

We can now use Parseval's relation to conclude that

$$|b_1|^2 + |b_{-1}|^2 = 1/2. \quad (3.83)$$

Substituting $b_1 = -b_{-1}$ in this equation, we obtain $|b_1| = 1/2$. Since b_1 is also known to be purely imaginary, it must be either $j/2$ or $-j/2$.

Now we can translate these conditions on b_0 and b_1 into equivalent statements on a_0 and a_1 . First, since $b_0 = 0$, Fact 4 implies that $a_0 = 0$. With $k = 1$, this condition implies that $a_1 = e^{-j\pi/2} b_{-1} = -j b_{-1} = j b_1$. Thus, if we take $b_1 = j/2$, then $a_1 = -1/2$, and therefore, from eq. (3.81), $x(t) = -\cos(\pi t/2)$. Alternatively, if we take $b_1 = -j/2$, then $a_1 = 1/2$, and therefore, $x(t) = \cos(\pi t/2)$.

3.6 FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS

In this section, we consider the Fourier series representation of discrete-time periodic signals. While the discussion closely parallels that of Section 3.3, there are some important differences. In particular, the Fourier series representation of a discrete-time periodic signal is a *finite* series, as opposed to the infinite series representation required for continuous-time periodic signals. As a consequence, there are no mathematical issues of convergence such as those discussed in Section 3.4.

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a discrete-time signal $x[n]$ is periodic with period N if

$$x[n] = x[n + N]. \quad (3.84)$$

The fundamental period is the smallest positive integer N for which eq. (3.84) holds, and $\omega_0 = 2\pi/N$ is the fundamental frequency. For example, the complex exponential $e^{j(2\pi/N)n}$ is periodic with period N . Furthermore, the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \dots \quad (3.85)$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related.

As mentioned in Section 1.3.3, there are only N distinct signals in the set given by eq. (3.85). This is a consequence of the fact that discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. Specifically, $\phi_0[n] = \phi_N[n]$, $\phi_1[n] = \phi_{N+1}[n]$, and, in general,

$$\phi_k[n] = \phi_{k+rN}[n]. \quad (3.86)$$

That is, when k is changed by any integer multiple of N , the identical sequence is generated. This differs from the situation in continuous time in which the signals $\phi_k(t)$ defined in eq. (3.24) are all different from one another.

We now wish to consider the representation of more general periodic sequences in terms of linear combinations of the sequences $\phi_k[n]$ in eq. (3.85). Such a linear combination has the form

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}. \quad (3.87)$$

Since the sequences $\phi_k[n]$ are distinct only over a range of N successive values of k , the summation in eq. (3.87) need only include terms over this range. Thus, the summation is on k , as k varies over a range of N successive integers, beginning with any value of k . We indicate this by expressing the limits of the summation as $k = \langle N \rangle$. That is,

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}. \quad (3.88)$$

For example, k could take on the values $k = 0, 1, \dots, N-1$, or $k = 3, 4, \dots, N+2$. In either case, by virtue of eq. (3.86), exactly the same set of complex exponential sequences appears in the summation on the right-hand side of eq. (3.88). Equation (3.88) is referred to as the *discrete-time Fourier series* and the coefficients a_k as the *Fourier series coefficients*.

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

Suppose now that we are given a sequence $x[n]$ that is periodic with fundamental period N . We would like to determine whether a representation of $x[n]$ in the form of eq. (3.88) exists and, if so, what the values of the coefficients a_k are. This question can be phrased in terms of finding a solution to a set of linear equations. Specifically, if we evaluate eq. (3.88) for N successive values of n corresponding to one period of $x[n]$, we obtain

$$\begin{aligned} x[0] &= \sum_{k=\langle N \rangle} a_k, \\ x[1] &= \sum_{k=\langle N \rangle} a_k e^{j2\pi k/N}, \\ &\vdots \\ x[N-1] &= \sum_{k=\langle N \rangle} a_k e^{j2\pi k(N-1)/N}. \end{aligned} \quad (3.89)$$

Thus, eq. (3.89) represents a set of N linear equations for the N unknown coefficients a_k as k ranges over a set of N successive integers. It can be shown that this set of equations is linearly independent and consequently can be solved to obtain the coefficients a_k in terms of the given values of $x[n]$. In Problem 3.32, we consider an example in which the Fourier series coefficients are obtained by explicitly solving the set of N equations given in eq. (3.89). However, by following steps parallel to those used in continuous time, it is possible to obtain a closed-form expression for the coefficients a_k in terms of the values of the sequence $x[n]$.

The basis for this result is the fact, shown in Problem 3.54, that

$$\sum_{n=\langle N \rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases} \quad (3.90)$$

Equation (3.90) states that the sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

Now consider the Fourier series representation of eq. (3.88). Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N \rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-r)(2\pi/N)n} \quad (3.91)$$

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N \rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-r)(2\pi/N)n} \quad (3.92)$$

From the identity in eq. (3.90), the innermost sum on n on the right-hand side of eq. (3.92) is zero, unless $k - r$ is zero or an integer multiple of N . Therefore, if we choose values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side of eq. (3.92) equals N if $k = r$ and 0 if $k \neq r$. The right-hand side of eq. (3.92) then reduces to Na_r , and we have

$$a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jr(2\pi/N)n} \quad (3.93)$$

This provides a closed-form expression for obtaining the Fourier series coefficients, and we have the *discrete-time Fourier series pair*

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \quad (3.94)$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk(2\pi/N)n}. \quad (3.95)$$

These equations play the same role for discrete-time periodic signals that eqs. (3.38) and (3.39) play for continuous-time periodic signals, with eq. (3.94) the *synthesis* equation and eq. (3.95) the *analysis* equation. As in continuous time, the discrete-time Fourier series coefficients a_k are often referred to as the *spectral coefficients* of $x[n]$. These coefficients specify a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials.

Referring to eq. (3.88), we see that if we take k in the range from 0 to $N - 1$, we have

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \dots + a_{N-1}\phi_{N-1}[n]. \quad (3.96)$$

Similarly, if k ranges from 1 to N , we obtain

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \dots + a_N\phi_N[n]. \quad (3.97)$$

From eq. (3.86), $\phi_0[n] = \phi_N[n]$, and therefore, upon comparing eqs. (3.96) and (3.97), we conclude that $a_0 = a_N$. Similarly, by letting k range over any set of N consecutive integers and using eq. (3.86), we can conclude that

$$a_k = a_{k+N}. \quad (3.98)$$

That is, if we consider more than N sequential values of k , the values a_k repeat periodically with period N . It is important that this fact be interpreted carefully. In particular, since there are only N distinct complex exponentials that are periodic with period N , the discrete-time Fourier series representation is a finite series with N terms. Therefore, if we fix the N consecutive values of k over which we define the Fourier series in eq. (3.94), we will obtain a set of exactly N Fourier coefficients from eq. (3.95). On the other hand, at times it will be convenient to use different sets of N values of k , and consequently, it is useful to regard eq. (3.94) as a sum over any *arbitrary* set of N successive values of k . For this reason, it is sometimes convenient to think of a_k as a sequence defined for all values of k , but where only N successive elements in the sequence will be used in the Fourier series representation. Furthermore, since the $\phi_k[n]$ repeat periodically with period N as we vary k [eq. (3.86)], so must the a_k [eq. (3.98)]. This viewpoint is illustrated in the next example.

Example 3.10

Consider the signal

$$x[n] = \sin \omega_0 n, \quad (3.99)$$

which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$ of Example 3.3. $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N , that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N , and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \quad (3.100)$$

Comparing eq. (3.100) with eq. (3.94), we see by inspection that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \quad (3.101)$$

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N ; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure 3.13. The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation (3.94).

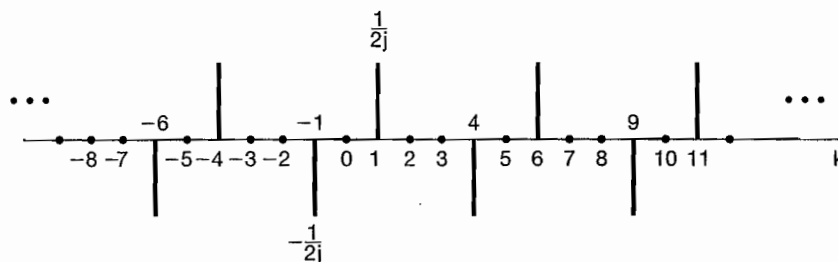


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is, when

$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N . Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n},$$

from which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in Figure 3.14. Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals $(-1/2j)$. Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

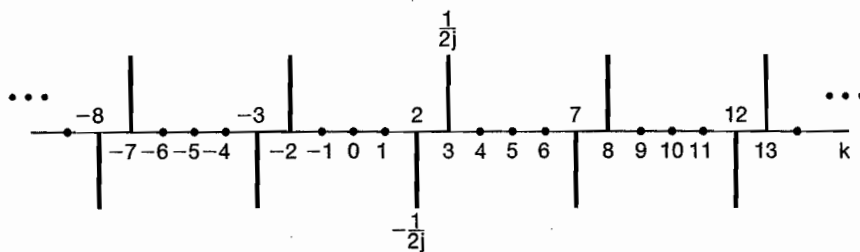


Figure 3.14 Fourier coefficients for $x[n] = \sin(3(2\pi/5)n)$.

Example 3.11

Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}n\right) + 3 \cos\left(\frac{2\pi}{N}n\right) + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

This signal is periodic with period N , and, as in Example 3.10, we can expand $x[n]$ directly in terms of complex exponentials to obtain

$$\begin{aligned} x[n] = 1 + \frac{1}{2j} [e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2} [e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] \\ + \frac{1}{2} [e^{j(4\pi n/N + \pi/2)} + e^{-j(4\pi n/N + \pi/2)}]. \end{aligned}$$

Collecting terms, we find that

$$\begin{aligned} x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right) e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right) e^{-j(2\pi/N)n} \\ + \left(\frac{1}{2} e^{j\pi/2}\right) e^{j2(2\pi/N)n} + \left(\frac{1}{2} e^{-j\pi/2}\right) e^{-j2(2\pi/N)n}. \end{aligned}$$

Thus the Fourier series coefficients for this example are

$$\begin{aligned} a_0 &= 1, \\ a_1 &= \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j, \\ a_{-1} &= \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j, \\ a_2 &= \frac{1}{2}j, \\ a_{-2} &= -\frac{1}{2}j, \end{aligned}$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation (3.94). Again, the Fourier coefficients are periodic with period N , so, for example, $a_N = 1$, $a_{3N-1} = \frac{3}{2} + \frac{1}{2}j$, and $a_{2-N} = \frac{1}{2}j$. In Figure 3.15(a) we have plotted the real and imaginary parts of these coefficients for $N = 10$, while the magnitude and phase of the coefficients are depicted in Figure 3.15(b).

Note that in Example 3.11, $a_{-k} = a_k^*$ for all values of k . In fact, this equality holds whenever $x[n]$ is real. The property is identical to one that we discussed in Section 3.3 for continuous-time periodic signals, and as in continuous time, one implication is that there are two alternative forms for the discrete-time Fourier series of real periodic sequences. These forms are analogous to the continuous-time Fourier series representations given in eqs. (3.31) and (3.32) and are examined in Problem 3.52. For our purposes, the exponential form of the Fourier series, as given in eqs. (3.94) and (3.95), is particularly convenient, and we will use it exclusively.

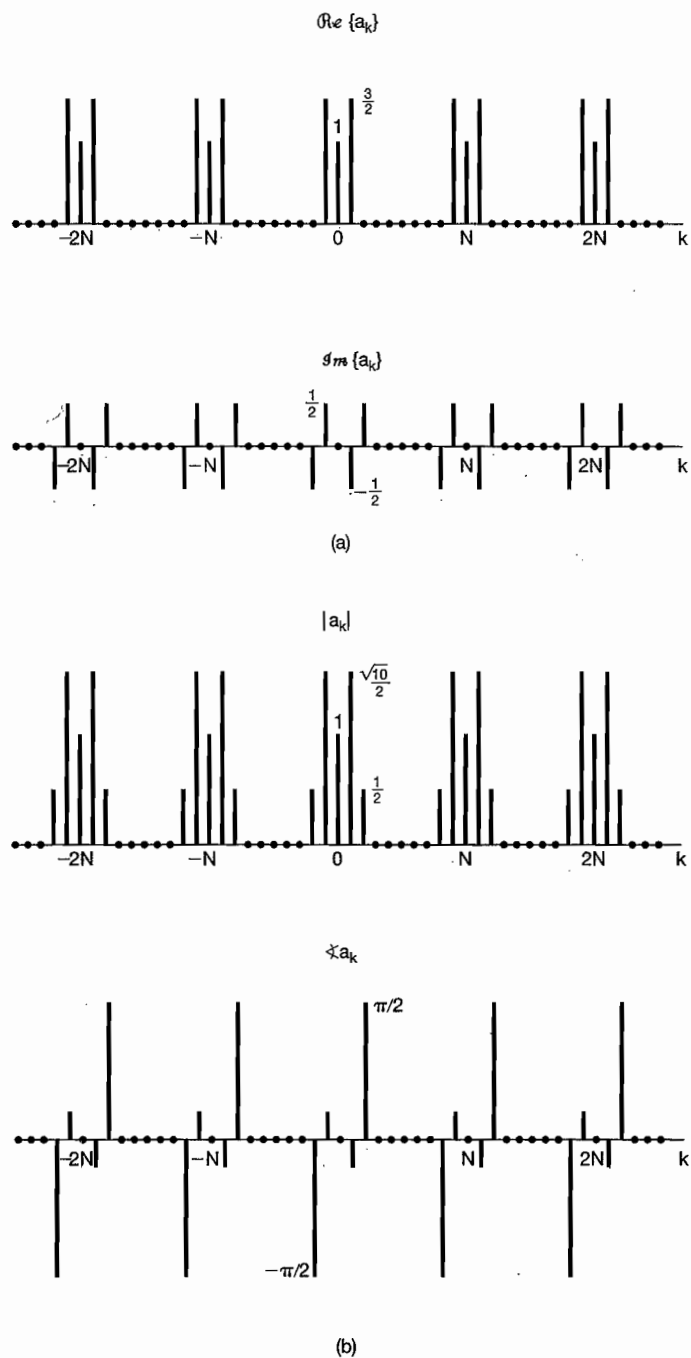


Figure 3.15 (a) Real and imaginary parts of the Fourier series coefficients in Example 3.11; (b) magnitude and phase of the same coefficients.

Example 3.12

In this example, we consider the discrete-time periodic square wave shown in Figure 3.16. We can evaluate the Fourier series for this signal using eq. (3.95). Because $x[n] = 1$ for $-N_1 \leq n \leq N_1$, it is particularly convenient to choose the length- N interval of summation in eq. (3.95) so that it includes the range $-N_1 \leq n \leq N_1$. In this case, we can express eq. (3.95) as

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}. \quad (3.102)$$

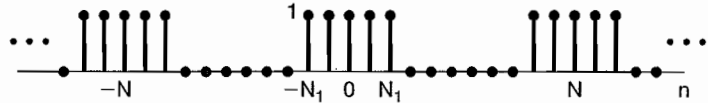


Figure 3.16 Discrete-time periodic square wave.

Letting $m = n + N_1$, we observe that eq. (3.102) becomes

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} \\ &= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}. \end{aligned} \quad (3.103)$$

The summation in eq. (3.103) consists of the sum of the first $2N_1 + 1$ terms in a geometric series, which can be evaluated using the result of Problem 1.54. This yields

$$\begin{aligned} a_k &= \frac{1}{N} e^{jk(2\pi/N)N_1} \left(\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right) \\ &= \frac{1}{N} \frac{e^{-jk(2\pi/2N)} [e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N}]}{e^{-jk(2\pi/2N)} [e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}]} \\ &= \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \dots \end{aligned} \quad (3.104)$$

and

$$a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \dots \quad (3.105)$$

The coefficients a_k for $2N_1 + 1 = 5$ are sketched for $N = 10, 20,$ and 40 in Figures 3.17(a), (b), and (c), respectively.

In discussing the convergence of the continuous-time Fourier series in Section 3.4, we considered the example of a symmetric square wave and observed how the finite sum in eq. (3.52) converged to the square wave as the number of terms approached infinity. In par-

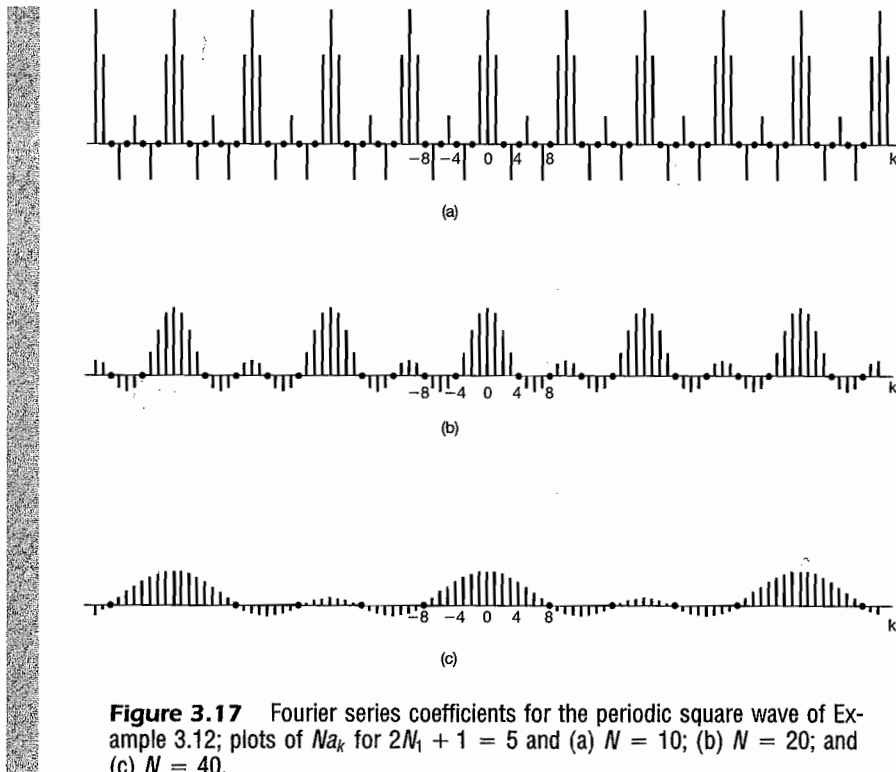


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

ticular, we observed the Gibbs phenomenon at the discontinuity, whereby, as the number of terms increased, the ripples in the partial sum (Figure 3.9) became compressed toward the discontinuity, with the peak amplitude of the ripples remaining constant independently of the number of terms in the partial sum. Let us consider the analogous sequence of partial sums for the discrete-time square wave, where, for convenience, we will assume that the period N is odd. In Figure 3.18, we have depicted the signals

$$\hat{x}[n] = \sum_{k=-M}^M a_k e^{jk(2\pi/N)n} \quad (3.106)$$

for the example of Figure 3.16 with $N = 9$, $2N_1 + 1 = 5$, and for several values of M . For $M = 4$, the partial sum exactly equals $x[n]$. We see in particular that in contrast to the continuous-time case, there are no convergence issues and there is no Gibbs phenomenon. In fact, there are no convergence issues with the discrete-time Fourier series in general. The reason for this stems from the fact that any discrete-time periodic sequence $x[n]$ is completely specified by a *finite* number N of parameters, namely, the values of the sequence over one period. The Fourier series analysis equation (3.95) simply transforms this set of N parameters into an equivalent set—the values of the N Fourier coefficients—and

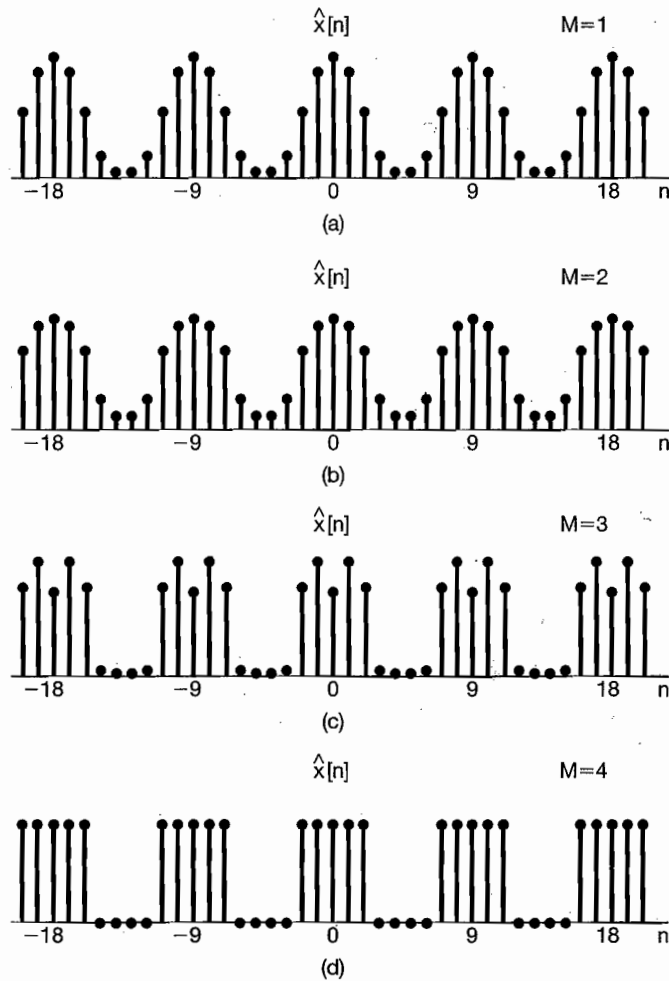


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

the synthesis equation (3.94) tells us how to recover the values of the original sequence in terms of a *finite* series. Thus, if N is odd and we take $M = (N - 1)/2$ in eq. (3.106), the sum includes exactly N terms, and consequently, from the synthesis equations, we have $\hat{x}[n] = x[n]$. Similarly, if N is even and we let

$$\hat{x}[n] = \sum_{k=-M+1}^M a_k e^{jk(2\pi/N)n}, \quad (3.107)$$

then with $M = N/2$, this sum consists of N terms, and again, we can conclude from eq. (3.94) that $\hat{x}[n] = x[n]$.

In contrast, a continuous-time periodic signal takes on a continuum of values over a single period, and an infinite number of Fourier coefficients are required to represent it.

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ } Periodic with period N and $y[n]$ } fundamental frequency $\omega_0 = 2\pi/N$	a_k } Periodic with b_k } period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a_{-k}^*
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=(N)} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=(N)} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^n x[k]$ (finite valued and periodic only) (if $a_0 = 0$)	$\left(\frac{1}{1 - e^{-jk(2\pi/N)}}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\begin{cases} \Re\{a_k\} \\ j\Im\{a_k\} \end{cases}$
Parseval's Relation for Periodic Signals		
$\frac{1}{N} \sum_{n=(N)} x[n] ^2 = \sum_{k=(N)} a_k ^2$		

The derivations of many of these properties are very similar to those of the corresponding properties for continuous-time Fourier series, and several such derivations are considered in the problems at the end of the chapter. In addition, in Chapter 5 we will see that most of the properties can be inferred from corresponding properties of the discrete-time Fourier transform. Consequently, we limit the discussion in the following subsections to only a few of these properties, including several that have important differences relative to those for continuous time. We also provide examples illustrating the usefulness of various discrete-time Fourier series properties for developing conceptual insights and helping to reduce the complexity of the evaluation of the Fourier series of many periodic sequences.

As with continuous time, it is often convenient to use a shorthand notation to indicate the relationship between a periodic signal and its Fourier series coefficients. Specifically, if $x[n]$ is a periodic signal with period N and with Fourier series coefficients denoted by a_k , then we will write

$$x[n] \xleftrightarrow{\text{FS}} a_k.$$

3.7.1 Multiplication

The multiplication property of the Fourier series representation is one example of a property that reflects the difference between continuous time and discrete time. From Table 3.1, the product of two continuous-time signals of period T results in a periodic signal with period T whose sequence of Fourier series coefficients is the *convolution* of the sequences of Fourier series coefficients of the two signals being multiplied. In discrete time, suppose that

$$x[n] \xleftrightarrow{\text{FS}} a_k$$

and

$$y[n] \xleftrightarrow{\text{FS}} b_k$$

are both periodic with period N . Then the product $x[n]y[n]$ is also periodic with period N , and, as shown in Problem 3.57, its Fourier coefficients, d_k , are given by

$$x[n]y[n] \xleftrightarrow{\text{FS}} d_k = \sum_{l=(N)} a_l b_{k-l}. \quad (3.108)$$

Equation (3.108) is analogous to the definition of convolution, except that the summation variable is now restricted to an interval of N consecutive samples. As shown in Problem 3.57, the summation can be taken over *any* set of N consecutive values of l . We refer to this type of operation as a *periodic convolution* between the two periodic sequences of Fourier coefficients. The usual form of the convolution sum (where the summation variable ranges from $-\infty$ to ∞) is sometimes referred to as *aperiodic convolution*, to distinguish it from periodic convolution.

3.7.2 First Difference

The discrete-time parallel to the differentiation property of the continuous-time Fourier series involves the use of the first-difference operation, which is defined as $x[n] - x[n-1]$.

If $x[n]$ is periodic with period N , then so is $y[n]$, since shifting $x[n]$ or linearly combining $x[n]$ with another periodic signal whose period is N always results in a periodic signal with period N . Also, if

$$x[n] \xleftrightarrow{\text{FS}} a_k,$$

then the Fourier coefficients corresponding to the first difference of $x[n]$ may be expressed as

$$x[n] - x[n-1] \xleftrightarrow{\text{FS}} (1 - e^{-jk(2\pi/N)})a_k, \quad (3.109)$$

which is easily obtained by applying the time-shifting and linearity properties in Table 3.2. A common use of this property is in situations where evaluation of the Fourier series coefficients is easier for the first difference than for the original sequence. (See Problem 3.31.)

3.7.3 Parseval's Relation for Discrete-Time Periodic Signals

As shown in Problem 3.57, Parseval's relation for discrete-time periodic signals is given by

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2, \quad (3.110)$$

where the a_k are the Fourier series coefficients of $x[n]$ and N is the period. As in the continuous-time case, the left-hand side of Parseval's relation is the average power in one period for the periodic signal $x[n]$. Similarly, $|a_k|^2$ is the average power in the k th harmonic component of $x[n]$. Thus, once again, Parseval's relation states that the average power in a periodic signal equals the sum of the average powers in all of its harmonic components. In discrete time, of course, there are only N distinct harmonic components, and since the a_k are periodic with period N , the sum on the right-hand side of eq. (3.110) can be taken over any N consecutive values of k .

3.7.4 Examples

In this subsection, we present several examples illustrating how properties of the discrete-time Fourier series can be used to characterize discrete-time periodic signals and to compute their Fourier series representations. Specifically, Fourier series properties, such as those listed in Table 3.2, may be used to simplify the process of determining the Fourier series coefficients of a given signal. This involves first expressing the given signal in terms of other signals whose Fourier series coefficients are already known or are simpler to compute. Then, using Table 3.2, we can express the Fourier series coefficients of the given signal in terms of the Fourier series coefficients of the other signals. This is illustrated in Example 3.13. Example 3.14 then illustrates the determination of a sequence from some partial information. In Example 3.15 we illustrate the use of the periodic convolution property in Table 3.2.

Example 3.13

Let us consider the problem of finding the Fourier series coefficients a_k of the sequence $x[n]$ shown in Figure 3.19(a). This sequence has a fundamental period of 5. We observe that $x[n]$ may be viewed as the sum of the square wave $x_1[n]$ in Figure 3.19(b) and the dc sequence $x_2[n]$ in Figure 3.19(c). Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k , we use the linearity property of Table 3.2 to conclude that

$$a_k = b_k + c_k. \quad (3.111)$$

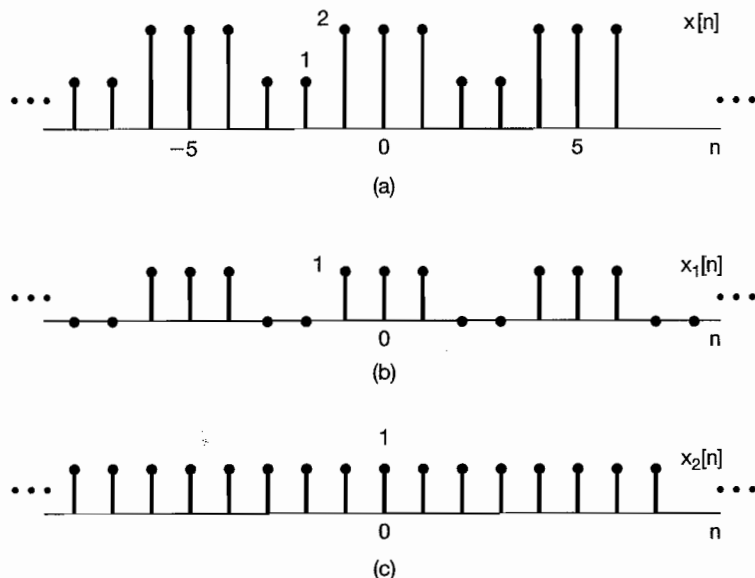


Figure 3.19 (a) Periodic sequence $x[n]$ for Example 3.13 and its representation as a sum of (b) the square wave $x_1[n]$ and (c) the dc sequence $x_2[n]$.

From Example 3.12 (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients b_k corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \frac{1}{5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases} \quad (3.112)$$

The sequence $x_2[n]$ has only a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5} \sum_{n=0}^4 x_2[n] = 1. \quad (3.113)$$

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5. The remaining coefficients of $x_2[n]$ must be zero, because $x_2[n]$ contains only a dc component. We can now substitute the expressions for b_k and c_k into eq. (3.111) to obtain

$$a_k = \begin{cases} b_k = \frac{1}{5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \frac{8}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases} \quad (3.114)$$

Example 3.14

Suppose we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^5 x[n] = 2$.
3. $\sum_{n=-2}^7 (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

Let us determine the sequence $x[n]$. We denote the Fourier series coefficients of $x[n]$ by a_k . From Fact 2, we conclude that $a_0 = 1/3$. Noting that $(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n}$, we see from Fact 3 that $a_3 = 1/6$. From Parseval's relation (see Table 3.2), the average power in $x[n]$ is

$$P = \sum_{k=0}^5 |a_k|^2. \tag{3.115}$$

Since each nonzero coefficient contributes a positive amount to P , and since the values of a_0 and a_3 are prespecified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It then follows that

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n, \tag{3.116}$$

which is sketched in Figure 3.20.

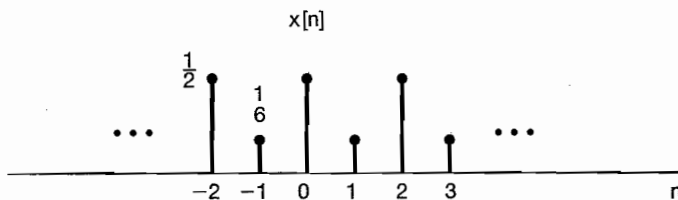


Figure 3.20 Sequence $x[n]$ that is consistent with the properties specified in Example 3.14.

Example 3.15

In this example we determine and sketch a periodic sequence, given an algebraic expression for its Fourier series coefficients. In the process, we will also exploit the periodic convolution property (see Table 3.2) of the discrete-time Fourier series. Specifically, as stated in the table and as shown in Problem 3.58, if $x[n]$ and $y[n]$ are periodic with period N , then the signal

$$w[n] = \sum_{r=(N)} x[r]y[n-r]$$

is also periodic with period N . Here, the summation may be taken over any set of N consecutive values of r . Furthermore, the Fourier series coefficients of $w[n]$ are equal to $N a_k b_k$, where a_k and b_k are the Fourier coefficients of $x[n]$ and $y[n]$, respectively.

Suppose now that we are told that a signal $w[n]$ is periodic with a fundamental period of $N = 7$ and with Fourier series coefficients

$$c_k = \frac{\sin^2(3\pi k/7)}{7 \sin^2(\pi k/7)}. \quad (3.117)$$

We observe that $c_k = 7d_k^2$, where d_k denotes the sequence of Fourier series coefficients of a square wave $x[n]$, as in Example 3.12, with $N_1 = 1$ and $N = 7$. Using the periodic convolution property, we see that

$$w[n] = \sum_{r \equiv 7} x[r]x[n-r] = \sum_{r=-3}^3 x[r]x[n-r], \quad (3.118)$$

where, in the last equality, we have chosen to sum over the interval $-3 \leq r \leq 3$. Except for the fact that the sum is limited to a finite interval, the product-and-sum method for evaluating convolution is applicable here. In fact, we can convert eq. (3.118) to an ordinary convolution by defining a signal $\hat{x}[n]$ that equals $x[n]$ for $-3 \leq n \leq 3$ and is zero otherwise. Then, from eq. (3.118),

$$w[n] = \sum_{r=-3}^3 \hat{x}[r]x[n-r] = \sum_{r=-\infty}^{+\infty} \hat{x}[r]x[n-r].$$

That is, $w[n]$ is the aperiodic convolution of the sequences $\hat{x}[n]$ and $x[n]$.

The sequences $x[r]$, $\hat{x}[r]$, and $x[n-r]$ are sketched in Figure 3.21 (a)–(c). From the figure we can immediately calculate $w[n]$. In particular we see that $w[0] = 3$; $w[-1] = w[1] = 2$; $w[-2] = w[2] = 1$; and $w[-3] = w[3] = 0$. Since $w[n]$ is periodic with period 7, we can then sketch $w[n]$ as shown in Figure 3.21(d).

3.8 FOURIER SERIES AND LTI SYSTEMS

In the preceding few sections, we have seen that the Fourier series representation can be used to construct any periodic signal in discrete time and essentially all periodic continuous-time signals of practical importance. In addition, in Section 3.2 we saw that the response of an LTI system to a linear combination of complex exponentials takes a particularly simple form. Specifically, in continuous time, if $x(t) = e^{st}$ is the input to a continuous-time LTI system, then the output is given by $y(t) = H(s)e^{st}$, where, from eq. (3.6),

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau, \quad (3.119)$$

in which $h(t)$ is the impulse response of the LTI system.

Similarly, if $x[n] = z^n$ is the input to a discrete-time LTI system, then the output is given by $y[n] = H(z)z^n$, where, from eq. (3.10),

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}, \quad (3.120)$$

in which $h[n]$ is the impulse response of the LTI system.

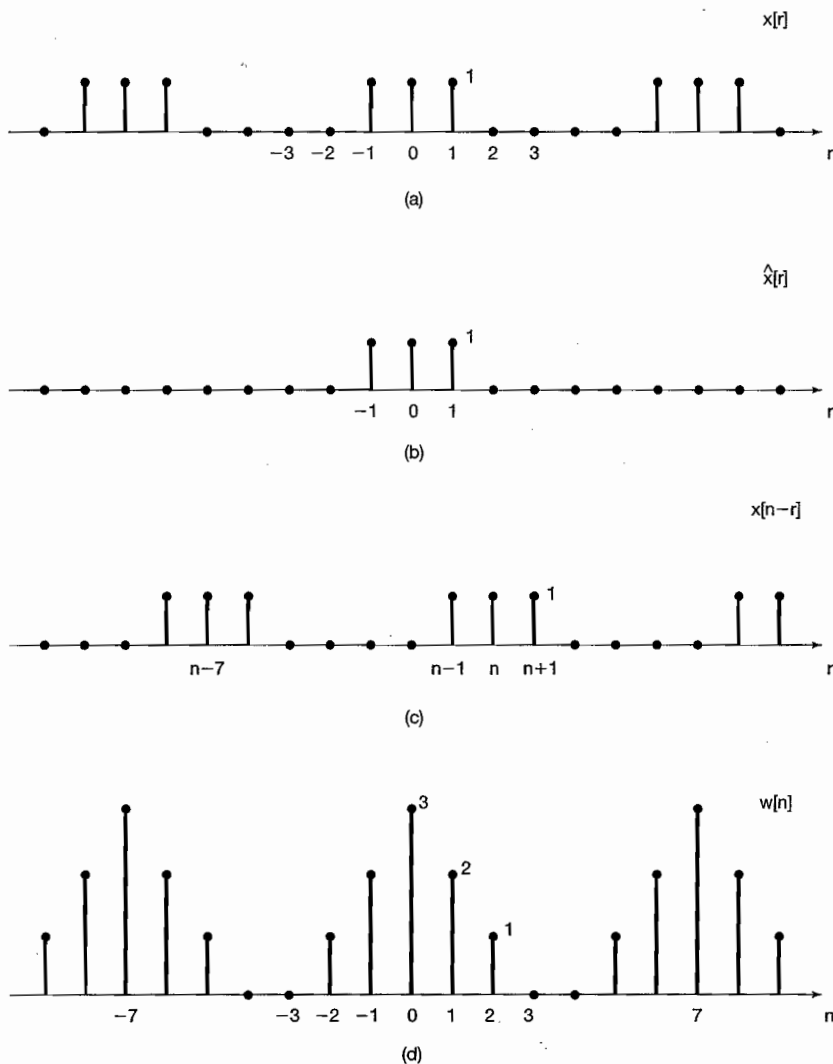


Figure 3.21 (a) The square-wave sequence $x[r]$ in Example 3.15; (b) the sequence $\hat{x}[r]$ equal to $x[r]$ for $-3 \leq r \leq 3$ and zero otherwise; (c) the sequence $x[n-r]$; (d) the sequence $w[n]$ equal to the periodic convolution of $x[n]$ with itself and to the aperiodic convolution of $\hat{x}[n]$ with $x[n]$.

When s or z are general complex numbers, $H(s)$ and $H(z)$ are referred to as the *system functions* of the corresponding systems. For continuous-time signals and systems in this and the following chapter, we focus on the specific case in which $\Re\{s\} = 0$, so that $s = j\omega$, and consequently, e^{st} is of the form $e^{j\omega t}$. This input is a complex exponential at frequency ω . The system function of the form $s = j\omega$ —i.e., $H(j\omega)$ viewed as a function of ω —is referred to as the *frequency response* of the system and is given by

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t} dt. \tag{3.121}$$