Accordingly, we note first that periodicity in the time domain implies that the Fourier transform is zero, except possibly for impulses located at various integer multiples of the fundamental frequency. This is not true for $X(e^{j\omega})$. We conclude, then, that $x[n]$ is *not* periodic.

Next, from the symmetry properties for Fourier transforms, we know that a real-valued sequence must have a Fourier transform of even magnitude and a phase function that is odd. This is true for the given $|X(e^{j\omega})|$ and $\sphericalangle X(e^{j\omega})$. We thus conclude that $x[n]$ is real.

Third, if $x[n]$ is an even function, then, by the symmetry properties for real signals, $X(e^{j\omega})$ must be real and even. However, since $X(e^{j\omega}) = |X(e^{j\omega})|e^{-j2\omega}$, $X(e^{j\omega})$ is not a real-valued function. Consequently, $x[n]$ is not even.

Finally, to test for the finite-energy property, we may use Parseval's relation,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega.$$

It is clear from Figure 5.16 that integrating $|X(e^{j\omega})|^2$ from $-\pi$ to π will yield a finite quantity. We conclude that $x[n]$ has finite energy.

In the next few sections, we consider several additional properties. The first two of these are the convolution and multiplication properties, similar to those discussed in Sections 4.4 and 4.5. The third is the property of duality, which is examined in Section 5.7, where we consider not only duality in the discrete-time domain, but also the duality that exists *between* the continuous-time and discrete-time domains.

5.4 THE CONVOLUTION PROPERTY

In Section 4.4, we discussed the importance of the continuous-time Fourier transform with regard to its effect on the operation of convolution and its use in dealing with continuous-time LTI systems. An identical relation applies in discrete time, and this is one of the principal reasons that the discrete-time Fourier transform is of such great value in representing and analyzing discrete-time LTI systems. Specifically, if $x[n]$, $h[n]$, and $y[n]$ are the input, impulse response, and output, respectively, of an LTI system, so that

$$y[n] = x[n] * h[n],$$

then

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}),} \tag{5.48}$$

where $X(e^{j\omega})$, $H(e^{j\omega})$, and $Y(e^{j\omega})$ are the Fourier transforms of $x[n]$, $h[n]$, and $y[n]$, respectively. Furthermore, comparing eqs. (3.122) and (5.9), we see that the frequency response of a discrete-time LTI system, as first defined in Section 3.8, is the Fourier transform of the impulse response of the system.

The derivation of eq. (5.48) exactly parallels that carried out in Section 4.4. In particular, as in continuous time, the Fourier synthesis equation (5.8) for $x[n]$ can be interpreted as a decomposition of $x[n]$ into a linear combination of complex exponentials with infinitesimal amplitudes proportional to $X(e^{j\omega})$. Each of these exponentials is an eigenfunction of the system. In Chapter 3, we used this fact to show that the Fourier series coefficients of the response of an LTI system to a periodic input are simply the Fourier coefficients of the input multiplied by the system's frequency response evaluated at the corresponding harmonic frequencies. The convolution property (5.48) represents the extension of this result to aperiodic inputs and outputs by using the Fourier transform rather than the Fourier series.

As in continuous time, eq. (5.48) maps the convolution of two signals to the simple algebraic operation of multiplying their Fourier transforms, a fact that both facilitates the analysis of signals and systems and adds significantly to our understanding of the way in which an LTI system responds to the input signals that are applied to it. In particular, from eq. (5.48), we see that the frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at each frequency ω. Thus, in frequency-selective filtering, for example, we want $H(e^{j\omega}) \approx 1$ over the range of frequencies corresponding to the desired passband and $H(e^{j\omega}) \approx 0$ over the band of frequencies to be eliminated or significantly attenuated.

5.4.1 Examples

To illustrate the convolution property, along with a number of other properties, we consider several examples in this section.

Example 5.11

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0].$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0]e^{-j\omega n} = e^{-j\omega n_0}.$$

Thus, for any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0}X(e^{j\omega}). \tag{5.49}$$

We note that, for this example, $y[n] = x[n - n_0]$ and eq. (5.49) is consistent with the time-shifting property. Note also that the frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ of a pure time shift has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is linear with frequency.

Example 5.12

Consider the discrete-time ideal lowpass filter introduced in Section 3.9.2. This system has the frequency response $H(e^{j\omega})$ illustrated in Figure 5.17(a). Since the impulse response and frequency response of an LTI system are a Fourier transform pair, we can determine the impulse response of the ideal lowpass filter from the frequency response using the Fourier transform synthesis equation (5.8). In particular, using $-\pi \le \omega \le \pi$ as the interval of integration in that equation, we see from Figure 5.17(a) that

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{\sin \omega_c n}{\pi n}, \end{aligned} \tag{5.50}$$

which is shown in Figure 5.17(b).

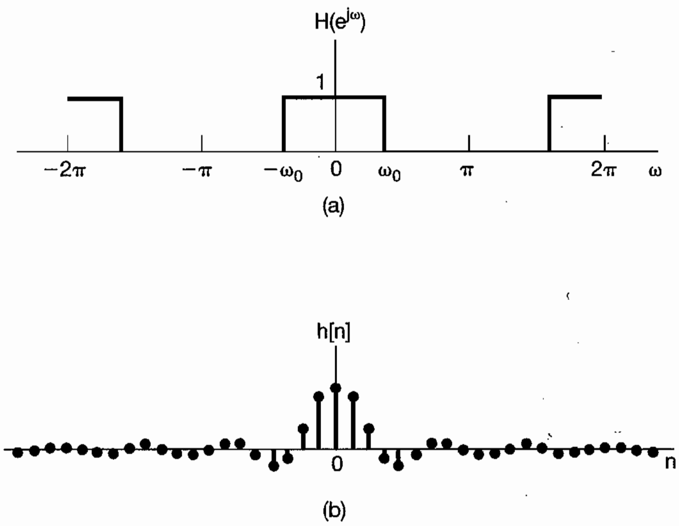

Figure 5.17 (a) Frequency response of a discrete-time ideal lowpass filter; (b) impulse response of the ideal lowpass filter.

In Figure 5.17, we come across many of the same issues that surfaced with the continuous-time ideal lowpass filter in Example 4.18. First, since $h[n]$ is not zero for $n < 0$, the ideal lowpass filter is not causal. Second, even if causality is not an important issue, there are other reasons, including ease of implementation and preferable time domain characteristics, that nonideal filters are generally used to perform frequency-selective filtering. In particular, the impulse response of the ideal lowpass filter in Figure 5.17(b) is oscillatory, a characteristic that is undesirable in some applications. In such cases, a trade-off between frequency-domain objectives such as frequency selectivity and time-domain properties such as nonoscillatory behavior must be made. In Chapter 6, we will discuss these and related ideas in more detail.

As the following example illustrates, the convolution property can also be of value in facilitating the calculation of convolution sums.

Example 5.13

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n],$$

with $|\alpha| < 1$, and suppose that the input to this system is

$$x[n] = \beta^n u[n],$$

with $|\beta| < 1$. Evaluating the Fourier transforms of $h[n]$ and $x[n]$, we have

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \tag{5.51}$$

and

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}, \tag{5.52}$$

so that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}. \tag{5.53}$$

As with Example 4.19 , determining the inverse transform of $Y(e^{j\omega})$ is most easily done by expanding $Y(e^{j\omega})$ by the method of partial fractions. Specifically, $Y(e^{j\omega})$ is a ratio of polynomials in powers of $e^{-j\omega}$, and we would like to express this as a sum of simpler terms of this type so that we can find the inverse transform of each term by inspection (together, perhaps, with the use of the frequency differentiation property of Section 5.3.8). The general algebraic procedure for rational transforms is described in the appendix. For this example, if $\alpha \neq \beta$, the partial fraction expansion of $Y(e^{j\omega})$ is of the form

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}. \tag{5.54}$$

Equating the right-hand sides of eqs (5.53) and (5.54), we find that

$$A = \frac{\alpha}{\alpha - \beta}, \quad B = -\frac{\beta}{\alpha - \beta}.$$

Therefore, from Example 5.1 and the linearity property, we can obtain the inverse transform of eq. (5.54) by inspection:

$$\begin{aligned} y[n] &= \frac{\alpha}{\alpha - \beta}\alpha^n u[n] - \frac{\beta}{\alpha - \beta}\beta^n u[n] \\ &= \frac{1}{\alpha - \beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]. \end{aligned} \tag{5.55}$$

For $\alpha = \beta$, the partial-fraction expansion in eq. (5.54) is not valid. However, in this case,

$$Y(e^{j\omega}) = \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)^2,$$

which can be expressed as

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right). \tag{5.56}$$

As in Example 4.19, we can use the frequency differentiation property, eq. (5.46), together with the Fourier transform pair

$$\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \alpha e^{-j\omega}},$$

to conclude that

$$n\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right).$$

To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{\mathcal{F}}{\longleftrightarrow} je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right),$$

and finally, accounting for the factor $1/\alpha$, in eq. (5.56), we obtain

$$y[n] = (n+1)\alpha^n u[n+1]. \tag{5.57}$$

It is worth noting that, although the right-hand side is multiplied by a step that begins at $n = -1$, the sequence $(n+1)\alpha^n u[n+1]$ is still zero prior to $n = 0$, since the factor $n+1$ is zero at $n = -1$. Thus, we can alternatively express $y[n]$ as

$$y[n] = (n+1)\alpha^n u[n]. \tag{5.58}$$

As illustrated in the next example, the convolution property, along with other Fourier transform properties, is often useful in analyzing system interconnections.

Example 5.14

Consider the system shown in Figure 5.18(a) with input $x[n]$ and output $y[n]$. The LTI systems with frequency response $H_{lp}(e^{j\omega})$ are ideal lowpass filters with cutoff frequency $\pi/4$ and unity gain in the passband.

Let us first consider the top path in Figure 5.18(a). The Fourier transform of the signal $w_1[n]$ can be obtained by noting that $(-1)^n = e^{j\pi n}$ so that $w_1[n] = e^{j\pi n}x[n]$. Using the frequency-shifting property, we then obtain

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)}).$$

The convolution property yields

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)}).$$

Since $w_3[n] = e^{j\pi n}w_2[n]$, we can again apply the frequency-shifting property to obtain

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}). \end{aligned}$$

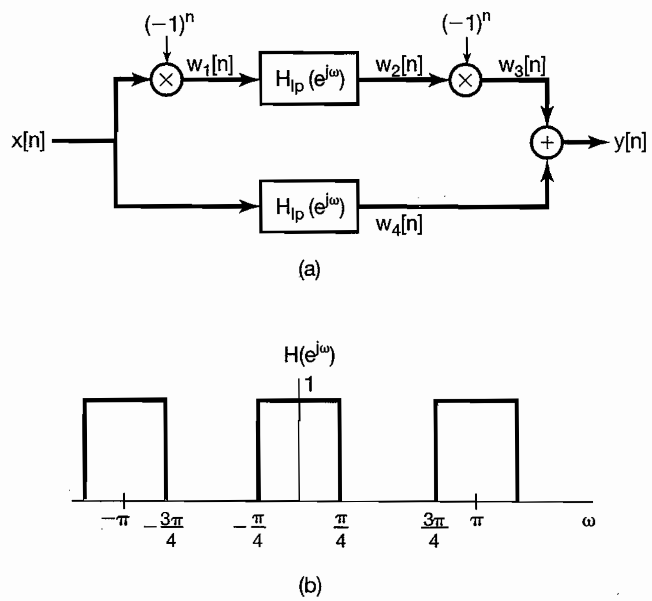

Figure 5.18 (a) System interconnection for Example 5.14; (b) the overall frequency response for this system.

Since discrete-time Fourier transforms are always periodic with period 2π,

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}).$$

Applying the convolution property to the lower path, we get

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega}).$$

From the linearity property of the Fourier transform, we obtain

$$\begin{aligned} Y(e^{j\omega}) &= W_3(e^{j\omega}) + W_4(e^{j\omega}) \\ &= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega}). \end{aligned}$$

Consequently, the overall system in Figure 5.18(a) has the frequency response

$$H(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]$$

which is shown in Figure 5.18(b).

As we saw in Example 5.7, $H_{lp}(e^{j(\omega-\pi)})$ is the frequency response of an ideal highpass filter. Thus, the overall system passes both low and high frequencies and stops frequencies between these two passbands. That is, the filter has what is often referred to as an *ideal bandstop characteristic,* where the stopband is the region $\pi/4 < |\omega| < 3\pi/4$.

It is important to note that, as in continuous time, not every discrete-time LTI system has a frequency response. For example, the LTI system with impulse response $h[n] = 2^n u[n]$ does not have a finite response to sinusoidal inputs, which is reflected in the fact that the Fourier transform analysis equation for $h[n]$ diverges. However, if an LTI system is stable, then, from Section 2.3.7, its impulse response is absolutely summable; that is,

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty. \tag{5.59}$$

Therefore, the frequency response always converges for stable systems. In using Fourier methods, we will be restricting ourselves to systems with impulse responses that have well-defined Fourier transforms. In Chapter 10, we will introduce an extension of the Fourier transform referred to as the z-transform that will allow us to use transform techniques for LTI systems for which the frequency response does not converge.

5.5 THE MULTIPLICATION PROPERTY

In Section 4.5, we introduced the multiplication property for continuous-time signals and indicated some of its applications through several examples. An analogous property exists for discrete-time signals and plays a similar role in applications. In this section, we derive this result directly and give an example of its use. In Chapters 7 and 8, we will use the multiplication property in the context of our discussions of sampling and communications.

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms. Then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n},$$

or since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta, \tag{5.60}$$

it follows that

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\} e^{-j\omega n}. \tag{5.61}$$

Interchanging the order of summation and integration, we obtain

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta. \tag{5.62}$$

The bracketed summation is $X_2(e^{j(\omega-\theta)})$, and consequently, eq. (5.62) becomes

$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.} \tag{5.63}$$

Equation (5.63) corresponds to a *periodic* convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluated over any interval of length 2π. The usual form of convolution (in which the integral ranges from $-\infty$ to $+\infty$) is often referred to as aperiodic convolution to distinguish it from periodic convolution. The mechanics of periodic convolution are most easily illustrated through an example.

Example 5.15

Consider the problem of finding the Fourier transform $X(e^{j\omega})$ of a signal $x[n]$ which is the product of two other signals; that is,

$$x[n] = x_1[n]x_2[n],$$

where

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

and

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

From the multiplication property given in eq. (5.63), we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, where the integral in eq. (5.63) can be taken over any interval of length 2π. Choosing the interval $-\pi < \theta \leq \pi$, we obtain

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta. \tag{5.64}$$

Equation (5.64) resembles aperiodic convolution, except for the fact that the integration is limited to the interval $-\pi < \theta \leq \pi$. However, we can convert the equation into an ordinary convolution by defining

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \text{for } -\pi < \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}.$$

Then, replacing $X_1(e^{j\theta})$ in eq. (5.64) by $\hat{X}_1(e^{j\theta})$, and using the fact that $\hat{X}_1(e^{j\theta})$ is zero for $|\theta| > \pi$, we see that

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta. \end{aligned}$$

Thus, $X(e^{j\omega})$ is $1/2\pi$ times the aperiodic convolution of the rectangular pulse $\hat{X}_1(e^{j\omega})$ and the periodic square wave $X_2(e^{j\omega})$, both of which are shown in Figure 5.19. The result of this convolution is the Fourier transform $X(e^{j\omega})$ shown in Figure 5.20.

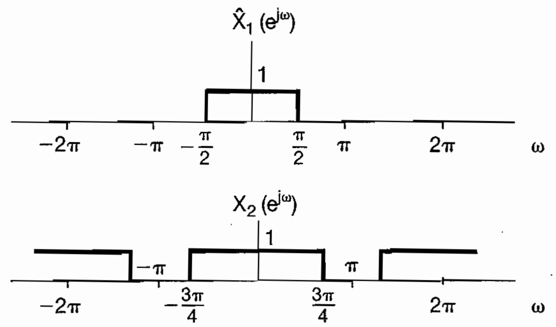

Figure 5.19 $\hat{X}_1(e^{j\omega})$ representing one period of $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$. The linear convolution of $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$ corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$.

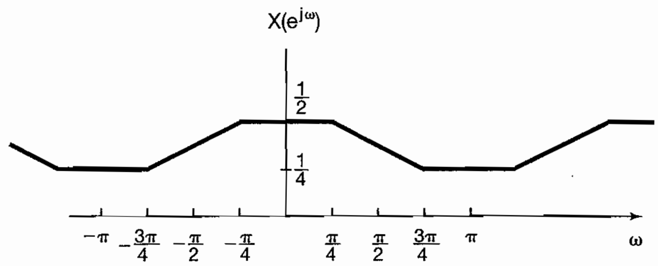

Figure 5.20 Result of the periodic convolution in Example 5.15.

5.6 TABLES OF FOURIER TRANSFORM PROPERTIES AND BASIC FOURIER TRANSFORM PAIRS

In Table 5.1, we summarize a number of important properties of the discrete-time Fourier transform and indicate the section of the text in which each is discussed. In Table 5.2, we summarize some of the basic and most important discrete-time Fourier transform pairs. Many of these have been derived in examples in the chapter.

5.7 DUALITY

In considering the continuous-time Fourier transform, we observed a symmetry or duality between the analysis equation (4.9) and the synthesis equation (4.8). No corresponding duality exists between the analysis equation (5.9) and the synthesis equation (5.8) for the discrete-time Fourier transform. However, there *is* a duality in the discrete-time Fourier *series* equations (3.94) and (3.95), which we develop in Section 5.7.1. In addition, there is

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

signal, reversed in time). To see this in more detail, consider two periodic sequences with period N, related through the summation

$$f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jr(2\pi/N)m}. \tag{5.65}$$

If we let $m = k$ and $r = n$, eq. (5.65) becomes

$$f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(2\pi/N)n}.$$

Comparing this with eq. (3.95), we see that the sequence $f[k]$ corresponds to the Fourier series coefficients of the signal $g[n]$. That is, if we adopt the notation

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

introduced in Chapter 3 for a periodic discrete-time signal and its set of Fourier coefficients, the two periodic sequences related through eq. (5.65) satisfy

$$g[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} f[k]. \tag{5.66}$$

Alternatively, if we let $m = n$ and $r = -k$, eq. (5.65) becomes

$$f[n] = \sum_{k=\langle N \rangle} \frac{1}{N} g[-k] e^{jk(2\pi/N)n}.$$

Comparing this with eq. (3.94), we find that $(1/N)g[-k]$ corresponds to the sequence of Fourier series coefficients of $f[n]$. That is,

$$f[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} \frac{1}{N} g[-k]. \tag{5.67}$$

As in continuous time, this duality implies that every property of the discrete-time Fourier series has a dual. For example, referring to Table 3.2, we see that the pair of properties

$$x[n - n_0] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k e^{-jk(2\pi/N)n_0} \tag{5.68}$$

and

$$e^{jm(2\pi/N)n} x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{k-m} \tag{5.69}$$

are dual. Similarly, from the same table, we can extract another pair of dual properties:

$$\sum_{r=\langle N \rangle} x[r] y[n-r] \stackrel{\mathcal{FS}}{\longleftrightarrow} N a_k b_k \tag{5.70}$$

and

$$x[n] y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} \sum_{l=\langle N \rangle} a_l b_{k-l}. \tag{5.71}$$

In addition to its consequences for the properties of discrete-time Fourier series, duality can often be useful in reducing the complexity of the calculations involved in determining Fourier series representations. This is illustrated in the following example.

Example 5.16

Consider the following periodic signal with a period of $N = 9$:

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \frac{5}{9}, & n = \text{multiple of } 9 \end{cases} \tag{5.72}$$

In Chapter 3, we found that a rectangular square wave has Fourier coefficients in a form much as in eq. (5.72). Duality, then, suggests that the coefficients for $x[n]$ must be in the form of a rectangular square wave. To see this more precisely, let $g[n]$ be a rectangular square wave with period $N = 9$ such that

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4. \end{cases}$$

The Fourier series coefficients b_k for $g[n]$ can be determined from Example 3.12 as

$$b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ \frac{5}{9}, & k = \text{multiple of } 9 \end{cases}$$

The Fourier series analysis equation (3.95) for $g[n]$ can now be written as

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-j2\pi nk/9}.$$

Interchanging the names of the variables k and n and noting that $x[n] = b_n$, we find that

$$x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-j2\pi nk/9}.$$

Letting $k' = -k$ in the sum on the right side, we obtain

$$x[n] = \frac{1}{9}\sum_{k'=-2}^{2}e^{+j2\pi nk'/9}.$$

Finally, moving the factor 1/9 inside the summation, we see that the right side of this equation has the form of the synthesis equation (3.94) for $x[n]$. We thus conclude that the Fourier coefficients of $x[n]$ are given by

$$a_k = \begin{cases} 1/9, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4, \end{cases}$$

and, of course, are periodic with period $N = 9$.

5.7.2 Duality between the Discrete-Time Fourier Transform and the Continuous-Time Fourier Series

In addition to the duality for the discrete Fourier series, there is a duality between the *discrete-time* Fourier *transform* and the *continuous-time* Fourier *series*. Specifically, let us compare the continuous-time Fourier series equations (3.38) and (3.39) with the discrete-time Fourier transform equations (5.8) and (5.9). We repeat these equations here for convenience:

[eq. (5.8)]
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega, \tag{5.73}$$

[eq. (5.9)]
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \tag{5.74}$$

[eq. (3.38)]
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{5.75}$$

[eq. (3.39)]
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \tag{5.76}$$

Note that eqs. (5.73) and (5.76) are very similar, as are eqs. (5.74) and (5.75), and in fact, we can interpret eqs. (5.73) and (5.74) as a *Fourier series* representation of the periodic frequency response $X(e^{j\omega})$. In particular, since $X(e^{j\omega})$ is a periodic function of ω with period 2π, it has a Fourier series representation as a weighted sum of harmonically related periodic exponential functions of ω, all of which have the common period of 2π. That is, $X(e^{j\omega})$ can be represented in a Fourier series as a weighted sum of the signals $e^{j\omega n}$, $n = 0, \pm 1, \pm 2, \ldots$. From eq. (5.74), we see that the nth Fourier coefficient in this expansion—i.e., the coefficient multiplying $e^{j\omega n}$—is $x[-n]$. Furthermore, since the period of $X(e^{j\omega})$ is 2π, eq. (5.73) can be interpreted as the Fourier series analysis equation for the Fourier series coefficient $x[n]$—i.e., for the coefficient multiplying $e^{-j\omega n}$ in the expression for $X(e^{j\omega})$ in eq. (5.74). The use of this duality relationship is best illustrated with an example.

Example 5.17

The duality between the discrete-time Fourier transform synthesis equation and the continuous-time Fourier series analysis equation may be exploited to determine the discrete-time Fourier transform of the sequence

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

To use duality, we first must identify a continuous-time signal $g(t)$ with period $T = 2\pi$ and Fourier coefficients $a_k = x[k]$. From Example 3.5, we know that if $g(t)$ is a periodic square wave with period 2π (or, equivalently, with fundamental frequency $\omega_0 = 1$) and with

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases},$$

then the Fourier series coefficients of $g(t)$ are

$$a_k = \frac{\sin(kT_1)}{k\pi}$$

Consequently, if we take $T_1 = \pi/2$, we will have $a_k = x[k]$. In this case the analysis equation for $g(t)$ is

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jkt}dt.$$

Renaming k as n and t as ω, we have

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jn\omega}d\omega. \tag{5.77}$$

Replacing n by $-n$ on both sides of eq. (5.77) and noting that the sinc function is even, we obtain

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{jn\omega}d\omega.$$

The right-hand side of this equation has the form of the Fourier transform synthesis equation for $x[n]$, where

$$X(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| \le \pi \end{cases}.$$

In Table 5.3, we present a compact summary of the Fourier series and Fourier transform expressions for both continuous-time and discrete-time signals, and we also indicate the duality relationships that apply in each case.

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality (Fourier Series: discrete time ↔ discrete frequency); duality (Fourier Series continuous time frequency domain ↔ Fourier Transform discrete time time domain); duality (Fourier Transform: continuous time ↔ continuous frequency)

5.8 SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

A general linear constant-coefficient difference equation for an LTI system with input $x[n]$ and output $y[n]$ is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5.78}$$

The class of systems described by such difference equations is quite an important and useful one. In this section, we take advantage of several of the properties of the discrete-time Fourier transform to determine the frequency response $H(e^{j\omega})$ for an LTI system described by such an equation. The approach we follow closely parallels the discussion in Section 4.7 for continuous-time LTI systems described by linear constant-coefficient differential equations.

There are two related ways in which to determine $H(e^{j\omega})$. The first of these, which we illustrated in Section 3.11 for several simple difference equations, explicitly uses the fact that complex exponentials are eigenfunctions of LTI systems. Specifically, if $x[n] = e^{j\omega n}$ is the input to an LTI system, then the output must be of the form $H(e^{j\omega})e^{j\omega n}$. Substituting these expressions into eq. (5.78) and performing some algebra allows us to solve for $H(e^{j\omega})$. In this section, we follow a second approach making use of the convolution, linearity, and time-shifting properties of the discrete-time Fourier transform. Let $X(e^{j\omega})$, $Y(e^{j\omega})$, and $H(e^{j\omega})$ denote the Fourier transforms of the input $x[n]$, output $y[n]$, and impulse response $h[n]$, respectively. The convolution property, eq. (5.48), of the discrete-time Fourier transform then implies that

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}. \tag{5.79}$$

Applying the Fourier transform to both sides of eq. (5.78) and using the linearity and time-shifting properties, we obtain the expression

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

or equivalently,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}. \tag{5.80}$$

Comparing eq. (5.80) with eq. (4.76), we see that, as in the case of continuous time, $H(e^{j\omega})$ is a ratio of polynomials, but in discrete time the polynomials are in the variable $e^{-j\omega}$. The coefficients of the numerator polynomial are the same coefficients as appear on the right side of eq. (5.78), and the coefficients of the denominator polynomial are the same as appear on the left side of that equation. Therefore, the frequency response of the LTI system specified by eq. (5.78) can be written down by inspection.

The difference equation (5.78) is generally referred to as an Nth-order difference equation, as it involves delays in the output $y[n]$ of up to N time steps. Also, the denominator of $H(e^{j\omega})$ in eq. (5.80) is an Nth-order polynomial in $e^{-j\omega}$.

Example 5.18

Consider the causal LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n], \tag{5.81}$$

with $|a| < 1$. From eq. (5.80), the frequency response of this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{5.82}$$

Comparing this with Example 5.1, we recognize it as the Fourier transform of the sequence $a^n u[n]$. Thus, the impulse response of the system is

$$h[n] = a^n u[n]. \tag{5.83}$$

Example 5.19

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]. \tag{5.84}$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}. \tag{5.85}$$

As a first step in obtaining the impulse response, we factor the denominator of eq. (5.85):

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}. \tag{5.86}$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example A.3 in the appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}. \tag{5.87}$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \tag{5.88}$$

The procedure followed in Example 5.19 is identical in style to that used in continuous time. Specifically, after expanding $H(e^{j\omega})$ by the method of partial fractions, we can find the inverse transform of each term by inspection. The same approach can be applied to the frequency response of any LTI system described by a linear constant-coefficient difference equation in order to determine the system impulse response. Also, as illustrated in the next example, if the Fourier transform $X(e^{j\omega})$ of the input to such a system is a ratio of polynomials in $e^{-j\omega}$, then $Y(e^{j\omega})$ is as well. In this case, we can use the same technique to find the response $y[n]$ to the input $x[n]$.

Example 5.20

Consider the LTI system of Example 5.19, and let the input to this system be

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Then, using eq. (5.80) and Example 5.1 or 5.18, we obtain

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$
$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-e\omega})^2}. \tag{5.89}$$

As described in the appendix, the form of the partial-fraction expansion in this case is

$$Y(e^{j\omega}) = \frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}, \tag{5.90}$$

where the constants B_{11}, B_{12}, and B_{21} can be determined using the techniques described in the appendix. This particular expansion is worked out in detail in Example A.4, and the values obtained are

$$B_{11} = -4, \quad B_{12} = -2, \quad B_{21} = 8,$$

so that

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}. \tag{5.91}$$

The first and third terms are of the same type as those encountered in Example 5.19, while the second term is of the same form as one seen in Example 5.13. Either from these examples or from Table 5.2, we can invert each of the terms in eq. (5.91) to obtain the inverse transform

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\} u[n]. \tag{5.92}$$

SUMMARY

In this chapter, we have paralleled Chapter 4 as we developed the Fourier transform for discrete-time signals and examined many of its important properties. Throughout the chapter, we have seen a great many similarities between continuous-time and discrete-time Fourier analysis, and we have also seen some important differences. For example, the relationship between Fourier series and Fourier transforms in discrete time is exactly analogous to that in continuous time. In particular, our derivation of the discrete-time Fourier transform for aperiodic signals from the discrete-time Fourier series representations is very much the same as the corresponding continuous-time derivation. Furthermore, many of the properties of continuous-time transforms have exact discrete-time counterparts. On the other hand, in contrast to the continuous-time case, the discrete-time Fourier transform of an aperiodic signal is always periodic with period 2π. In addition to similarities and differences such as these, we have described the duality relationships among the Fourier representations of continuous-time and discrete-time signals.

The most important similiarities between continuous- and discrete-time Fourier analysis are in their uses in analyzing and representing signals and LTI systems. Specifically, the convolution property provides us with the basis for the frequency-domain analysis of LTI systems. We have already seen some of the utility of this approach in our discussion of