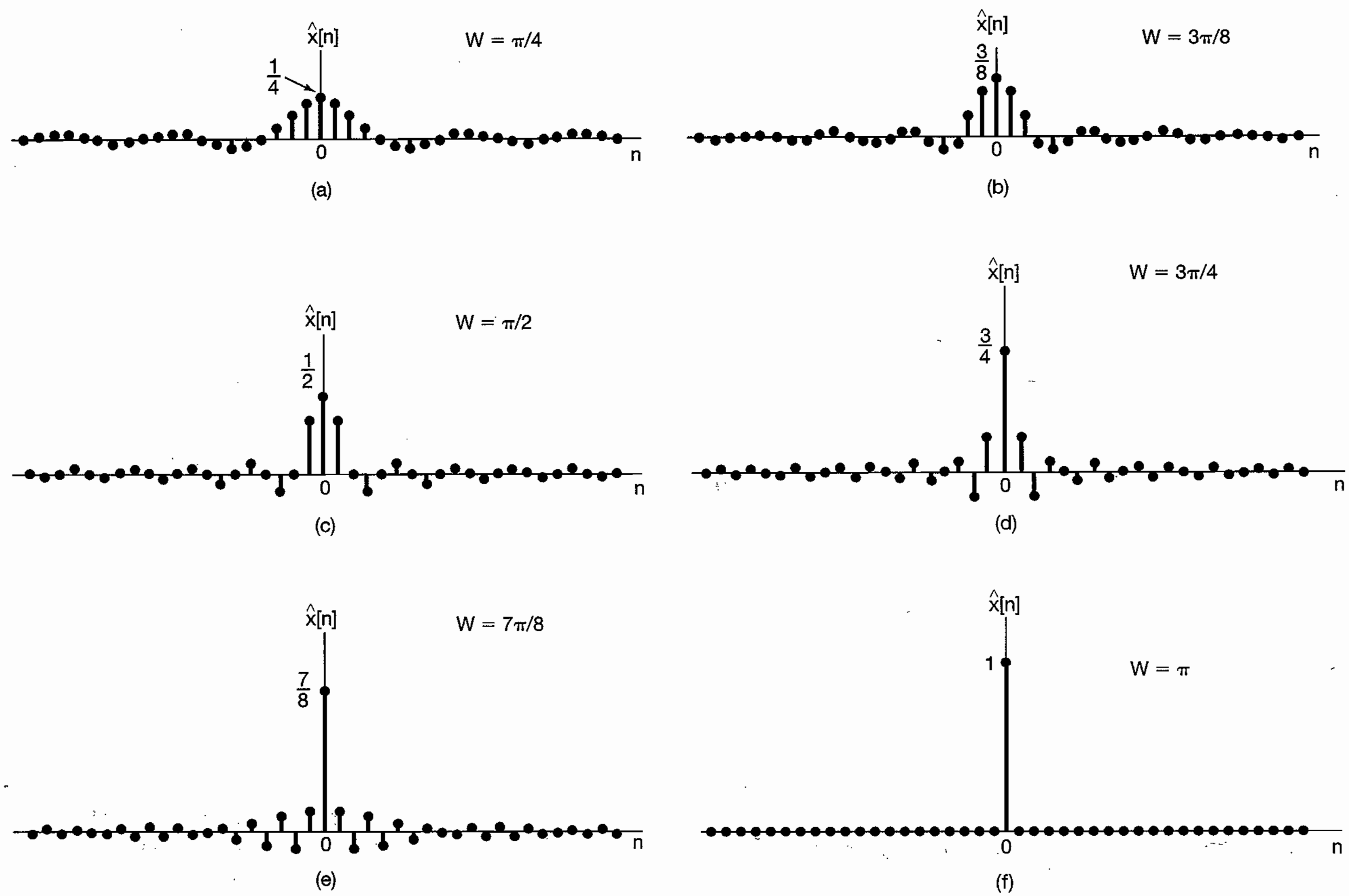


Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

which is illustrated in Figure 5.8. In order to check the validity of this expression, we must evaluate its inverse transform. Substituting eq. (5.18) into the synthesis equation (5.8), we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)e^{j\omega n}\,d\omega.$$

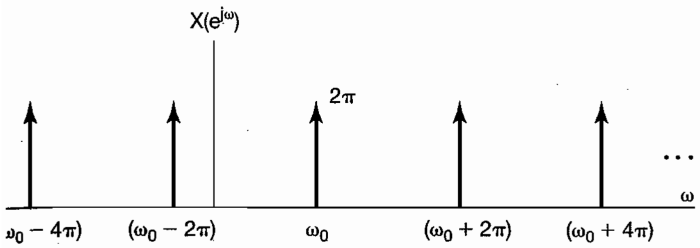


Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

Note that any interval of length 2π includes exactly one impulse in the summation given in eq. (5.18). Therefore, if the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}.$$

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \tag{5.19}$$

In this case, the Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right), \tag{5.20}$$

so that the Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients.

To verify that eq. (5.20) is in fact correct, note that $x[n]$ in eq. (5.19) is a linear combination of signals of the form in eq. (5.17), and thus the Fourier transform of $x[n]$ must be a linear combination of transforms of the form of eq. (5.18). In particular, suppose that we choose the interval of summation in eq. (5.19) as $k = 0, 1, \ldots, N-1$, so that

$$\begin{aligned} x[n] = {} & a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} \\ & + \cdots + a_{N-1}e^{j(N-1)(2\pi/N)n}. \end{aligned} \tag{5.21}$$

Thus, $x[n]$ is a linear combination of signals, as in eq. (5.17), with $\omega_0 = 0, 2\pi/N, 4\pi/N, \ldots, (N-1)2\pi/N$. The resulting Fourier transform is illustrated in Figure 5.9. In Figure 5.9(a), we have depicted the Fourier transform of the first term on the right-hand side of eq. (5.21): The Fourier transform of the constant signal $a_0 = a_0 e^{j0\cdot n}$ is a periodic impulse train, as in eq. (5.18), with $\omega_0 = 0$ and a scaling of $2\pi a_0$ on each of the impulses. Moreover, from Chapter 4 we know that the Fourier series coefficients a_k are periodic with period N, so that $2\pi a_0 = 2\pi a_N = 2\pi a_{-N}$. In Figure 5.9(b) we have illustrated the Fourier transform of the second term in eq. (5.21), where we have again used eq. (5.18),

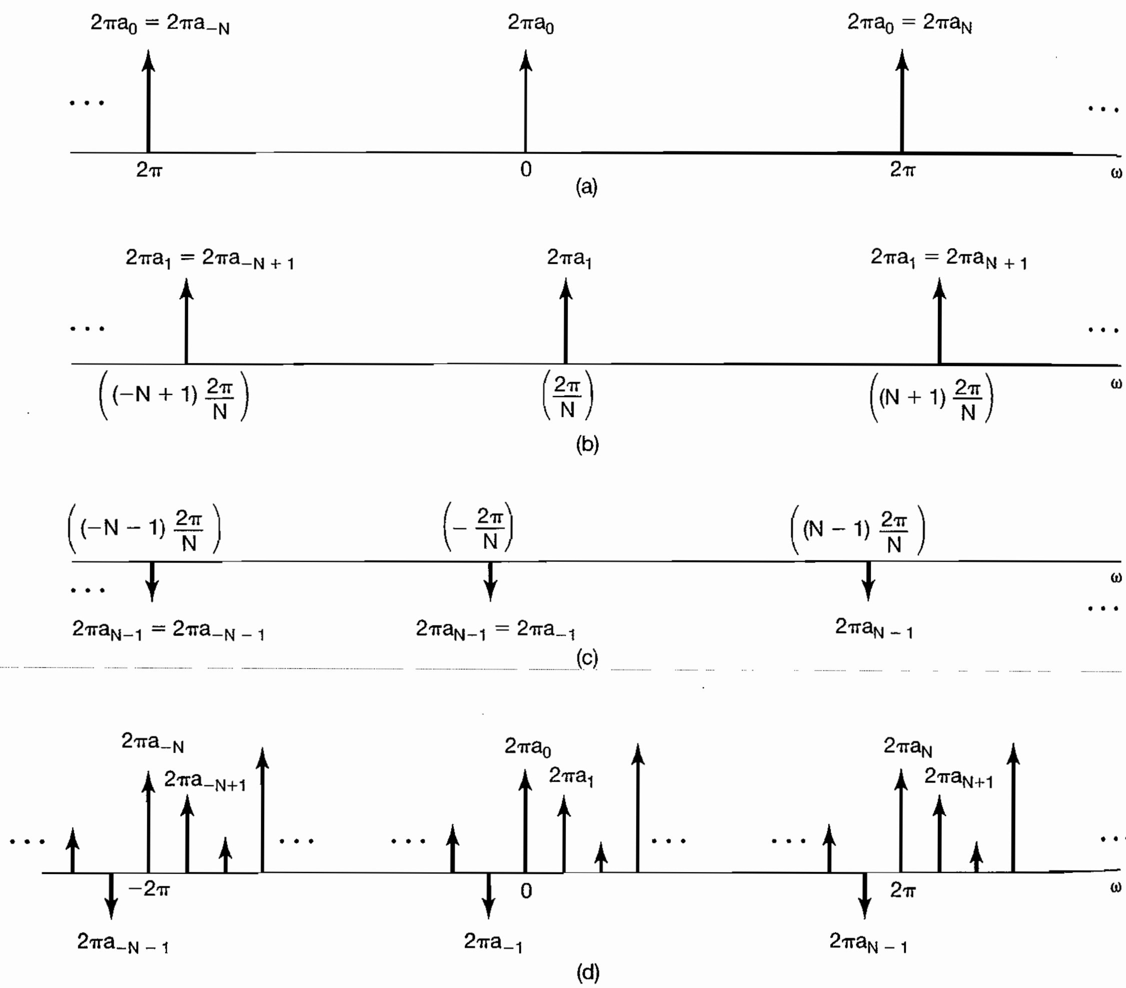


Figure 5.9 Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

in this case for $a_1 e^{j(2\pi/N)n}$, and the fact that $2\pi a_1 = 2\pi a_{N+1} = 2\pi a_{-N+1}$. Similarly, Figure 5.9(c) depicts the final term. Finally, Figure 5.9(d) depicts the entire expression for $X(e^{j\omega})$. Note that because of the periodicity of the a_k, $X(e^{j\omega})$ can be interpreted as a train of impulses occurring at multiples of the fundamental frequency $2\pi/N$, with the area of the impulse located at $\omega = 2\pi k/N$ being $2\pi a_k$, which is exactly what is stated in eq. (5.20).

Example 5.5

Consider the periodic signal

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad \text{with} \quad \omega_0 = \frac{2\pi}{5}. \tag{5.22}$$

From eq. (5.18), we can immediately write

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right). \tag{5.23}$$

That is,

$$X(e^{j\omega}) = \pi\,\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\,\delta\left(\omega + \frac{2\pi}{5}\right), \qquad -\pi \le \omega < \pi, \tag{5.24}$$

and $X(e^{j\omega})$ repeats periodically with a period of 2π, as illustrated in Figure 5.10.

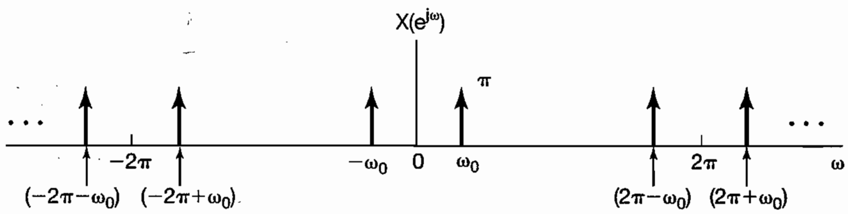


Figure 5.10 Discrete-time Fourier transform of $x[n] = \cos\omega_0 n$.

Example 5.6

The discrete-time counterpart of the periodic impulse train of Example 4.8 is the sequence

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN], \tag{5.25}$$

as sketched in Figure 5.11(a). The Fourier series coefficients for this signal can be calculated directly from eq. (3.95):

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

Choosing the interval of summation as $0 \leq n \leq N-1$, we have

$$a_k = \frac{1}{N}. \tag{5.26}$$

Using eqs. (5.26) and (5.20), we can then represent the Fourier transform of the signal as

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right), \tag{5.27}$$

which is illustrated in Figure 5.11(b).

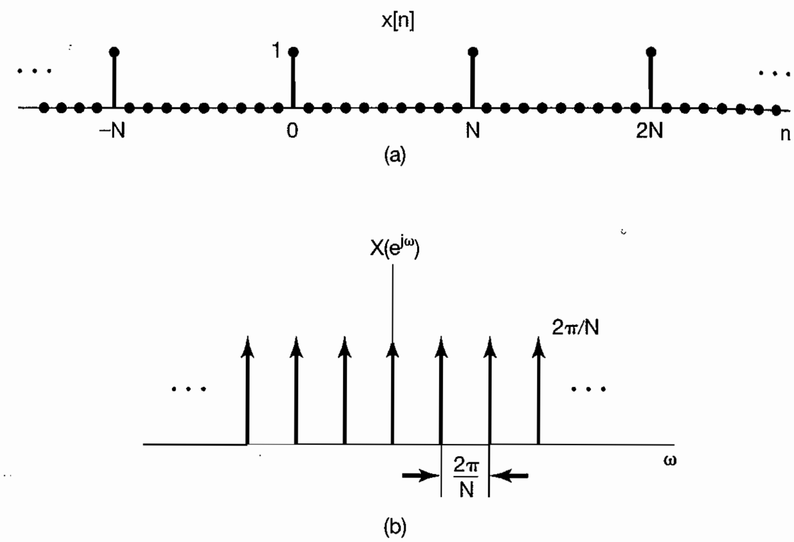


Figure 5.11 (a) Discrete-time periodic impulse train; (b) its Fourier transform.

5.3 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

As with the continuous-time Fourier transform, a variety of properties of the discrete-time Fourier transform provide further insight into the transform and, in addition, are often useful in reducing the complexity in the evaluation of transforms and inverse transforms. In this and the following two sections we consider these properties, and in Table 5.1 we present a concise summary of them. By comparing this table with Table 4.1, we can get a clear picture of some of the similarities and differences between continuous-time and discrete-time Fourier transform properties. When the derivation or interpretation of a discrete-time Fourier transform property is essentially identical to its continuous-time counterpart, we will simply state the property. Also, because of the close relationship between the Fourier series and the Fourier transform, many of the transform properties translate directly into corresponding properties for the discrete-time Fourier series, which we summarized in Table 3.2 and briefly discussed in Section 3.7.

In the following discussions, it will be convenient to adopt notation similar to that used in Section 4.3 to indicate the pairing of a signal and its transform. That is,

$$X(e^{j\omega}) = \mathfrak{F}\{x[n]\},$$

$$x[n] = \mathfrak{F}^{-1}\{X(e^{j\omega})\},$$

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}).$$

5.3.1 Periodicity of the Discrete-Time Fourier Transform

As we discussed in Section 5.1, the discrete-time Fourier transform is *always* periodic in ω with period 2π; i.e.,

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega}).} \tag{5.28}$$

This is in contrast to the continuous-time Fourier transform, which in general is not periodic.

5.3.2 Linearity of the Fourier Transform

If

$$x_1[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

then

$$\boxed{ax_1[n] + bx_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega}).} \tag{5.29}$$

5.3.3 Time Shifting and Frequency Shifting

If

$$x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x[n-n_0] \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega n_0}X(e^{j\omega})} \tag{5.30}$$

and

$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}). \tag{5.31}$$

Equation (5.30) can be obtained by direct substitution of $x[n - n_0]$ into the analysis equation (5.9), while eq. (5.31) is derived by substituting $X(e^{j(\omega-\omega_0)})$ into the synthesis equation (5.8).

As a consequence of the periodicity and frequency-shifting properties of the discrete-time Fourier transform, there exists a special relationship between ideal lowpass and ideal highpass discrete-time filters. This is illustrated in the next example.

Example 5.7

In Figure 5.12(a) we have depicted the frequency response $H_{lp}(e^{j\omega})$ of a lowpass filter with cutoff frequency ω_c, while in Figure 5.12(b) we have displayed $H_{lp}(e^{j(\omega-\pi)})$—that is, the frequency response $H_{lp}(e^{j\omega})$ shifted by one-half period, i.e., by π. Since high frequencies in discrete time are concentrated near π (and other odd multiples of π), the filter in Figure 5.12(b) is an ideal highpass filter with cutoff frequency $\pi - \omega_c$. That is,

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}). \tag{5.32}$$

As we can see from eq. (3.122), and as we will discuss again in Section 5.4, the frequency response of an LTI system is the Fourier transform of the impulse response of the system. Thus, if $h_{lp}[n]$ and $h_{hp}[n]$ respectively denote the impulse responses of

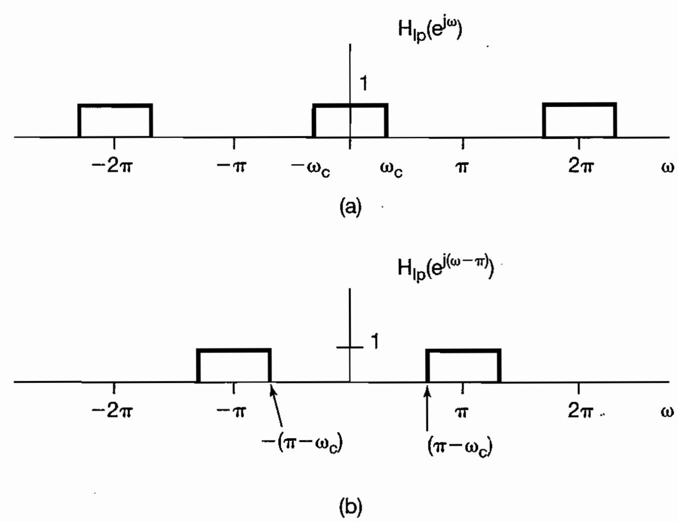


Figure 5.12 (a) Frequency response of a lowpass filter; (b) frequency response of a highpass filter obtained by shifting the frequency response in (a) by $\omega = \pi$ corresponding to one-half period.

Figure 5.12, eq. (5.32) and the frequency-shifting property imply that the lowpass and highpass filters in

$$h_{\text{hp}}[n] = e^{j\pi n} h_{\text{lp}}[n] \tag{5.33}$$

$$= (-1)^n h_{\text{lp}}[n]. \tag{5.34}$$

5.3.4 Conjugation and Conjugate Symmetry

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x^*[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega}).} \tag{5.35}$$

Also, if $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is,

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{real}].} \tag{5.36}$$

From this, it follows that $\mathcal{R}e\{X(e^{j\omega})\}$ is an even function of ω and $\mathcal{I}m\{X(e^{j\omega})\}$ is an odd function of ω. Similarly, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function. Furthermore,

$$\mathcal{E}v\{x[n]\} \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}$$

and

$$\mathcal{O}d\{x[n]\} \stackrel{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\},$$

where $\mathcal{E}v$ and $\mathcal{O}d$ denote the even and odd parts, respectively, of $x[n]$. For example, if $x[n]$ is real and even, its Fourier transform is also real and even. Example 5.2 illustrates this symmetry for $x[n] = a^{|n|}$.

5.3.5 Differencing and Accumulation

In this subsection, we consider the discrete-time counterpart of integration—that is, accumulation—and its inverse, first differencing. Let $x[n]$ be a signal with Fourier transform $X(e^{j\omega})$. Then, from the linearity and time-shifting properties, the Fourier transform pair for the first-difference signal $x[n] - x[n-1]$ is given by

$$\boxed{x[n] - x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega}).} \tag{5.37}$$

Next, consider the signal

$$y[n] = \sum_{m=-\infty}^{n} x[m]. \tag{5.38}$$

Since $y[n] - y[n-1] = x[n]$, we might conclude that the transform of $y[n]$ should be related to the transform of $x[n]$ by division by $(1 - e^{-j\omega})$. This is partly correct, but as with the continuous-time integration property given by eq. (4.32), there is more involved. The precise relationship is

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k). \tag{5.39}$$

The impulse train on the right-hand side of eq. (5.39) reflects the dc or average value that can result from summation.

Example 5.8

Let us derive the Fourier transform $X(e^{j\omega})$ of the unit step $x[n] = u[n]$ by making use of the accumulation property and the knowledge that

$$g[n] = \delta[n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{j\omega}) = 1.$$

From Section 1.4.1 we know that the unit step is the running sum of the unit impulse. That is,

$$x[n] = \sum_{m=-\infty}^{n} g[m].$$

Taking the Fourier transform of both sides and using accumulation yields

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{(1-e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k). \end{aligned}$$

5.3.6 Time Reversal

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$, and consider the transform $Y(e^{j\omega})$ of $y[n] = x[-n]$. From eq. (5.9),

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n}. \tag{5.40}$$

Substituting $m = -n$ into eq. (5.40), we obtain

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega}). \tag{5.41}$$

That is,

$$x[-n] \overset{\mathfrak{F}}{\longleftrightarrow} X(e^{-j\omega}). \tag{5.42}$$

5.3.7 Time Expansion

Because of the discrete nature of the time index for discrete-time signals, the relation between time and frequency scaling in discrete time takes on a somewhat different form from its continuous-time counterpart. Specifically, in Section 4.3.5 we derived the continuous-time property

$$x(at) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right). \tag{5.43}$$

However, if we try to define the signal $x[an]$, we run into difficulties if a is not an integer. Therefore, we cannot slow down the signal by choosing $a < 1$. On the other hand, if we let a be an integer other than ± 1—for example, if we consider $x[2n]$—we do not merely speed up the original signal. That is, since n can take on only integer values, the signal $x[2n]$ consists of the even samples of $x[n]$ alone.

There is a result that does closely parallel eq. (5.43), however. Let k be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases} \tag{5.44}$$

As illustrated in Figure 5.13 for $k = 3$, $x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal. Intuitively, we can think of $x_{(k)}[n]$ as a slowed-down version of $x[n]$. Since $x_{(k)}[n]$ equals 0 unless n is a multiple of k, i.e., unless $n = rk$, we see that the Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}.$$

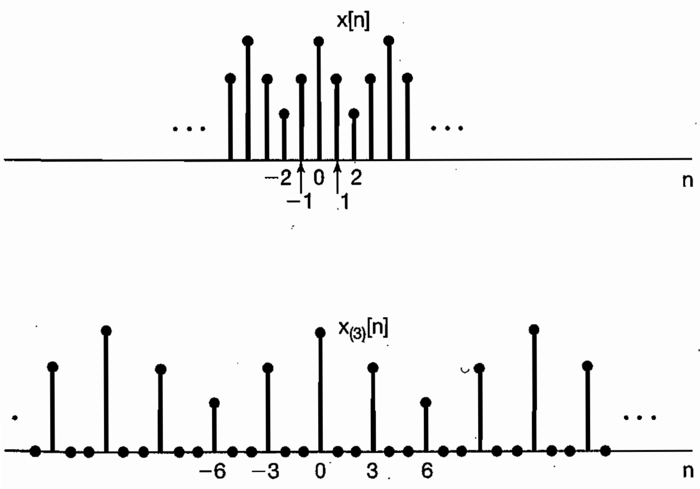


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

Furthermore, since $x_{(k)}[rk] = x[r]$, we find that

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jk\omega}). \tag{5.45}$$

Note that as the signal is spread out and slowed down in time by taking $k > 1$, its Fourier transform is compressed. For example, since $X(e^{j\omega})$ is periodic with period 2π, $X(e^{jk\omega})$ is periodic with period $2\pi/k$. This property is illustrated in Figure 5.14 for a rectangular pulse.

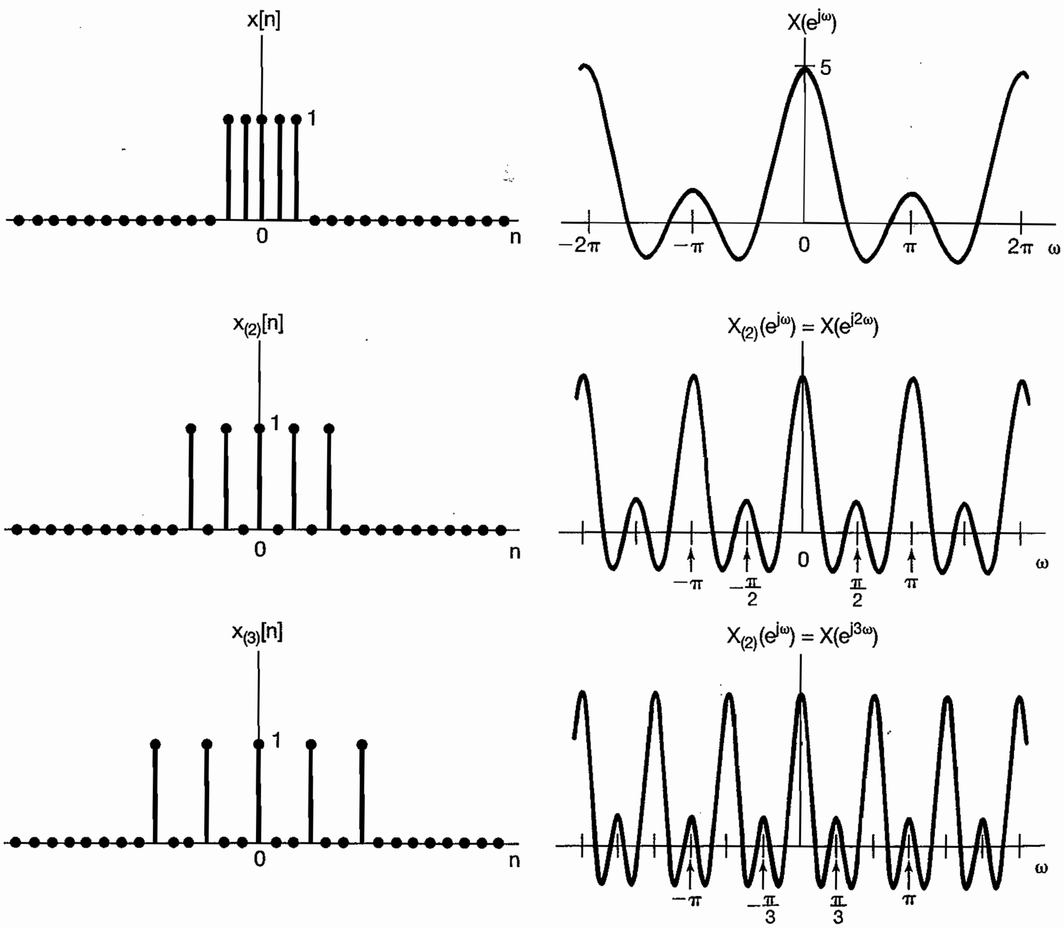


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Example 5.9

As an illustration of the usefulness of the time-expansion property in determining Fourier transforms, let us consider the sequence $x[n]$ displayed in Figure 5.15(a). This sequence can be related to the simpler sequence $y[n]$ depicted in Figure 5.15(b). In particular

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

where

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

and $y_{(2)}[n-1]$ represents $y_{(2)}[n]$ shifted one unit to the right. The signals $y_{(2)}[n]$ and $2y_{(2)}[n-1]$ are depicted in Figures 5.15(c) and (d), respectively.

Next, note that $y[n] = g[n-2]$, where $g[n]$ is a rectangular pulse as considered in Example 5.3 (with $N_1 = 2$) and as depicted in Figure 5.6(a). Consequently, from Example 5.3 and the time-shifting property, we see that

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

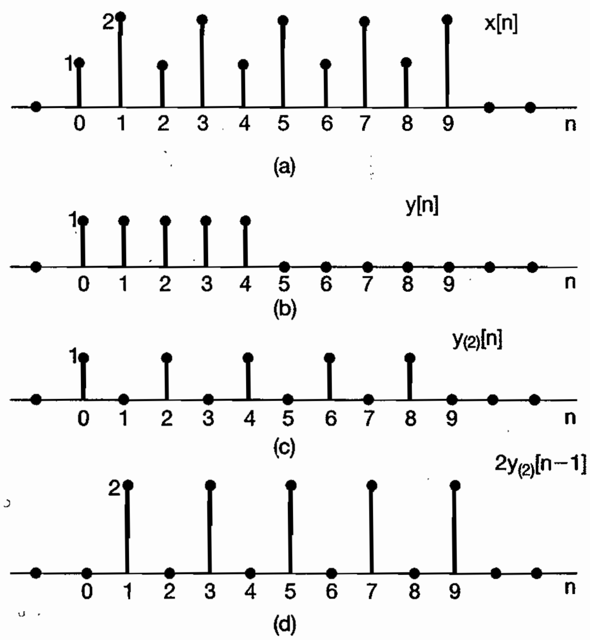


Figure 5.15 (a) The signal $x[n]$ in Example 5.9; (b) the signal $y[n]$; (c) the signal $y_{(2)}[n]$ obtained by inserting one zero between successive values of $y[n]$; and (d) the signal $2y_{(2)}[n-1]$.

Using the time-expansion property, we then obtain

$$y_{(2)}[n] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)},$$

and using the linearity and time-shifting properties, we get

$$2y_{(2)}[n-1] \overset{\mathcal{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}.$$

Combining these two results, we have

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$

5.3.8 Differentiation in Frequency

Again, let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}).$$

If we use the definition of $X(e^{j\omega})$ in the analysis equation (5.9) and differentiate both sides, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of this equation is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

$$\boxed{nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.} \tag{5.46}$$

The usefulness of this property will be illustrated in Example 5.13 in Section 5.4.

5.3.9 Parseval's Relation

If $x[n]$ and $X(e^{j\omega})$ are a Fourier transform pair, then

$$\boxed{\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.} \tag{5.47}$$

We note that this is similar to eq. (4.43), and the derivation proceeds in a similar manner. The quantity on the left-hand side of eq. (5.47) is the total energy in the signal $x[n]$, and

Parseval's relation states that this energy can also be determined by integrating the energy per unit frequency, $|X(e^{j\omega})|^2/2\pi$, over a full 2π interval of distinct discrete-time frequencies. In analogy with the continuous-time case, $|X(e^{j\omega})|^2$ is referred to as the *energy-density spectrum* of the signal $x[n]$. Note also that eq. (5.47) is the counterpart for aperiodic signals of Parseval's relation, eq. (3.110), for periodic signals, which equates the average power in a periodic signal with the sum of the average powers of its individual harmonic components.

Given the Fourier transform of a sequence, it is possible to use Fourier transform properties to determine whether a particular sequence has a number of different properties. To illustrate this idea, we present the following example.

Example 5.10

Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is depicted for $-\pi \leq \omega \leq \pi$ in Figure 5.16. We wish to determine whether or not, in the time domain, $x[n]$ is periodic, real, even, and/or of finite energy.

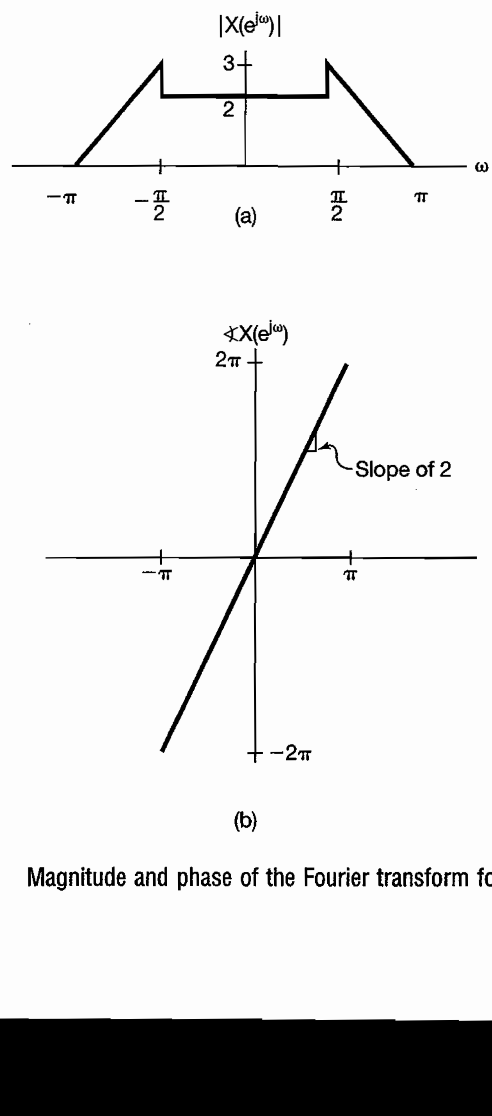


Figure 5.16 Magnitude and phase of the Fourier transform for Example 5.10.