particularly central roles in the study of LTI systems and their applications. The first of these, discussed in Section 4.4, is referred to as the *convolution property,* which is central to many signals and systems applications, including filtering. The second, discussed in Section 4.5, is referred to as the *multiplication property,* and it provides the foundation for our discussion of sampling in Chapter 7 and amplitude modulation in Chapter 8. In Section 4.6, we summarize the properties of the Fourier transform.

4.4 THE CONVOLUTION PROPERTY

As we saw in Chapter 3, if a periodic signal is represented in a Fourier series—i.e., as a linear combination of harmonically related complex exponentials, as in eq. (3.38)—then the response of an LTI system to this input can also be represented by a Fourier series. Because complex exponentials are eigenfunctions of LTI systems, the Fourier series coefficients of the output are those of the input multiplied by the frequency response of the system evaluated at the corresponding harmonic frequencies.

In this section, we extend this result to the situation in which the signals are aperiodic. We first derive the property somewhat informally, to build on the intuition we developed for periodic signals in Chapter 3, and then provide a brief, formal derivation starting directly from the convolution integral.

Recall our interpretation of the Fourier transform synthesis equation as an expression for $x(t)$ as a linear combination of complex exponentials. Specifically, referring back to eq. (4.7), $x(t)$ is expressed as the limit of a sum; that is,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \quad (4.47)$$

As developed in Sections 3.2 and 3.8, the response of a linear system with impulse response $h(t)$ to a complex exponential $e^{jk\omega_0 t}$ is $H(jk\omega_0)e^{jk\omega_0 t}$, where

$$H(jk\omega_0) = \int_{-\infty}^{+\infty} h(t)e^{-jk\omega_0 t}dt. \quad (4.48)$$

We can recognize the frequency response $H(j\omega)$, as defined in eq. (3.121), as the Fourier transform of the system impulse response. In other words, the Fourier transform of the impulse response (evaluated at $\omega = k\omega_0$) is the complex scaling factor that the LTI system applies to the eigenfunction $e^{jk\omega_0 t}$. From superposition [see eq. (3.124)], we then have

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0,$$

and thus, from eq. (4.47), the response of the linear system to $x(t)$ is

$$\begin{aligned} y(t) &= \lim_{\omega_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega. \end{aligned} \quad (4.49)$$

Since $y(t)$ and its Fourier transform $Y(j\omega)$ are related by

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)e^{j\omega t}d\omega, \tag{4.50}$$

we can identify $Y(j\omega)$ from eq. (4.49), yielding

$$Y(j\omega) = X(j\omega)H(j\omega). \tag{4.51}$$

As a more formal derivation, we consider the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \tag{4.52}$$

We desire $Y(j\omega)$, which is

$$Y(j\omega) = \mathfrak{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt. \tag{4.53}$$

Interchanging the order of integration and noting that $x(\tau)$ does not depend on t, we have

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau. \tag{4.54}$$

By the time-shift property, eq. (4.27), the bracketed term is $e^{-j\omega\tau}H(j\omega)$. Substituting this into eq. (4.54) yields

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau. \tag{4.55}$$

The integral is $X(j\omega)$, and hence,

$$Y(j\omega) = H(j\omega)X(j\omega).$$

That is,

$$\boxed{y(t) = h(t) * x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).} \tag{4.56}$$

Equation (4.56) is of major importance in signal and system analysis. As expressed in this equation, the Fourier transform maps the convolution of two signals into the product of their Fourier transforms. $H(j\omega)$, the Fourier transform of the impulse response, is the frequency response as defined in eq. (3.121) and captures the change in complex amplitude of the Fourier transform of the input at each frequency ω. For example, in frequency-selective filtering we may want to have $H(j\omega) \approx 1$ over one range of frequencies, so that the frequency components in this band experience little or no attenuation or change due to the system, while over another range of frequencies we may want to have $H(j\omega) \approx 0$, so that components in this range are eliminated or significantly attenuated.

The frequency response $H(j\omega)$ plays as important a role in the analysis of LTI systems as does its inverse transform, the unit impulse response. For one thing, since $h(t)$ completely characterizes an LTI system, then so must $H(j\omega)$. In addition, many of the properties of LTI systems can be conveniently interpreted in terms of $H(j\omega)$. For example, in Section 2.3, we saw that the impulse response of the cascade of two LTI systems is the convolution of the impulse responses of the individual systems and that the overall impulse response does not depend on the order in which the systems are cascaded. Using eq. (4.56), we can rephrase this in terms of frequency responses. As illustrated in Figure 4.19, since the impulse response of the cascade of two LTI systems is the convolution of the individual impulse responses, the convolution property then implies that the overall frequency response of the cascade of two systems is simply the product of the individual frequency responses. From this observation, it is then clear that the overall frequency response does not depend on the order of the cascade.

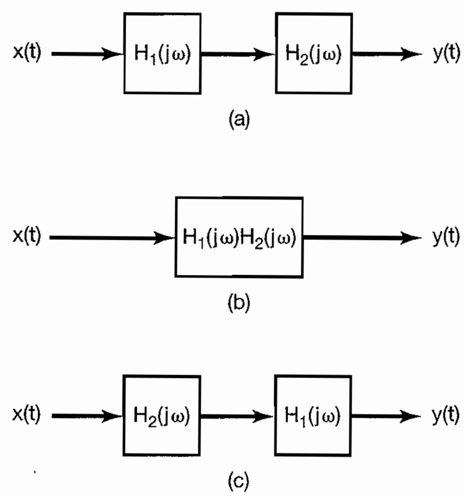

Figure 4.19 Three equivalent LTI systems. Here, each block represents an LTI system with the indicated frequency response.

As discussed in Section 4.1.2, convergence of the Fourier transform is guaranteed only under certain conditions, and consequently, the frequency response cannot be defined for every LTI system. If, however, an LTI system is stable, then, as we saw in Section 2.3.7 and Problem 2.49, its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty. \tag{4.57}$$

Equation (4.57) is one of the three Dirichlet conditions that together guarantee the existence of the Fourier transform $H(j\omega)$ of $h(t)$. Thus, assuming that $h(t)$ satisfies the other two conditions, as essentially all signals of physical or practical significance do, we see that a stable LTI system has a frequency response $H(j\omega)$.

In using Fourier analysis to study LTI systems, we will be restricting ourselves to systems whose impulse responses possess Fourier transforms. In order to use transform techniques to examine unstable LTI systems we will develop a generalization of the continuous-time Fourier transform, the Laplace transform. We defer this discussion to Chapter 9, and until then we will consider the many problems and practical applications that we can analyze using the Fourier transform.

4.4.1 Examples

To illustrate the convolution property and its applications further, let us consider several examples.

Example 4.15

Consider a continuous-time LTI system with impulse response

$$h(t) = \delta(t - t_0). \tag{4.58}$$

The frequency response of this system is the Fourier transform of $h(t)$ and is given by

$$H(j\omega) = e^{-j\omega t_0}. \tag{4.59}$$

Thus, for any input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}X(j\omega). \end{aligned} \tag{4.60}$$

This result, in fact, is consistent with the time-shift property of Section 4.3.2. Specifically, a system for which the impulse response is $\delta(t - t_0)$ applies a time shift of t_0 to the input—that is,

$$y(t) = x(t - t_0).$$

Thus, the shifting property given in eq. (4.27) also yields eq. (4.60). Note that, either from our discussion in Section 4.3.2 or directly from eq. (4.59), the frequency response of a system that is a pure time shift has unity magnitude at all frequencies (i.e., $|e^{-j\omega t_0}| = 1$) and has a phase characteristic $-\omega t_0$ that is a linear function of ω.

Example 4.16

As a second example, let us examine a differentiator—that is, an LTI system for which the input $x(t)$ and the output $y(t)$ are related by

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property of Section 4.3.4,

$$Y(j\omega) = j\omega X(j\omega). \tag{4.61}$$

Consequently, from eq. (4.56), it follows that the frequency response of a differentiator is

$$H(j\omega) = j\omega. \tag{4.62}$$

Example 4.17

Consider an integrator—that is, an LTI system specified by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The impulse response for this system is the unit step $u(t)$, and therefore, from Example 4.11 and eq. (4.33), the frequency response of the system is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Then using eq. (4.56), we have

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega), \end{aligned}$$

which is consistent with the integration property of eq. (4.32).

Example 4.18

As we discussed in Section 3.9.2, frequency-selective filtering is accomplished with an LTI system whose frequency response $H(j\omega)$ passes the desired range of frequencies and significantly attenuates frequencies outside that range. For example, consider the ideal lowpass filter introduced in Section 3.9.2, which has the frequency reponse illustrated in Figure 4.20 and given by

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}. \tag{4.63}$$

Now that we have developed the Fourier transform representation, we know that the impulse response $h(t)$ of this ideal filter is the inverse transform of eq. (4.63). Using the result in Example 4.5, we then have

$$h(t) = \frac{\sin \omega_c t}{\pi t}, \tag{4.64}$$

which is plotted in Figure 4.21.

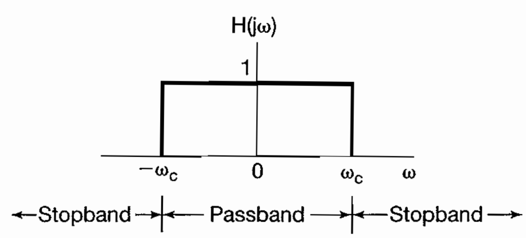

Figure 4.20 Frequency response of an ideal lowpass filter.

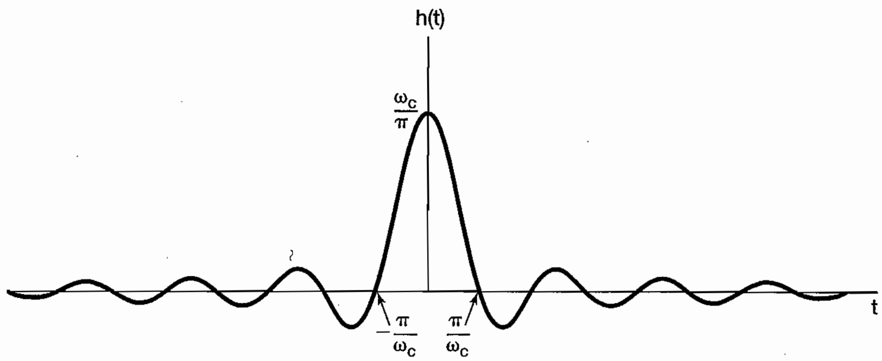

Figure 4.21 Impulse response of an ideal lowpass filter.

From Example 4.18, we can begin to see some of the issues that arise in filter design that involve looking in both the time and frequency domains. In particular, while the ideal lowpass filter does have perfect frequency selectivity, its impulse response has some characteristics that may not be desirable. First, note that $h(t)$ is not zero for $t < 0$. Consequently, the ideal lowpass filter is not causal, and thus, in applications requiring causal systems, the ideal filter is not an option. Moreover, as we discuss in Chapter 6, even if causality is not an essential constraint, the ideal filter is not easy to approximate closely, and non-ideal filters that are more easily implemented are typically preferred. Furthermore, in some applications (such as the automobile suspension system discussed in Section 6.7.1), oscillatory behavior in the impulse response of a lowpass filter may be undesirable. In such applications the time domain characteristics of the ideal lowpass filter, as shown in Figure 4.21, may be unacceptable, implying that we may need to trade off frequency-domain characteristics such as ideal frequency selectivity with time-domain properties.

For example, consider the LTI system with impulse response

$$h(t) = e^{-t}u(t). \tag{4.65}$$

The frequency response of this system is

$$H(j\omega) = \frac{1}{j\omega + 1}. \tag{4.66}$$

Comparing eqs. (3.145) and (4.66), we see that this system can be implemented with the simple *RC* circuit discussed in Section 3.10. The impulse response and the magnitude of the frequency response are shown in Figure 4.22. While the system does not have the strong frequency selectivity of the ideal lowpass filter, it is causal and has an impulse response that decays monotonically, i.e., without oscillations. This filter or somewhat more complex ones corresponding to higher order differential equations are quite frequently preferred to ideal filters because of their causality, ease of implementation, and flexibility in allowing trade-offs, among other design considerations such as frequency selectivity and oscillatory behavior in the time domain. Many of these issues will be discussed in more detail in Chapter 6.

The convolution property is often useful in evaluating the convolution integral—i.e., in computing the response of LTI systems. This is illustrated in the next example.

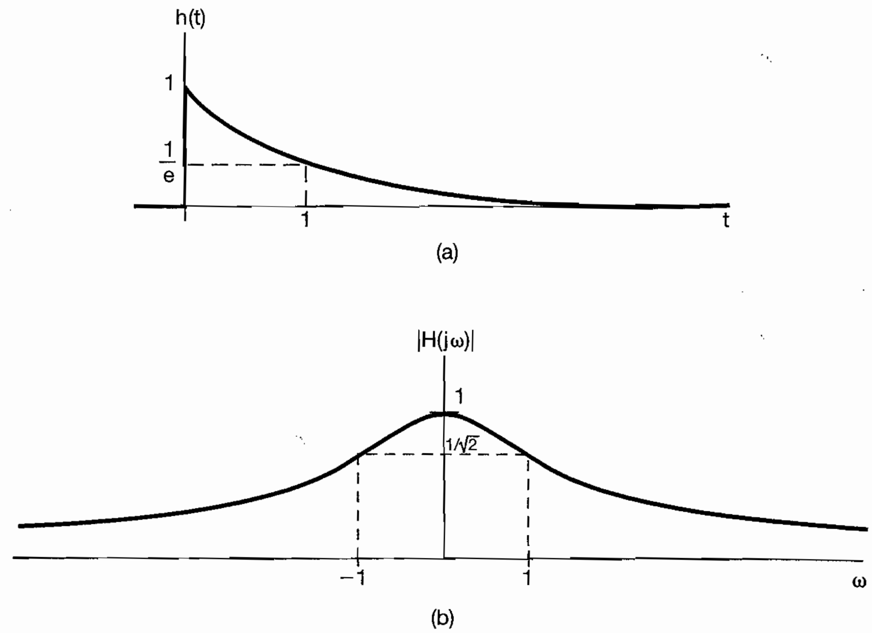

Figure 4.22 (a) Impulse response of the LTI system in eq. (4.65); (b) magnitude of the frequency response of the system.

Example 4.19

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, let us transform the problem into the frequency domain. From Example 4.1, the Fourier transforms of $x(t)$ and $h(t)$ are

$$X(j\omega) = \frac{1}{b + j\omega}$$

and

$$H(j\omega) = \frac{1}{a + j\omega}.$$

Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}. \tag{4.67}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion. Such expansions are extremely useful in evaluating inverse transforms, and the general method for performing a partial-fraction expansion is developed in the appendix. For this example, assuming that $b \neq a$, the partial fraction expansion for $Y(j\omega)$ takes the form

$$Y(j\omega) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega}, \tag{4.68}$$

where A and B are constants to be determined. One way to find A and B is to equate the right-hand sides of eqs. (4.67) and (4.68), multiply both sides by $(a+j\omega)(b+j\omega)$, and solve for A and B. Alternatively, in the appendix we present a more general and efficient method for computing the coefficients in partial-fraction expansions such as eq. (4.68). Using either of these approaches, we find that

$$A = \frac{1}{b-a} = -B,$$

and therefore,

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]. \tag{4.69}$$

The inverse transform for each of the two terms in eq. (4.69) can be recognized by inspection. Using the linearity property of Section 4.3.1, we have

$$y(t) = \frac{1}{b-a}[e^{-at}u(t) - e^{-bt}u(t)].$$

When $b = a$, the partial fraction expansion of eq. (4.69) is not valid. However, with $b = a$, eq. (4.67) becomes

$$Y(j\omega) = \frac{1}{(a+j\omega)^2}.$$

Recognizing this as

$$\frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right],$$

we can use the dual of the differentiation property, as given in eq. (4.40). Thus,

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

$$te^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right] = \frac{1}{(a+j\omega)^2},$$

and consequently,

$$y(t) = te^{-at}u(t).$$

Example 4.20

As another illustration of the usefulness of the convolution property, let us consider the problem of determining the response of an ideal lowpass filter to an input signal $x(t)$ that has the form of a sinc function. That is,

$$x(t) = \frac{\sin \omega_i t}{\pi t}.$$

Of course, the impulse response of the ideal lowpass filter is of a similar form, namely,

$$h(t) = \frac{\sin \omega_c t}{\pi t}.$$

The filter output $y(t)$ will therefore be the convolution of two sinc functions, which, as we now show, also turns out to be a sinc function. A particularly convenient way of deriving this result is to first observe that

$$Y(j\omega) = X(j\omega)H(j\omega),$$

where

$$X(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_i \\ 0 & \text{elsewhere} \end{cases}$$

and

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{elsewhere} \end{cases}.$$

Therefore,

$$Y(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{elsewhere} \end{cases},$$

where ω_0 is the smaller of the two numbers ω_i and ω_c. Finally, the inverse Fourier transform of $Y(j\omega)$ is given by

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} & \text{if } \omega_c \leq \omega_i \\ \dfrac{\sin \omega_i t}{\pi t} & \text{if } \omega_i \leq \omega_c \end{cases}.$$

That is, depending upon which of ω_c and ω_i is smaller, the output is equal to either $x(t)$ or $h(t)$.

4.5 THE MULTIPLICATION PROPERTY

The convolution property states that convolution in the *time* domain corresponds to multiplication in the *frequency* domain. Because of duality between the time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta \tag{4.70}$$

This can be shown by exploiting duality as discussed in Section 4.3.6, together with the convolution property, or by directly using the Fourier transform relations in a manner analogous to the procedure used in deriving the convolution property.

Multiplication of one signal by another can be thought of as using one signal to scale or *modulate* the amplitude of the other, and consequently, the multiplication of two signals is often referred to as *amplitude modulation*. For this reason, eq. (4.70) is sometimes referred to as the *modulation property.* As we shall see in Chapters 7 and 8, this property has several very important applications. To illustrate eq. (4.70), and to suggest one of the applications that we will discuss in subsequent chapters, let us consider several examples.

Example 4.21

Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is depicted in Figure 4.23(a). Also, consider the signal

$$p(t) = \cos \omega_0 t.$$

Then

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0),$$

as sketched in Figure 4.23(b), and the spectrum $R(j\omega)$ of $r(t) = s(t)p(t)$ is obtained by

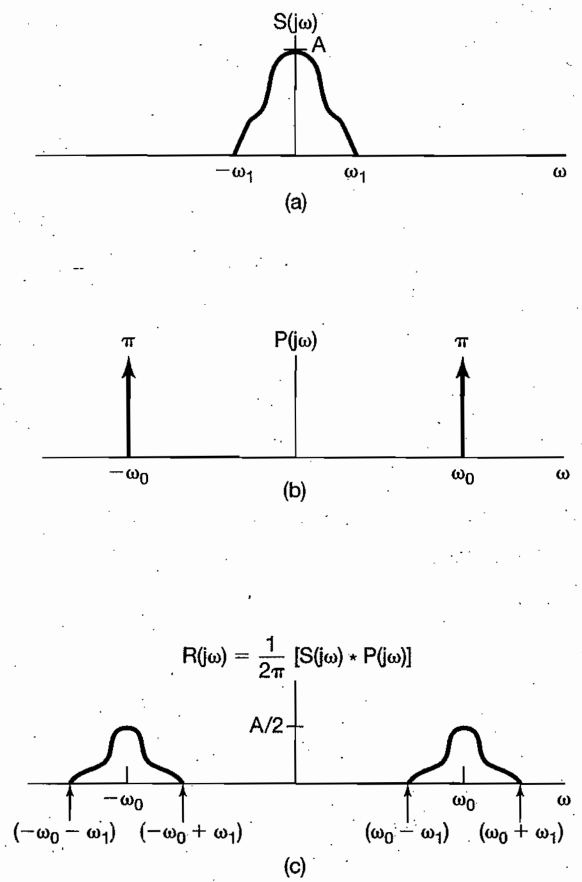

Figure 4.23 Use of the multiplication property in Example 4.21: (a) the Fourier transform of a signal $s(t)$; (b) the Fourier transform of $p(t) = \cos \omega_0 t$; (c) the Fourier transform of $r(t) = s(t)p(t)$.

an application of eq. (4.70), yielding

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta$$
$$= \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0)), \quad (4.71)$$

which is sketched in Figure 4.23(c). Here we have assumed that $\omega_0 > \omega_1$, so that the two nonzero portions of $R(j\omega)$ do not overlap. Clearly, the spectrum of $r(t)$ consists of the sum of two shifted and scaled versions of $S(j\omega)$.

From eq. (4.71) and from Figure 4.23, we see that all of the information in the signal $s(t)$ is preserved when we multiply this signal by a sinusoidal signal, although the information has been shifted to higher frequencies. This fact forms the basis for sinusoidal amplitude modulation systems for communications. In the next example, we learn how we can recover the original signal $s(t)$ from the amplitude-modulated signal $r(t)$.

Example 4.22

Let us now consider $r(t)$ as obtained in Example 4.21, and let

$$g(t) = r(t)p(t),$$

where, again, $p(t) = \cos\omega_0 t$. Then, $R(j\omega)$, $P(j\omega)$, and $G(j\omega)$ are as shown in Figure 4.24.

From Figure 4.24(c) and the linearity of the Fourier transform, we see that $g(t)$ is the sum of $(1/2)s(t)$ and a signal with a spectrum that is nonzero only at higher frequen-

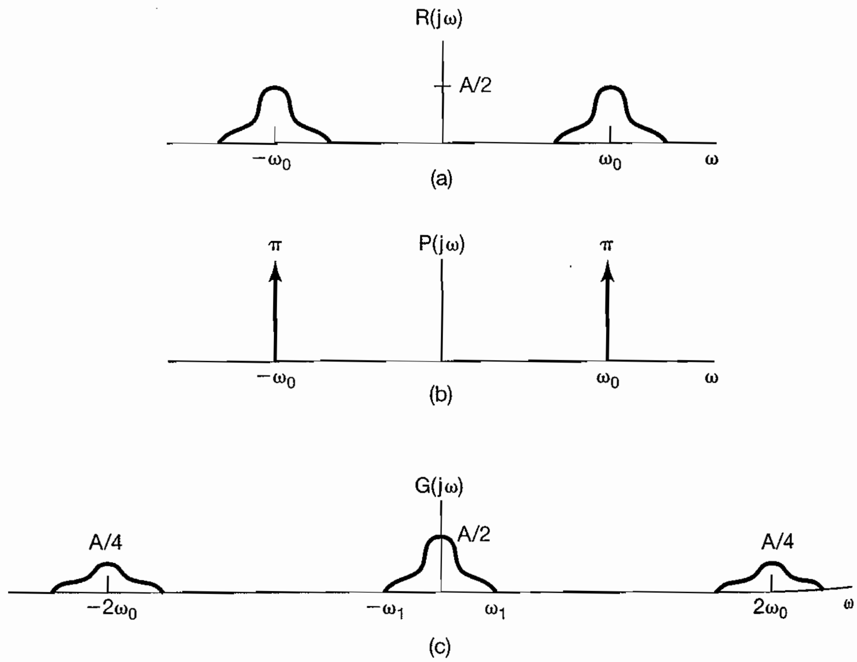

Figure 4.24 Spectra of signals considered in Example 4.22: (a) $R(j\omega)$; (b) $P(j\omega)$; (c) $G(j\omega)$.

cies (centered around $\pm 2\omega_0$). Suppose then that we apply the signal $g(t)$ as the input to a frequency-selective lowpass filter with frequency response $H(j\omega)$ that is constant at low frequencies (say, for $|\omega| < \omega_1$) and zero at high frequencies (for $|\omega| > \omega_1$). Then the output of this system will have as its spectrum $H(j\omega)G(j\omega)$, which, because of the particular choice of $H(j\omega)$, will be a scaled replica of $S(j\omega)$. Therefore, the output itself will be a scaled version of $s(t)$. In Chapter 8, we expand significantly on this idea as we develop in detail the fundamentals of amplitude modulation.

Example 4.23

Another illustration of the usefulness of the Fourier transform multiplication property is provided by the problem of determining the Fourier transform of the signal

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}.$$

The key here is to recognize $x(t)$ as the product of two sinc functions:

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right).$$

Applying the multiplication property of the Fourier transform, we obtain

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}.$$

Noting that the Fourier transform of each sinc function is a rectangular pulse, we can proceed to convolve those pulses to obtain the function $X(j\omega)$ displayed in Figure 4.25.

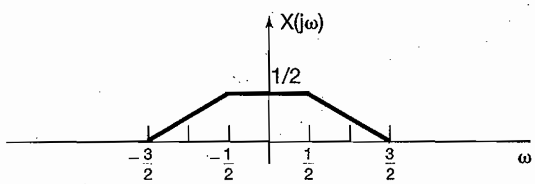

Figure 4.25 The Fourier transform of $x(t)$ in Example 4.23.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

As suggested in Examples 4.21 and 4.22 and developed more fully in Chapter 8, one of the important applications of the multiplication property is amplitude modulation in communication systems. Another important application is in the implementation of frequency-selective bandpass filters with tunable center frequencies that can be adjusted by the simple turn of a dial. In a frequency-selective bandpass filter built with elements such as resistors, operational amplifiers, and capacitors, the center frequency depends on a number of element values, all of which must be varied simultaneously in the correct way if the center frequency is to be adjusted directly. This is generally difficult and cumbersome in comparison with building a filter whose characteristics are fixed. An alternative to directly varying the filter characteristics is to use a fixed frequency-selective filter and shift the spectrum of the signal appropriately, using the principles of sinusoidal amplitude modulation.

For example, consider the system shown in Figure 4.26. Here, an input signal $x(t)$ is multiplied by the complex exponential signal $e^{j\omega_c t}$. The resulting signal is then passed through a lowpass filter with cutoff frequency ω_0, and the output is multiplied by $e^{-j\omega_c t}$. The spectra of the signals $x(t)$, $y(t)$, $w(t)$, and $f(t)$ are illustrated in Figure 4.27.

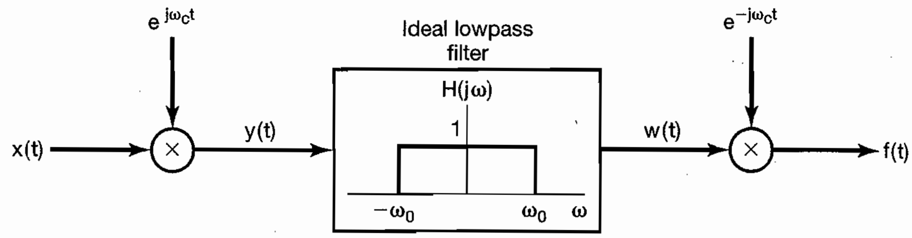

Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

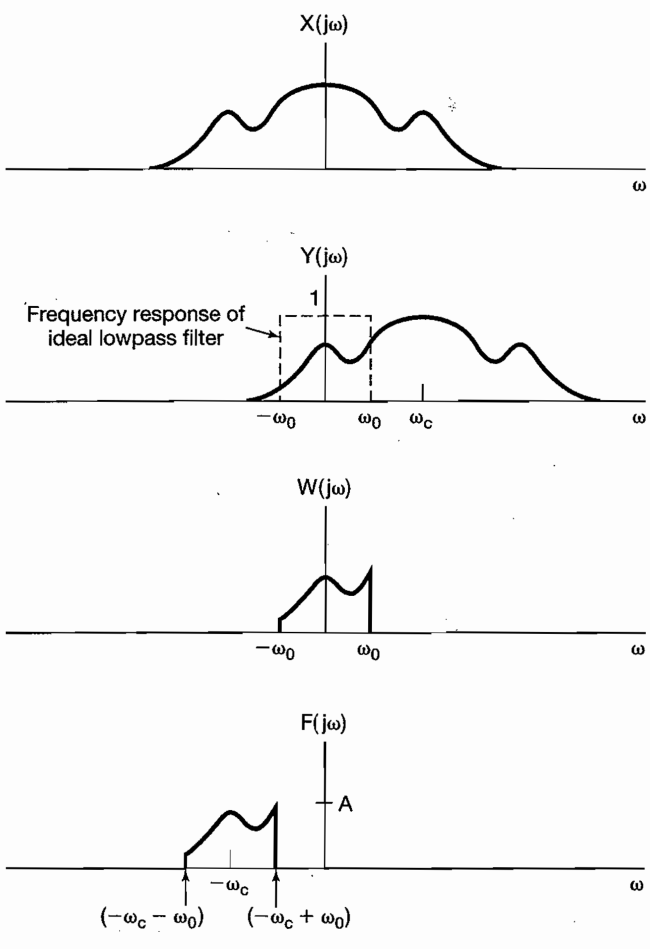

Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Specifically, from either the multiplication property or the frequency-shifting property it follows that the Fourier transform of $y(t) = e^{j\omega_c t}x(t)$ is

$$Y(j\omega) = \int_{-\infty}^{+\infty} \delta(\theta - \omega_c)X(\omega - \theta)d\theta$$

so that $Y(j\omega)$ equals $X(j\omega)$ shifted to the right by ω_c and frequencies in $X(j\omega)$ near $\omega = \omega_c$ have been shifted into the passband of the lowpass filter. Similarly, the Fourier transform of $f(t) = e^{-j\omega_c t}w(t)$ is

$$F(j\omega) = W(j(\omega + \omega_0)),$$

so that the Fourier transform of $F(j\omega)$ is $W(j\omega)$ shifted to the left by ω_c. From Figure 4.27, we observe that the overall system of Figure 4.26 is equivalent to an ideal bandpass filter with center frequency $-\omega_c$ and bandwidth $2\omega_0$, as illustrated in Figure 4.28. As the frequency ω_c of the complex exponential oscillator is varied, the center frequency of the bandpass filter varies.

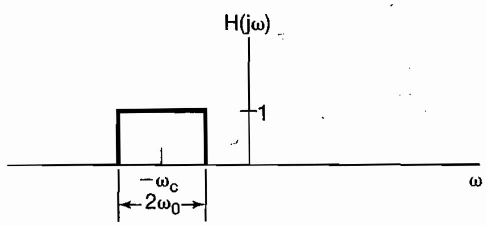

Figure 4.28 Bandpass filter equivalent of Figure 4.26.

In the system of Figure 4.26 with $x(t)$ real, the signals $y(t)$, $w(t)$, and $f(t)$ are all complex. If we retain only the real part of $f(t)$, the resulting spectrum is that shown in Figure 4.29, and the equivalent bandpass filter passes bands of frequencies centered around ω_c and $-\omega_c$, as indicated in Figure 4.30. Under certain conditions, it is also possible to use sinusoidal rather than complex exponential modulation to implement the system of the latter figure. This is explored further in Problem 4.46.

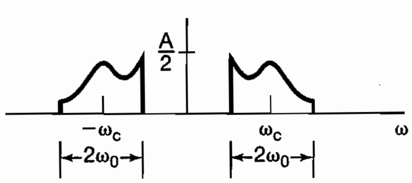

Figure 4.29 Spectrum of $\mathcal{R}e\{f(t)\}$ associated with Figure 4.26.

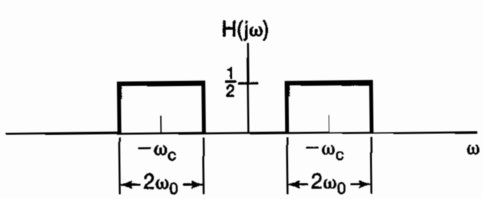

Figure 4.30 Equivalent bandpass filter for $\mathcal{R}e\{f(t)\}$ in Figure 4.29.

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \; \mathcal{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \; \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier analysis in our examination of signals and systems. All of the transform pairs, except for the last one in the table, have been considered in examples in the preceding sections. The last pair is considered in Problem 4.40. In addition, note that several of the signals in Table 4.2 are periodic, and for these we have also listed the corresponding Fourier series coefficients.

4.7 SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENTIAL EQUATIONS

As we have discussed on several occasions, a particularly important and useful class of continuous-time LTI systems is those for which the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{4.72}$$

In this section, we consider the question of determining the frequency response of such an LTI system. Throughout the discussion we will always assume that the frequency response of the system exists, i.e., that eq. (3.121) converges.

There are two closely related ways in which to determine the frequency response $H(j\omega)$ for an LTI system described by the differential equation (4.72). The first of these, which relies on the fact that complex exponential signals are eigenfunctions of LTI systems, was used in Section 3.10 in our analysis of several simple, nonideal filters. Specifically, if $x(t) = e^{j\omega t}$, then the output must be $y(t) = H(j\omega)e^{j\omega t}$. Substituting these expressions into the differential equation (4.72) and performing some algebra, we can then solve for $H(j\omega)$. In this section we use an alternative approach to arrive at the same answer, making use of the differentiation property, eq. (4.31), of Fourier transforms.

Consider an LTI system characterized by eq. (4.72). From the convolution property,

$$Y(j\omega) = H(j\omega)X(j\omega),$$

or equivalently,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \tag{4.73}$$

where $X(j\omega)$, $Y(j\omega)$, and $H(j\omega)$ are the Fourier transforms of the input $x(t)$, output $y(t)$, and impulse response $h(t)$, respectively. Next, consider applying the Fourier transform to both sides of eq. (4.72) to obtain

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}. \tag{4.74}$$

From the linearity property, eq. (4.26), this becomes

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\}, \tag{4.75}$$

and from the differentiation property, eq. (4.31),

$$\sum_{k=0}^{N} a_k(j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k(j\omega)^k X(j\omega),$$

or equivalently,

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k(j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k(j\omega)^k\right].$$

Thus, from eq. (4.73),

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k}. \tag{4.76}$$

Observe that $H(j\omega)$ is thus a rational function; that is, it is a ratio of polynomials in $(j\omega)$. The coefficients of the numerator polynomial are the same coefficients as those that appear on the right-hand side of eq. (4.72), and the coefficients of the denominator polynomial are the same coefficients as appear on the left side of eq. (4.72). Hence, the frequency response given in eq. (4.76) for the LTI system characterized by eq. (4.72) can be written down directly by inspection.

The differential equation (4.72) is commonly referred to as an Nth-order differential equation, as the equation involves derivatives of the output $y(t)$ up through the Nth derivative. Also, the denominator of $H(j\omega)$ in eq. (4.76) is an Nth-order polynomial in $(j\omega)$.

Example 4.24

Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{4.77}$$

with $a > 0$. From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}. \tag{4.78}$$

Comparing this with the result of Example 4.1, we see that eq. (4.78) is the Fourier transform of $e^{-at}u(t)$. The impulse response of the system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example 4.25

Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}. \tag{4.79}$$

To determine the corresponding impulse response, we require the inverse Fourier transform of $H(j\omega)$. This can be found using the technique of partial-fraction expansion employed in Example 4.19 and discussed in detail in the appendix. (In particular, see Example A.1, in which the details of the calculations for the partial-fraction expansion of eq. (4.79) are worked out.) As a first step, we factor the denominator of the right-hand side of eq. (4.79) into a product of lower order terms:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}. \tag{4.80}$$

Then, using the method of partial-fraction expansion, we find that

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}.$$

The inverse transform of each term can be recognized from Example 4.24, with the result that

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

The procedure used in Example 4.25 to obtain the inverse Fourier transform is generally useful in inverting transforms that are ratios of polynomials in $j\omega$. In particular, we can use eq. (4.76) to determine the frequency response of any LTI system described by a linear constant-coefficient differential equation and then can calculate the impulse response by performing a partial-fraction expansion that puts the frequency response into a form in which the inverse transform of each term can be recognized by inspection. In addition, if the Fourier transform $X(j\omega)$ of the input to such a system is also a ratio of polynomials in $j\omega$, then so is $Y(j\omega) = H(j\omega)X(j\omega)$. In this case we can use the same technique to solve the differential equation—that is, to find the response $y(t)$ to the input $x(t)$. This is illustrated in the next example.

Example 4.26

Consider the system of Example 4.25, and suppose that the input is

$$x(t) = e^{-t}u(t).$$

Then, using eq. (4.80), we have

$$\begin{aligned} Y(j\omega) = H(j\omega)X(j\omega) &= \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}\right]\left[\frac{1}{j\omega + 1}\right] \\ &= \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}. \end{aligned} \tag{4.81}$$

As discussed in the appendix, in this case the partial-fraction expansion takes the form

$$Y(j\omega) = \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3}, \tag{4.82}$$

where A_{11}, A_{12}, and A_{21} are constants to be determined. In Example A.2 in the appendix, the technique of partial-fraction expansion is used to determine these constants. The values obtained are

$$A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4},$$

so that

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}. \tag{4.83}$$

Again, the inverse Fourier transform for each term in eq. (4.83) can be obtained by inspection. The first and third terms are of the same type that we have encountered in the preceding two examples, while the inverse transform of the second term can be obtained from Table 4.2 or, as was done in Example 4.19, by applying the dual of the differentiation property, as given in eq. (4.40), to $1/(j\omega + 1)$. The inverse transform of eq. (4.83) is then found to be

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

From the preceding examples, we see how the techniques of Fourier analysis allow us to reduce problems concerning LTI systems characterized by differential equations to straightforward algebraic problems. This important fact is illustrated further in a number of the problems at the end of the chapter. In addition (see Chapter 6), the algebraic structure of the rational transforms encountered in dealing with LTI systems described by differential equations greatly facilitate the analysis of their frequency-domain properties and the development of insights into both the time-domain and frequency-domain characteristics of this important class of systems.

SUMMARY

In this chapter, we have developed the Fourier transform representation for continous-time signals and have examined many of the properties that make this transform so useful. In particular, by viewing an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large, we derived the Fourier transform representation for aperiodic signals from the Fourier series representation for periodic signals developed in Chapter 3. In addition, periodic signals themselves can be represented using Fourier transforms consisting of trains of impulses located at the harmonic frequencies of the periodic signal and with areas proportional to the corresponding Fourier series coefficients.

The Fourier transform possesses a wide variety of important properties that describe how different characteristics of signals are reflected in their transforms, and in this chapter we have derived and examined many of these properties. Among them are two that have particular significance for our study of signals and systems. The first is the convolution property, which is a direct consequence of the eigenfunction property of complex exponential signals and which leads to the description of an LTI system in terms of its frequency response. This description plays a fundamental role in the frequency-domain approach to the analysis of LTI systems, which we will continue to explore in subsequent chapters. The second property of the Fourier transform that has extremely important implications is the multiplication property, which provides the basis for the frequency-domain analysis of sampling and modulation systems. We examine these systems further in Chapters 7 and 8.

We have also seen that the tools of Fourier analysis are particularly well suited to the examination of LTI systems characterized by linear constant-coefficient differential equations. Specifically, we have found that the frequency response for such a system can be determined by inspection and that the technique of partial-fraction expansion can then be used to facilitate the calculation of the impulse response of the system. In subsequent chapters, we will find that the convenient algebraic structure of the frequency responses of these systems allows us to gain considerable insight into their characteristics in both the time and frequency domains.

Chapter 4 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

4.1. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $e^{-2(t-1)}u(t-1)$ **(b)** $e^{-2|t-1|}$

Sketch and label the magnitude of each Fourier transform.

4.2. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $\delta(t+1) + \delta(t-1)$ **(b)** $\frac{d}{dt}\{u(-2-t) + u(t-2)\}$

Sketch and label the magnitude of each Fourier transform.

4.3. Determine the Fourier transform of each of the following periodic signals:

(a) $\sin(2\pi t + \frac{\pi}{4})$ **(b)** $1 + \cos(6\pi t + \frac{\pi}{8})$

4.4. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transforms of:

(a) $X_1(j\omega) = 2\pi\,\delta(\omega) + \pi\,\delta(\omega - 4\pi) + \pi\,\delta(\omega + 4\pi)$

(b) $X_2(j\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 2 \end{cases}$

4.5. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$, where

$$|X(j\omega)| = 2\{u(\omega+3) - u(\omega-3)\},$$

$$\sphericalangle X(j\omega) = -\frac{3}{2}\omega + \pi.$$

Use your answer to determine the values of t for which $x(t) = 0$.

4.6. Given that $x(t)$ has the Fourier transform $X(j\omega)$, express the Fourier transforms of the signals listed below in terms of $X(j\omega)$. You may find useful the Fourier transform properties listed in Table 4.1.

(a) $x_1(t) = x(1-t) + x(-1-t)$

(b) $x_2(t) = x(3t-6)$

(c) $x_3(t) = \frac{d^2}{dt^2}x(t-1)$

4.7. For each of the following Fourier transforms, use Fourier transform properties (Table 4.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X_1(j\omega) = u(\omega) - u(\omega-2)$

(b) $X_2(j\omega) = \cos(2\omega)\sin(\frac{\omega}{2})$

(c) $X_3(j\omega) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = (\sin 2\omega)/\omega$ and $B(\omega) = 2\omega + \frac{\pi}{2}$

(d) $X(j\omega) = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{|k|}\,\delta(\omega - \frac{k\pi}{4})$

4.8. Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2}. \\ 1, & t > \frac{1}{2} \end{cases}$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

4.9. Consider the signal

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t+1)/2, & -1 \le t \le 1 \end{cases}$$

(a) With the help of Tables 4.1 and 4.2, determine the closed-form expression for $X(j\omega)$.

(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$.

(c) What is the Fourier transform of the odd part of $x(t)$?