or equivalently,

$$x[n] = \frac{-(-a)^n}{n} u[n-1].$$

In Problem 10.63 we consider a related example with region of convergence $|z| < |a|$.

10.4 GEOMETRIC EVALUATION OF THE FOURIER TRANSFORM FROM THE POLE-ZERO PLOT

In Section 10.1 we noted that the z-transform reduces to the Fourier transform for $|z| = 1$ (i.e., for the contour in the z-plane corresponding to the unit circle), provided that the ROC of the z-transform includes the unit circle, so that the Fourier transform converges. In a similar manner, we saw in Chapter 9 that, for continuous-time signals, the Laplace transform reduces to the Fourier transform on the $j\omega$-axis in the s-plane. In Section 9.4, we also discussed the geometric evaluation of the continuous-time Fourier transform from the pole-zero plot. In the discrete-time case, the Fourier transform can again be evaluated geometrically by considering the pole and zero vectors in the z-plane. However, since in this case the rational function is to be evaluated on the contour $|z| = 1$, we consider the vectors from the poles and zeros to the unit circle rather than to the imaginary axis. To illustrate the procedure, let us consider first-order and second-order systems, as discussed in Section 6.6.

10.4.1 First-Order Systems

The impulse response of a first-order causal discrete-time system is of the general form

$$h[n] = a^n u[n], \tag{10.64}$$

and from Example 10.1, its z-transform is

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|. \tag{10.65}$$

For $|a| < 1$, the ROC includes the unit circle, and consequently, the Fourier transform of $h[n]$ converges and is equal to $H(z)$ for $z = e^{j\omega}$. Thus, the frequency response for the first-order system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{10.66}$$

Figure 10.13(a) depicts the pole-zero plot for $H(z)$ in eq. (10.65), including the vectors from the pole (at $z = a$) and zero (at $z = 0$) to the unit circle. With this plot, the geometric evaluation of $H(z)$ can be carried out using the same procedure as described in Section 9.4. In particular, if we wish to evaluate the frequency response in eq. (10.65), we perform the evaluation for values of z of the form $z = e^{j\omega}$. The magnitude of the frequency response at frequency ω is the ratio of the length of the vector $\mathbf{v}_1$ to the length of the vector $\mathbf{v}_2$ shown in Figure 10.13(a). The phase of the frequency response is the angle of $\mathbf{v}_1$ with respect to the real axis minus the angle of $\mathbf{v}_2$. Furthermore, the vector $\mathbf{v}_1$ from

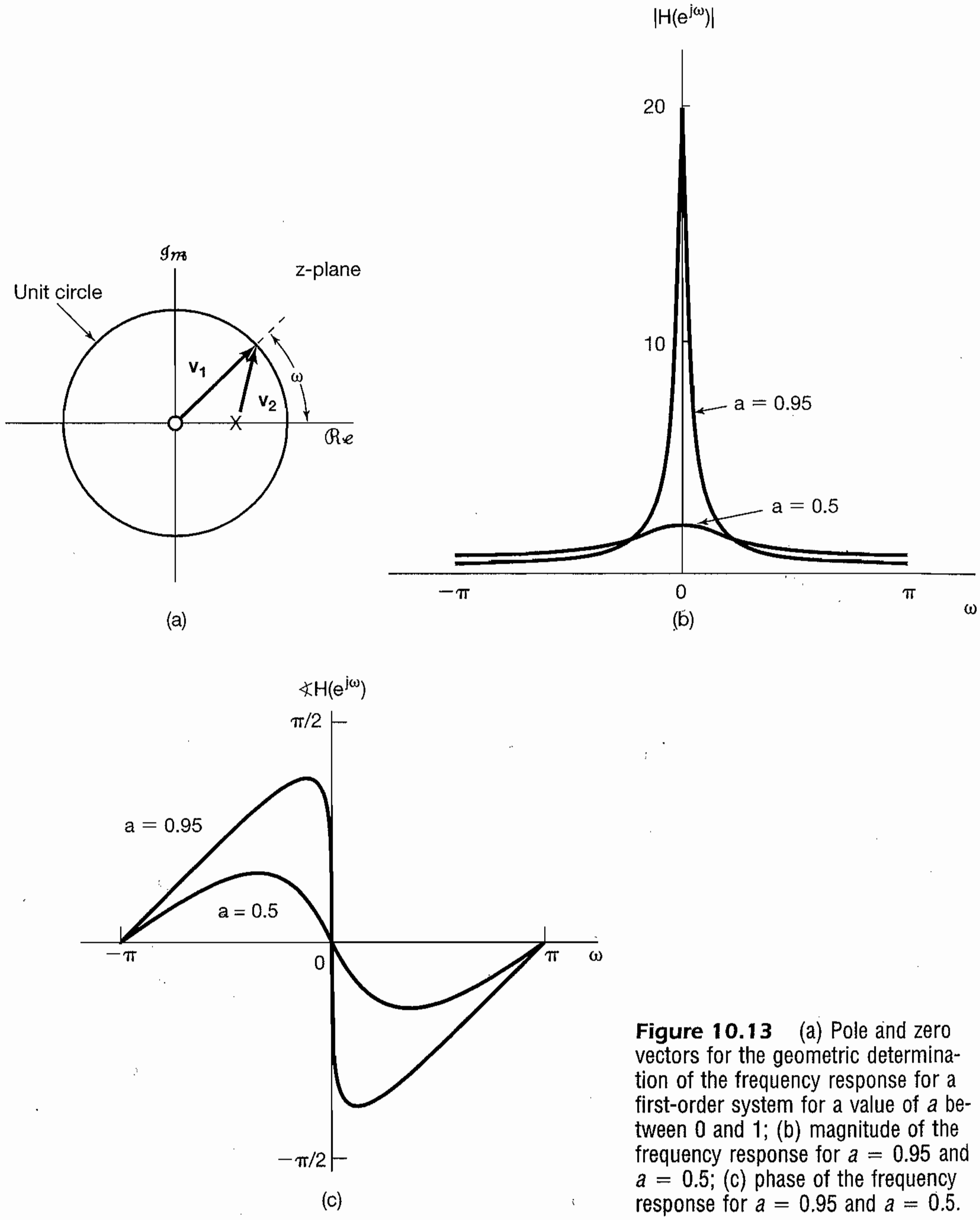

Figure 10.13 (a) Pole and zero vectors for the geometric determination of the frequency response for a first-order system for a value of a between 0 and 1; (b) magnitude of the frequency response for $a = 0.95$ and $a = 0.5$; (c) phase of the frequency response for $a = 0.95$ and $a = 0.5$.

the zero at the origin to the unit circle has a constant length of unity and thus has n effect on the magnitude of $H(e^{j\omega})$. The phase contributed to $H(e^{j\omega})$ by the zero is th angle of the zero vector with respect to the real axis, which we see is equal to ω. Fo $0 < a < 1$, the pole vector has minimum length at $\omega = 0$ and monotonically increases i length as ω increases from zero to π. Thus, the magnitude of the frequency response will b maximum at $\omega = 0$ and will decrease monotonically as ω increases from 0 to π. The angle of the pole vector begins at zero and increases monotonically as ω increases from zero to π. The resulting magnitude and phase of $H(e^{j\omega})$ are shown in Figures 10.13(b) and (c), respectively, for two values of a.

The magnitude of the parameter a in the discrete-time first-order system plays a role similar to that of the time constant τ for the continuous-time first-order system of Section 9.4.1. Note first that, as illustrated in Figure 10.13, the magnitude of the peak of $H(e^{j\omega})$ at $\omega = 0$ decreases as $|a|$ decreases toward 0. Also, as was discussed in Section 6.6.1 and illustrated in Figures 6.26 and 6.27, as $|a|$ decreases, the impulse response decays more sharply and the step response settles more quickly. With multiple poles, the speed of response associated with each pole is related to its distance from the origin, with those closest to the origin contributing the most rapidly decaying terms in the impulse response. This is further illustrated in the case of second-order systems, which we consider next.

10.4.2 Second-Order Systems

Next, let us consider the class of second-order systems as discussed in Section 6.6.2, with impulse response and frequency response given in eqs. (6.64) and (6.60), which we respectively repeat here as

$$h[n] = r^n \frac{\sin(n+1)\theta}{\sin\theta} u[n] \tag{10.67}$$

and

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}, \tag{10.68}$$

where $0 < r < 1$ and $0 \le \theta \le \pi$. Since $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$, we can infer from eq. (10.68) that the system function, corresponding to the z-transform of the system impulse response, is

$$H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}. \tag{10.69}$$

The poles of $H(z)$ are located at

$$z_1 = re^{j\theta}, \quad z_2 = re^{-j\theta}, \tag{10.70}$$

and there is a double zero at $z = 0$. The pole-zero plot and the pole and zero vectors with $0 < \theta < \pi/2$ are illustrated in Figure 10.14(a). In this case, the magnitude of the frequency response equals the square of the magnitude of $\mathbf{v}_1$ (since there is a double zero at the origin) divided by the product of the magnitudes of $\mathbf{v}_2$ and $\mathbf{v}_3$. Because the length of the vector $\mathbf{v}_1$ from the zero at the origin is 1 for all values of ω, the magnitude of the frequency response equals the reciprocal of the product of the lengths of the two pole vectors $\mathbf{v}_2$ and $\mathbf{v}_3$. Also, the phase of the frequency response equals twice the angle of $\mathbf{v}_1$ with respect to the real axis minus the sums of the angles of $\mathbf{v}_2$ and $\mathbf{v}_3$. In Figure 10.14(b) we show the magnitude of the frequency response for $r = 0.95$ and $r = 0.75$, while in Figure 10.14(c) we display the phase of $H(e^{j\omega})$ for the same two values of r. We note in particular that, as we move

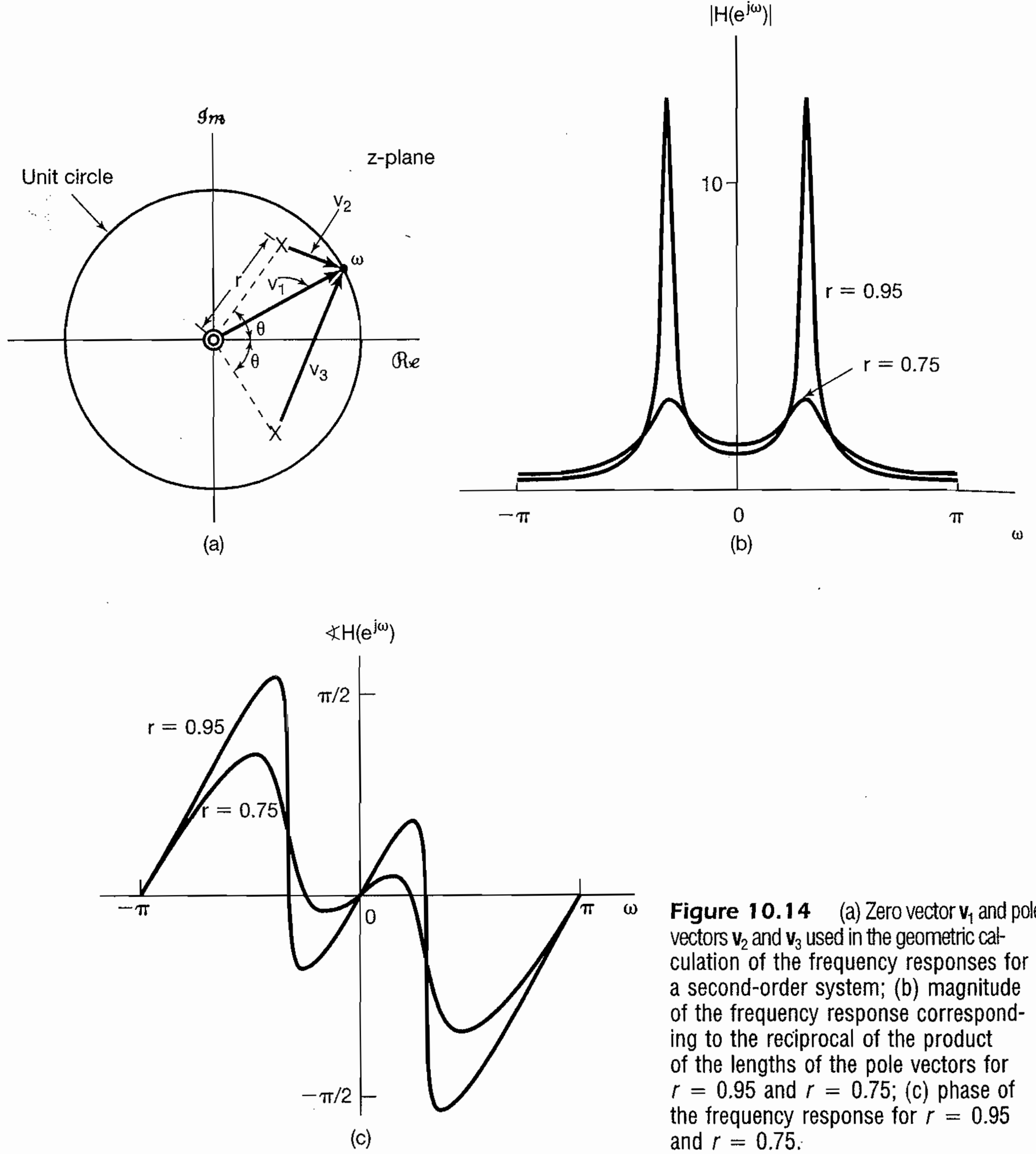

Figure 10.14 (a) Zero vector $\mathbf{v}_1$ and pole vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ used in the geometric calculation of the frequency responses for a second-order system; (b) magnitude of the frequency response corresponding to the reciprocal of the product of the lengths of the pole vectors for $r = 0.95$ and $r = 0.75$; (c) phase of the frequency response for $r = 0.95$ and $r = 0.75$.

along the unit circle from $\omega = 0$ toward $\omega = \pi$, the length of the vector $\mathbf{v}_2$ first decreases and then increases, with a minimum length in the vicinity of the pole location, at $\omega = \theta$. This is consistent with the fact that the magnitude of the frequency response peaks for ω near θ when the length of the vector $\mathbf{v}_2$ is small. Based on the behavior of the pole vectors, it is also evident that as r increases toward unity, the minimum length of the pole vectors will decrease, causing the frequency response to peak more sharply with increasing r. Also, for r near unity, the angle of the vector $\mathbf{v}_2$ changes sharply for ω in the vicinity of θ. Furthermore, from the form of the impulse response [eq. (10.67) and Figure 6.29] or the step response [eq. (6.67) and Figure 6.30], we see, as we did with the first-order system, that as the poles move closer to the origin, corresponding to r decreasing, the impulse response decays more rapidly and the step response settles more quickly.

10.5 PROPERTIES OF THE z-TRANSFORM

As with the other transforms we have developed, the z-transform possesses a number of properties that make it an extremely valuable tool in the study of discrete-time signals and systems. In this section, we summarize many of these properties. Their derivations are analogous to the derivations of properties for the other transforms, and thus, many are left as exercises at the end of the chapter. (See Problems 10.43 and 10.51–10.54.)

10.5.1 Linearity

If

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad \text{with ROC} = R_1,$$

and

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad \text{with ROC} = R_2,$$

then

$$\boxed{ax_1[n] + bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z) + bX_2(z), \quad \text{with ROC containing } R_1 \cap R_2.} \tag{10.71}$$

As indicated, the ROC of the linear combination is at least the intersection of R_1 and R_2. For sequences with rational z-transforms, if the poles of $aX_1(z) + bX_2(z)$ consist of all of the poles of $X_1(z)$ and $X_2(z)$ (i.e., if there is no pole-zero cancellation), then the region of convergence will be exactly equal to the overlap of the individual regions of convergence. If the linear combination is such that some zeros are introduced that cancel poles, then the region of convergence may be larger. A simple example of this occurs when $x_1[n]$ and $x_2[n]$ are both of infinite duration, but the linear combination is of finite duration. In this case the region of convergence of the linear combination is the entire z-plane, with the possible exception of zero and/or infinity. For example, the sequences $a^n u[n]$ and $a^n u[n-1]$ both have a region of convergence defined by $|z| > |a|$, but the sequence corresponding to the difference $(a^n u[n] - a^n u[n-1]) = \delta[n]$ has a region of convergence that is the entire z-plane.

10.5.2 Time Shifting

If

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$x[n-n_0] \stackrel{Z}{\longleftrightarrow} z^{-n_0}X(z), \quad \text{with ROC} = R\text{, except for the possible addition or deletion of the origin or infinity.} \tag{10.72}$$

Because of the multiplication by z^{-n_0}, for $n_0 > 0$ poles will be introduced at $z = 0$, which may cancel corresponding zeros of $X(z)$ at $z = 0$. Consequently, $z = 0$ may be a pole of $z^{-n_0}X(z)$ while it may not be a pole of $X(z)$. In this case the ROC for $z^{-n_0}X(z)$ equals the ROC of $X(z)$ but with the origin deleted. Similarly, if $n_0 < 0$, zeros will be introduced at $z = 0$, which may cancel corresponding poles of $X(z)$ at $z = 0$. Consequently, $z = 0$ may be a zero of $z^{-n_0}X(z)$ while it may not be a pole of $X(z)$. In this case $z = \infty$ is a pole of $z^{-n_0}X(z)$, and thus the ROC for $z^{-n_0}X(z)$ equals the ROC of $X(z)$ but with the $z = \infty$ deleted.

10.5.3 Scaling in the z-Domain

If

$$x[n] \stackrel{Z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$z_0^n x[n] \stackrel{Z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \quad \text{with ROC} = |z_0|R, \tag{10.73}$$

where $|z_0|R$ is the scaled version of R. That is, if z is a point in the ROC of $X(z)$, then the point $|z_0|z$ is in the ROC of $X(z/z_0)$. Also, if $X(z)$ has a pole (or zero) at $z = a$, then $X(z/z_0)$ has a pole (or zero) at $z = z_0 a$.

An important special case of eq. (10.73) is when $z_0 = e^{j\omega_0}$. In this case, $|z_0|R = R$ and

$$e^{j\omega_0 n}x[n] \stackrel{Z}{\longleftrightarrow} X(e^{-j\omega_0}z). \tag{10.74}$$

The left-hand side of eq. (10.74) corresponds to multiplication by a complex exponential sequence. The right-hand side can be interpreted as a rotation in the z-plane; that is, all pole-zero locations rotate in the z-plane by an angle of ω_0, as illustrated in Figure 10.15. This can be seen by noting that if $X(z)$ has a factor of the form $1 - az^{-1}$, then $X(e^{-j\omega_0}z)$ will have a factor $1 - ae^{j\omega_0}z^{-1}$, and thus, a pole or zero at $z = a$ in $X(z)$ will become a pole or zero at $z = ae^{j\omega_0}$ in $X(e^{-j\omega_0}z)$. The behavior of the z-transform on the unit circle will then also shift by an angle of ω_0. This is consistent with the frequency-shifting property set forth in Section 5.3.3, where multiplication with a complex exponential in the time domain was shown to correspond to a shift in frequency of the Fourier transform. Also, in the more general case when $z_0 = r_0 e^{j\omega_0}$ in eq. (10.73), the pole and zero locations are rotated by ω_0 and scaled in magnitude by a factor of r_0.

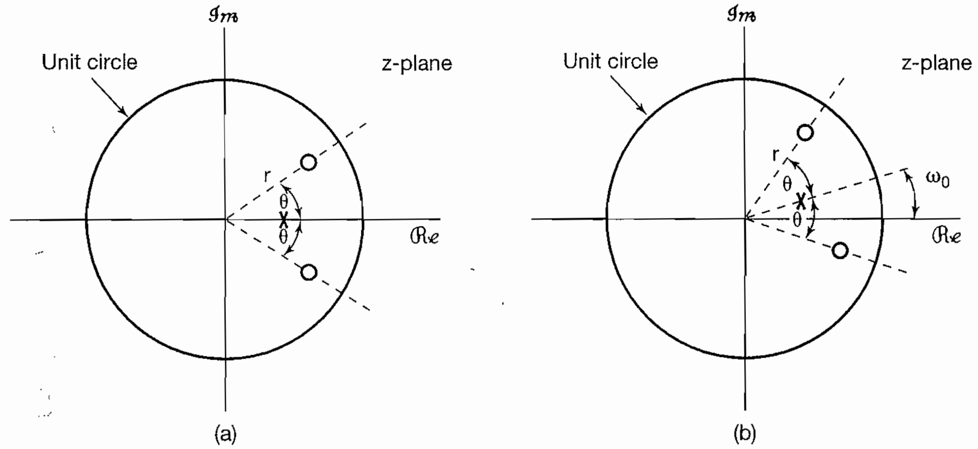

Figure 10.15 Effect on the pole-zero plot of time-domain multiplication by a complex exponential sequence $e^{j\omega_0 n}$: (a) pole-zero pattern for the z-transform for a signal $x[n]$; (b) pole-zero pattern for the z-transform of $x[n]e^{j\omega_0 n}$.

10.5.4 Time Reversal

If

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$\boxed{x[-n] \stackrel{z}{\longleftrightarrow} X(\tfrac{1}{z}), \quad \text{with ROC} = \tfrac{1}{R}.} \tag{10.75}$$

That is, if z_0 is in the ROC for $x[n]$, then $1/z_0$ is in the ROC for $x[-n]$.

10.5.5 Time Expansion

As we discussed in Section 5.3.7, the continuous-time concept of time scaling does not directly extend to discrete time, since the discrete-time index is defined only for integer values. However, the discrete-time concept of time expansion—i.e., of inserting a number of zeros between successive values of a discrete-time sequence $x[n]$—can be defined and does play an important role in discrete-time signal and system analysis. Specifically, the sequence $x_{(k)}[n]$, introduced in Section 5.3.7 and defined as

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases} \tag{10.76}$$

has $k-1$ zeros inserted between successive values of the original signal. In this case, if

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$x_{(k)}[n] \stackrel{z}{\longleftrightarrow} X(z^k), \quad \text{with ROC} = R^{1/k}. \tag{10.77}$$

That is, if z is in the ROC of $X(z)$, then the point $z^{1/k}$ is in the ROC of $X(z^k)$. Also, if $X(z)$ has a pole (or zero) at $z = a$, then $X(z^k)$ has a pole (or zero) at $z = a^{1/k}$.

The interpretation of this result follows from the power-series form of the z-transform, from which we see that the coefficient of the term z^{-n} equals the value of the signal at time n. That is, with

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n},$$

it follows that

$$X(z^k) = \sum_{n=-\infty}^{+\infty} x[n](z^k)^{-n} = \sum_{n=-\infty}^{+\infty} x[n]z^{-kn}. \tag{10.78}$$

Examining the right-hand side of eq. (10.78), we see that the only terms that appear are of the form z^{-kn}. In other words, the coefficient of the term z^{-m} in this power series equals 0 if m is not a multiple of k and equals $x[m/k]$ if m is a multiple of k. Thus, the inverse transform of eq. (10.78) is $x_{(k)}[n]$.

10.5.6 Conjugation

If

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R, \tag{10.79}$$

then

$$x^*[n] \stackrel{z}{\longleftrightarrow} X^*(z^*), \quad \text{with ROC} = R. \tag{10.80}$$

Consequently, if $x[n]$ is real, we can conclude from eq. (10.80) that

$$X(z) = X^*(z^*).$$

Thus, if $X(z)$ has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^*$. For example, the transform $X(z)$ for the real signal $x[n]$ in Example 10.4 has poles at $z = (1/3)e^{\pm j\pi/4}$.

10.5.7 The Convolution Property

If

$$x_1[n] \stackrel{z}{\longleftrightarrow} X_1(z), \quad \text{with ROC} = R_1,$$

and

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad \text{with ROC} = R_2,$$

then

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \quad \text{with ROC containing } R_1 \cap R_2. \tag{10.81}$$

Just as with the convolution property for the Laplace transform, the ROC of $X_1(z)X_2(z)$ includes the intersection of R_1 and R_2 and may be larger if pole-zero cancellation occurs in the product. The convolution property for the z-transform can be derived in a variety of different ways. A formal derivation is developed in Problem 10.56. A derivation can also be carried out analogous to that used for the convolution property for the continuous-time Fourier transform in Section 4.4, which relied on the interpretation of the Fourier transform as the change in amplitude of a complex exponential through an LTI system.

For the z-transform, there is another often useful interpretation of the convolution property. From the definition in eq. (10.3), we recognize the z-transform as a series in z^{-1} where the coefficient of z^{-n} is the sequence value $x[n]$. In essence, the convolution property equation (10.81) states that when two polynomials or power series $X_1(z)$ and $X_2(z)$ are multiplied, the coefficients in the polynomial representing the product are the convolution of the coefficients in the polynomials $X_1(z)$ and $X_2(z)$. (See Problem 10.57).

Example 10.15

Consider an LTI system for which

$$y[n] = h[n] * x[n], \tag{10.82}$$

where

$$h[n] = \delta[n] - \delta[n-1].$$

Note that

$$\delta[n] - \delta[n-1] \overset{z}{\longleftrightarrow} 1 - z^{-1}, \tag{10.83}$$

with ROC equal to the entire z-plane except the origin. Also, the z-transform in eq. (10.83) has a zero at $z = 1$. From eq. (10.81), we see that if

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$y[n] \overset{z}{\longleftrightarrow} (1 - z^{-1})X(z), \tag{10.84}$$

with ROC equal to R, with the possible deletion of $z = 0$ and/or addition of $z = 1$.

Note that for this system

$$y[n] = [\delta[n] - \delta[n-1]] * x[n] = x[n] - x[n-1].$$

That is, $y[n]$ is the first difference of the sequence $x[n]$. Since the first-difference operation is commonly thought of as a discrete-time counterpart to differentiation, eq. (10.83) can be thought of as the z-transform counterpart of the Laplace transform differentiation property presented in Section 9.5.7.

Example 10.16

Suppose we now consider the inverse of first differencing, namely, accumulation or summation. Specifically, let $w[n]$ be the running sum of $x[n]$:

$$w[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]. \tag{10.85}$$

Then, using eq. (10.81) together with the z-transform of the unit step in Example 10.1, we see that

$$w[n] = \sum_{k=-\infty}^{n} x[k] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}} X(z), \tag{10.86}$$

with ROC including at least the intersection of R with $|z| > 1$. Eq. (10.86) is the discrete-time z-transform counterpart of the integration property in Section 9.5.9.

10.5.8 Differentiation in the z-Domain

If

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \quad \text{with ROC} = R. \tag{10.87}$$

This property follows in a straightforward manner by differentiating both sides of the expression for the z-transform given in eq. (10.3). As an example of the use of this property, let us apply it to determining the inverse z-transform considered in Example 10.14.

Example 10.17

If

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|, \tag{10.88}$$

then

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}}, \quad |z| > |a|. \tag{10.89}$$

By differentiating, we have converted the z-transform to a rational expression. The inverse z-transform of the right-hand side of eq. (10.89) can be obtained by using Example 10.1 together with the time-shifting property, eq. (10.72), set forth in Section 10.5.2.

Specifically, from Example 10.1 and the linearity property,

$$a(-a)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{a}{1+az^{-1}}, \quad |z| > |a|. \tag{10.90}$$

Combining this with the time-shifting property yields

$$a(-a)^{n-1} u[n-1] \stackrel{z}{\longleftrightarrow} \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|.$$

Consequently,

$$x[n] = \frac{-(-a)^n}{n} u[n-1]. \tag{10.91}$$

Example 10.18

As another example of the use of the differentiation property, consider determining the inverse z-transform for

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|. \tag{10.92}$$

From Example 10.1,

$$a^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|, \tag{10.93}$$

and hence,

$$na^n u[n] \stackrel{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|. \tag{10.94}$$

10.5.9 The Initial-Value Theorem

If $x[n] = 0$, $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z). \tag{10.95}$$

This property follows by considering the limit of each term individually in the expression for the z-transform, with $x[n]$ zero for $n < 0$. With this constraint,

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

As $z \to \infty$, $z^{-n} \to 0$ for $n > 0$, whereas for $n = 0$, $z^{-n} = 1$. Thus, eq. (10.95) follows.

As one consequence of the initial-value theorem, for a causal sequence, if $x[0]$ is finite, then $\lim_{z\to\infty} X(z)$ is finite. Consequently, with $X(z)$ expressed as a ratio of polynomials in z, the order of the numerator polynomial cannot be greater than the order of the denominator polynomial; or, equivalently, the number of finite zeros of $X(z)$ cannot be greater than the number of finite poles.

Example 10.19

The initial-value theorem can also be useful in checking the correctness of the z-transform calculation for a signal. For example, consider the signal $x[n]$ in Example 10.3. From eq. (10.12), we see that $x[0] = 1$. Also, from eq. (10.14),

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1 - \frac{3}{2}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})} = 1,$$

which is consistent with the initial-value theorem.

10.5.10 Summary of Properties

In Table 10.1, we summarize the properties of the z-transform.

10.6 SOME COMMON z-TRANSFORM PAIRS

As with the inverse Laplace transform, the inverse z-transform can often be easily evaluated by expressing $X(z)$ as a linear combination of simpler terms, the inverse transforms of which are recognizable. In Table 10.2, we have listed a number of useful z-transform pairs. Each of these can be developed from previous examples in combination with the properties of the z-transform listed in Table 10.1. For example, transform pairs 2 and 5 follow directly from Example 10.1, and transform pair 7 is developed in Example 10.18. These, together with the time-reversal and time-shifting properties set forth in Sections 10.5.4 and 10.5.2, respectively, then lead to transform pairs 3, 6, and 8. Transform pairs 9 and 10 can be developed using transform pair 2 together with the linearity and scaling properties developed in Sections 10.5.1 and 10.5.3, respectively.

10.7 ANALYSIS AND CHARACTERIZATION OF LTI SYSTEMS USING z-TRANSFORMS

The z-transform plays a particularly important role in the analysis and representation of discrete-time LTI systems. From the convolution property presented in Section 10.5.7,

$$Y(z) = H(z)X(z), \tag{10.96}$$

where $X(z)$, $Y(z)$, and $H(z)$ are the z-transforms of the system input, output, and impulse response, respectively. $H(z)$ is referred to as the *system function* or *transfer function* of the system. For z evaluated on the unit circle (i.e., for $z = e^{j\omega}$), $H(z)$ reduces to the frequency response of the system, provided that the unit circle is in the ROC for $H(z)$. Also, from our discussion in Section 3.2, we know that if the input to an LTI system is the complex exponential signal $x[n] = z^n$, then the output will be $H(z)z^n$. That is, z^n is an eigenfunction of the system with eigenvalue given by $H(z)$, the z-transform of the impulse response.

Many properties of a system can be tied directly to characteristics of the poles, zeros, and region of convergence of the system function, and in this section we illustrate some of these relationships by examining several important system properties and an important class of systems.

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$			

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the z-plane. For some systems, e.g., if $h[n] = \delta[n]$, so that $H(z) = 1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n] = \delta[n+1]$, then $H(z) = z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

does not include any positive powers of z. Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

> A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.

If $H(z)$ is rational, then, from Property 8 in Section 10.2, for the system to be causal, the ROC must be outside the outermost pole and infinity must be in the ROC. Equivalently, the limit of $H(z)$ as $z \longrightarrow \infty$ must be finite. As we discussed in Section 10.5.9, this is equivalent to the numerator of $H(z)$ having degree no larger than the denominator when both are expressed as polynomials in z. That is:

> A discrete-time LTI system with rational system function $H(z)$ is causal if and only if: (a) the ROC is the exterior of a circle outside the outermost pole; and (b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Example 10.20

Consider a system with system function whose algebraic expression is

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}.$$

Without even knowing the ROC for this system, we can conclude that the system is not causal, because the numerator of $H(z)$ is of higher order than the denominator.

Example 10.21

Consider a system with system function

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2 \tag{10.97}$$

Since the ROC for this system function is the exterior of a circle outside the outermost pole, we know that the impulse response is right-sided. To determine if the system is causal, we then need only check the other condition required for causality, namely that $H(z)$, when expressed as a ratio of polynomials in z, has numerator degree no larger than the denominator. For this example,

$$H(z) = \frac{2 - \frac{5}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1}, \tag{10.98}$$

so that the numerator and denominator of $H(z)$ are both of degree two, and consequently we can conclude that the system is causal. This can also be verified by calculating the inverse transform of $H(z)$. In particular, using transform pair 5 in Table 10.2, we find that the impulse response of this system is

$$h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n]. \tag{10.99}$$

Since $h[n] = 0$ for $n < 0$, we can confirm that the system is causal.

10.7.2 Stability

As we discussed in Section 2.3.7, the stability of a discrete-time LTI system is equivalent to its impulse response being absolutely summable. In this case the Fourier transform of $h[n]$

converges, and consequently, the ROC of $H(z)$ must include the unit circle. Summarizing, we obtain the following result:

> An LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$.

Example 10.22

Consider again the system function in eq. (10.97). Since the associated ROC is the region $|z| > 2$, which does not include the unit circle, the system is not stable. This can also be seen by noting that the impulse response in eq. (10.99) is not absolutely summable. If, however, we consider a system whose system function has the same algebraic expression as in eq. (10.97) but whose ROC is the region $1/2 < |z| < 2$, then the ROC does contain the unit circle, so that the corresponding system is noncausal but stable. In this case, using transform pairs 5 and 6 from Table 10.2, we find that the corresponding impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1], \tag{10.100}$$

which is absolutely summable.

Also, for the third possible choice of ROC associated with the algebraic expression for $H(z)$ in eq. (10.97), namely, $|z| < 1/2$, the corresponding system is neither causal (since the ROC is not outside the outermost pole) nor stable (since the ROC does not include the unit circle). This can also be seen from the impulse response, which (using transform pair 6 in Table 10.2) is

$$h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right]u[-n-1].$$

As Example 10.22 illustrates, it is perfectly possible for a system to be stable but not causal. However, if we focus on causal systems, stability can easily be checked by examining the locations of the poles. Specifically, for a causal system with rational system function, the ROC is outside the outermost pole. For this ROC to include the unit circle, $|z| = 1$, all of the poles of the system must be inside the unit circle. That is:

> A causal LTI system with rational system function $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the unit circle—i.e., they must all have magnitude smaller than 1.

Example 10.23

Consider a causal system with system function

$$H(z) = \frac{1}{1 - az^{-1}},$$

which has a pole at $z = a$. For this system to be stable, its pole must be inside the unit circle, i.e., we must have $|a| < 1$. This is consistent with the condition for the absolute summability of the corresponding impulse response $h[n] = a^n u[n]$.

Example 10.24

The system function for a second-order system with complex poles was given in eq. (10.69), specifically,

$$H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}, \tag{10.101}$$

with poles located at $z_1 = re^{j\theta}$ and $z_2 = re^{-j\theta}$. Assuming causality, we see that the ROC is outside the outermost pole (i.e., $|z| > |r|$). The pole-zero plot and ROC for this system are shown in Figure 10.16 for $r < 1$ and $r > 1$. For $r < 1$, the poles are inside the unit circle, the ROC includes the unit circle, and therefore, the system is stable. For $r > 1$, the poles are outside the unit circle, the ROC does not include the unit circle, and the system is unstable.

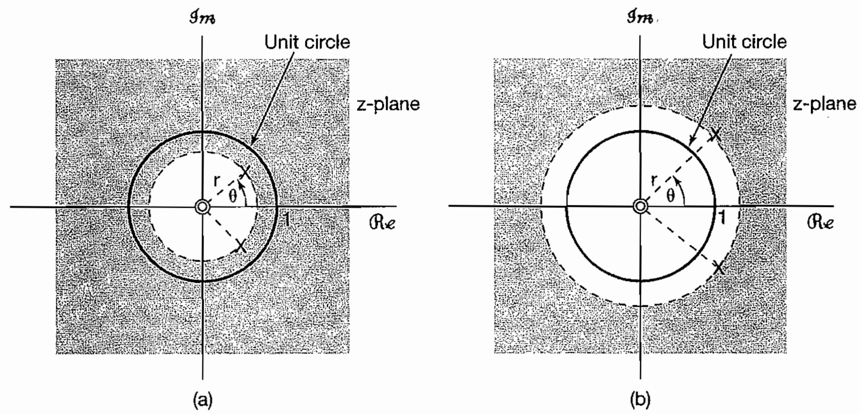

Figure 10.16 Pole-zero plot for a second-order system with complex poles: (a) $r < 1$; (b) $r > 1$.

10.7.3 LTI Systems Characterized by Linear Constant-Coefficient Difference Equations

For systems characterized by linear constant-coefficient difference equations, the properties of the z-transform provide a particularly convenient procedure for obtaining the system function, frequency response, or time-domain response of the system. Let us illustrate this with an example.

Example 10.25

Consider an LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]. \tag{10.102}$$

Applying the z-transform to both sides of eq. (10.102), and using the linearity property set forth in Section 10.5.1 and the time-shifting property presented in Section 10.5.2, we obtain

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z),$$

or

$$Y(z) = X(z)\left[\frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]. \tag{10.103}$$

From eq. (10.96), then,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}. \tag{10.104}$$

This provides the algebraic expression for $H(z)$, but not the region of convergence. In fact, there are two distinct impulse responses that are consistent with the difference equation (10.102), one right sided and the other left sided. Correspondingly, there are two different choices for the ROC associated with the algebraic expression (10.104). One, $|z| > 1/2$, is associated with the assumption that $h[n]$ is right sided, and the other, $|z| < 1/2$, is associated with the assumption that $h[n]$ is left sided.

Consider first the choice of ROC equal to $|z| > 1$. Writing

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right)\frac{1}{1 - \frac{1}{2}z^{-1}},$$

we can use transform pair 5 in Table 10.2, together with the linearity and time-shifting properties, to find the corresponding impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

For the other choice of ROC, namely, $|z| < 1$, we can use transform pair 6 in Table 10.2 and the linearity and time-shifting properties, yielding

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n].$$

In this case, the system is anticausal ($h[n] = 0$ for $n > 0$) and unstable.

For the more general case of an Nth-order difference equation, we proceed in a manner similar to that in Example 10.25, applying the z-transform to both sides of the equation and using the linearity and time-shifting properties. In particular, consider an LTI system for which the input and output satisfy a linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{10.105}$$

Then taking z-transforms of both sides of eq. (10.105) and using the linearity and time-shifting properties, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

or

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k},$$

so that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \qquad (10.106)$$

We note in particular that the system function for a system satisfying a linear constant-coefficient difference equation is always rational. Consistent with our previous example and with the related discussion for the Laplace transform, the difference equation by itself does not provide information about which ROC to associate with the algebraic expression $H(z)$. An additional constraint, such as the causality or stability of the system, however, serves to specify the region of convergence. For example, if we know in addition that the system is causal, the ROC will be outside the outermost pole. If the system is stable, the ROC must include the unit circle.

10.7.4 Examples Relating System Behavior to the System Function

As the previous subsections illustrate, many properties of discrete-time LTI systems can be directly related to the system function and its characteristics. In this section, we give several additional examples to show how z-transform properties can be used in analyzing systems.

Example 10.26

Suppose that we are given the following information about an LTI system:

1. If the input to the system is $x_1[n] = (1/6)^n u[n]$, then the output is

$$y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n \right] u[n],$$

where a is a real number.

2. If $x_2[n] = (-1)^n$, then the output is $y_2[n] = \frac{7}{4}(-1)^n$. As we now show, from these two pieces of information, we can determine the system function $H(z)$ for this system, including the value of the number a, and can also immediately deduce a number of other properties of the system.

The z-transforms of the signals specified in the first piece of information are

$$X_1(z) = \frac{1}{1 - \frac{1}{6}z^{-1}}, \quad |z| > \frac{1}{6}, \tag{10.107}$$

$$\begin{aligned} Y_1(z) &= \frac{a}{1 - \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{3}z^{-1}} \\ &= \frac{(a+10) - (5 + \frac{a}{3})z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{2}. \end{aligned} \tag{10.108}$$

From eq. (10.96), it follows that the algebraic expression for the system function is

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{[(a+10) - (5 + \frac{a}{3})z^{-1}][1 - \frac{1}{6}z^{-1}]}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}. \tag{10.109}$$

Furthermore, we know that the response to $x_2[n] = (-1)^n$ must equal $(-1)^n$ multiplied by the system function $H(z)$ evaluated at $z = -1$. Thus from the second piece of information given, we see that

$$\frac{7}{4} = H(-1) = \frac{[(a+10) + 5 + \frac{a}{3}][\frac{7}{6}]}{(\frac{3}{2})(\frac{4}{3})}. \tag{10.110}$$

Solving eq. (10.110), we find that $a = -9$, so that

$$H(z) = \frac{(1 - 2z^{-1})(1 - \frac{1}{6}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}, \tag{10.111}$$

or

$$H(z) = \frac{1 - \frac{13}{6}z^{-1} + \frac{1}{3}z^{-2}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}, \tag{10.112}$$

or, finally,

$$H(z) = \frac{z^2 - \frac{13}{6}z + \frac{1}{3}}{z^2 - \frac{5}{6}z + \frac{1}{6}}. \tag{10.113}$$

Also, from the convolution property, we know that the ROC of $Y_1(z)$ must include at least the intersections of the ROCs of $X_1(z)$ and $H(z)$. Examining the three possible ROCs for $H(z)$ (namely, $|z| < 1/3$, $1/3 < |z| < 1/2$, and $|z| > 1/2$), we find that the only choice that is consistent with the ROCs of $X_1(z)$ and $Y_1(z)$ is $|z| > 1/2$.

Since the ROC for the system includes the unit circle, we know that the system is stable. Furthermore, from eq. (10.113) with $H(z)$ viewed as a ratio of polynomials in z, the order of the numerator does not exceed that of the denominator, and thus we can conclude that the LTI system is causal. Also, using eqs. (10.112) and (10.106), we can write the difference equation that, together with the condition of initial rest, characterizes the system:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{13}{6}x[n-1] + \frac{1}{3}x[n-2].$$

Example 10.27

Consider a stable and causal system with impulse response $h[n]$ and rational system function $H(z)$. Suppose it is known that $H(z)$ contains a pole at $z = 1/2$ and a zero somewhere on the unit circle. The precise number and locations of all of the other poles and zeros are unknown. For each of the following statements, let us determine whether we can definitely say that it is true, whether we can definitely say that it is false, or whether there is insufficient information given to determine if it is true or not:

(a) $\mathfrak{F}\{(1/2)^n h[n]\}$ converges.

(b) $H(e^{j\omega}) = 0$ for some ω.

(c) $h[n]$ has finite duration.

(d) $h[n]$ is real.

(e) $g[n] = n[h[n] * h[n]]$ is the impulse response of a stable system.

Statement (a) is true. $\mathfrak{F}\{(1/2)^n h[n]\}$ corresponds to the value of the z-transform of $h[n]$ at $z = 2$. Thus, its convergence is equivalent to the point $z = 2$ being in the ROC. Since the system is stable and causal, all of the poles of $H(z)$ are inside the unit circle, and the ROC includes all the points outside the unit circle, including $z = 2$.

Statement (b) is true because there is a zero on the unit circle.

Statement (c) is false because a finite-duration sequence must have an ROC that includes the entire z-plane, except possibly $z = 0$ and/or $z = \infty$. This is not consistent with having a pole at $z = 1/2$.

Statement (d) requires that $H(z) = H^*(z^*)$. This in turn implies that if there is a pole (zero) at a nonreal location $z = z_0$, there must also be a pole (zero) at $z = z_0^*$. Insufficient information is given to validate such a conclusion.

Statement (e) is true. Since the system is causal, $h[n] = 0$ for $n < 0$. Consequently, $h[n] * h[n] = 0$ for $n < 0$; i.e., the system with $h[n] * h[n]$ as its impulse response is causal. The same is then true for $g[n] = n[h[n] * h[n]]$. Furthermore, by the convolution property set forth in Section 10.5.7, the system function corresponding to the impulse response $h[n] * h[n]$ is $H^2(z)$, and by the differentiation property presented in Section 10.5.8, the system function corresponding to $g[n]$ is

$$G(z) = -z\frac{d}{dz}H^2(z) = -2zH(z)\left[\frac{d}{dz}H(z)\right]. \tag{10.114}$$

From eq. (10.114), we can conclude that the poles of $G(z)$ are at the same locations as those of $H(z)$, with the possible exception of the origin. Therefore, since $H(z)$ has all its poles inside the unit circle, so must $G(z)$. It follows that $g[n]$ is the impulse response of a causal and stable system.

10.8 SYSTEM FUNCTION ALGEBRA AND BLOCK DIAGRAM REPRESENTATIONS

Just as with the Laplace transform in continuous time, the z-transform in discrete time allows us to replace time-domain operations such as convolution and time shifting with algebraic operations. This was exploited in Section 10.7.3, where we were able to replace the difference-equation description of an LTI system with an algebraic description. The use of the z-transform to convert system descriptions to algebraic equations is also helpful in analyzing interconnections of LTI systems and in representing and synthesizing systems as interconnections of basic system building blocks.

10.8.1 System Functions for Interconnections of LTI Systems

The system function algebra for analyzing discrete-time block diagrams such as series, parallel, and feedback interconnections is exactly the same as that for the corresponding continuous-time systems in Section 9.8.1. For example, the system function for the cascade of two discrete-time LTI systems is the product of the system functions for the individual systems in the cascade. Also, consider the feedback interconnection of two systems, as shown in Figure 10.17. It is relatively involved to determine the difference equation or impulse response for the overall system working directly in the time domain. However, with the systems and sequences expressed in terms of their z-transforms, the analysis involves only algebraic equations. The specific equations for the interconnection of Figure 10.17 exactly parallel eqs. (9.159)–(9.163), with the final result that the overall system function for the feedback system of Figure 10.17 is

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}. \tag{10.115}$$

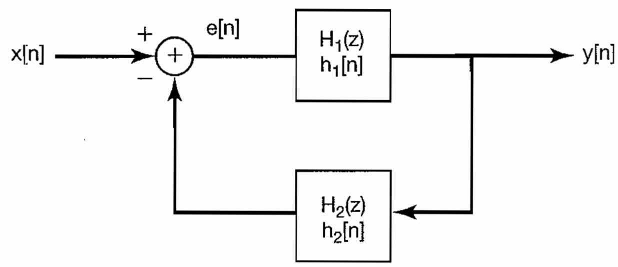

Figure 10.17 Feedback interconnection of two systems.

10.8.2 Block Diagram Representations for Causal LTI Systems Described by Difference Equations and Rational System Functions

As in Section 9.8.2, we can represent causal LTI systems described by difference equations using block diagrams involving three basic operations—in this case, addition, multiplication by a coefficient, and a unit delay. In Section 2.4.3, we described such a block diagram for a first-order difference equation. We first revisit that example, this time using system function algebra, and then consider several slightly more complex examples to illustrate the basic ideas in constructing block diagram representations.

Example 10.28

Consider the causal LTI system with system function

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}. \tag{10.116}$$

Using the results in Section 10.7.3, we find that this system can also be described by the difference equation

$$y[n] - \frac{1}{4}y[n-1] = x[n],$$

together with the condition of initial rest. In Section 2.4.3 we constructed a block diagram representation for a first-order system of this form, and an equivalent block diagram (corresponding to Figure 2.28 with $a = -1/4$ and $b = 1$) is shown in Figure 10.18(a). Here, z^{-1} is the system function of a unit delay. That is, from the time-shifting property, the input and output of this system are related by

$$w[n] = y[n-1].$$

The block diagram in Figure 10.18(a) contains a feedback loop much as for the system considered in the previous subsection and pictured in Figure 10.17. In fact, with some minor modifications, we can obtain the equivalent block diagram shown in Figure 10.18(b), which is exactly in the form shown in Figure 10.17, with $H_1(z) = 1$ and $H_2(z) = -1/4z^{-1}$. Then, applying eq. (10.115), we can verify that the system function of the system in Figure 10.18 is given by eq. (10.116).

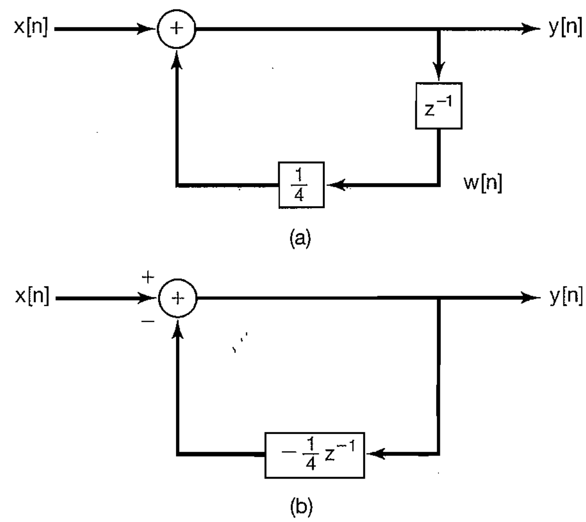

Figure 10.18 (a) Block diagram representations of the causal LTI system in Example 10.28; (b) equivalent block diagram representation.

Example 10.29

Suppose we now consider the causal LTI system with system function

$$H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)(1-2z^{-1}). \tag{10.117}$$

As eq. (10.117) suggests, we can think of this system as the cascade of a system with system function $1/[1-(1/4)z^{-1}]$ and one with system function $1-2z^{-1}$. We have illustrated the cascade in Figure 10.19(a), in which we have used the block diagram in Figure 10.18(a) to represent $1/[1-(1/4)z^{-1}]$. We have also represented $1-2z^{-1}$ using a unit delay, an adder, and a coefficient multiplier. Using the time-shifting property, we then see that the input $v[n]$ and output $y[n]$ of the system with system function $1-2z^{-1}$

are related by

$$y[n] = v[n] - 2v[n-1].$$

While the block diagram in Figure 10.19(a) is certainly a valid representation of the system in eq. (10.117), it has an inefficiency whose elimination leads to an alternative block-diagram representation. To see this, note that the input to both unit delay elements in Figure 10.19(a) is $v[n]$, so that the outputs of these elements are identical; i.e.,

$$w[n] = s[n] = v[n-1].$$

Consequently, we need not keep both of these delay elements, and we can simply use the output of one of them as the signal to be fed to both coefficient multipliers. The result is the block diagram representation in Figure 10.19(b). Since each unit delay element requires a memory register to store the preceding value of its input, the representation in Figure 10.19(b) requires less memory than that in Figure 10.19(a).

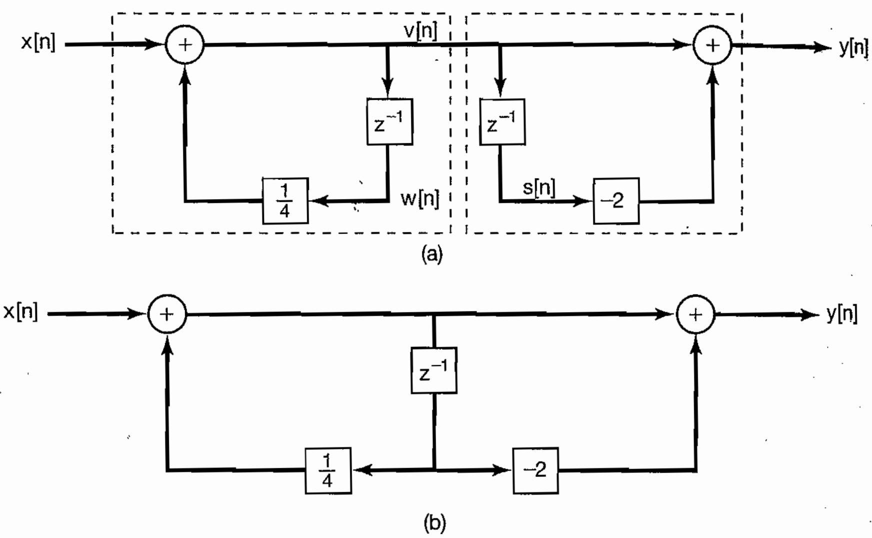

Figure 10.19 (a) Block-diagram representations for the system in Example 10.29; (b) equivalent block-diagram representation using only one unit delay element.

Example 10.30

Next, consider the second-order system function

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}, \tag{10.118}$$

which is also described by the difference equation

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]. \tag{10.119}$$

Using the same ideas as in Example 10.28, we obtain the block-diagram representation for this system shown in Figure 10.20(a). Specifically, since the two system function blocks in this figure with system function z^{-1} are unit delays, we have

$$f[n] = y[n-1],$$
$$e[n] = f[n-1] = y[n-2],$$

so that eq. (10.119) can be rewritten as

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n],$$

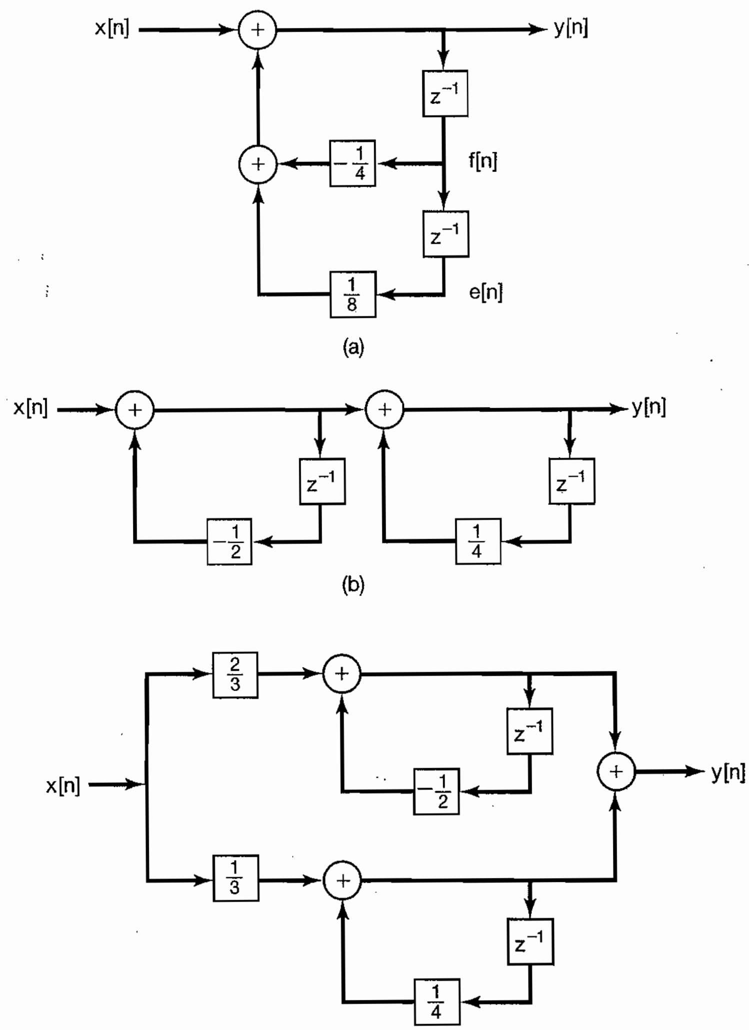

Figure 10.20 Block-diagram representations for the system in Example 10.30: (a) direct form; (b) cascade form; (c) parallel form.

or

$$y[n] = -\frac{1}{4}f[n] + \frac{1}{8}e[n] + x[n],$$

which is exactly what the figure represents.

The block diagram in Figure 10.20(a) is commonly referred to as a *direct-form* representation, since the coefficients appearing in the diagram can be determined by inspection from the coefficients appearing in the difference equation or, equivalently, the system function. Alternatively, as in continuous time, we can obtain both *cascade-form* and *parallel-form* block diagrams with the aid of a bit of system function algebra. Specifically, we can rewrite eq. (10.118) as

$$H(z) = \left(\frac{1}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \tag{10.120}$$

which suggests the cascade-form representation depicted in Figure 10.20(b) in which the system is represented as the cascade of two systems corresponding to the two factors in eq. (10.120).

Also, by performing a partial-fraction expansion, we obtain

$$H(z) = \frac{\frac{2}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1 - \frac{1}{4}z^{-1}},$$

which leads to the parallel-form representation depicted in Figure 10.20(c).

Example 10.31

Finally, consider the system function

$$H(z) = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}. \tag{10.121}$$

Writing

$$H(z) = \left(\frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}\right)\left(1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}\right) \tag{10.122}$$

suggests representing the system as the cascade of the system in Figure 10.20(a) and the system with system function $1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}$. However, as in Example 10.29, the unit delay elements needed to implement the first term in eq. (10.122) also produce the delayed signals needed in computing the output of the second system. The result is the direct-form block diagram shown in Figure 10.21, the details of the construction of which are examined in Problem 10.38. The coefficients in the direct-form representation can be determined by inspection from the coefficients in the system function of eq. (10.121).

We can also write $H(z)$ in the forms

$$H(z) = \left(\frac{1 + \frac{1}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}}\right) \tag{10.123}$$

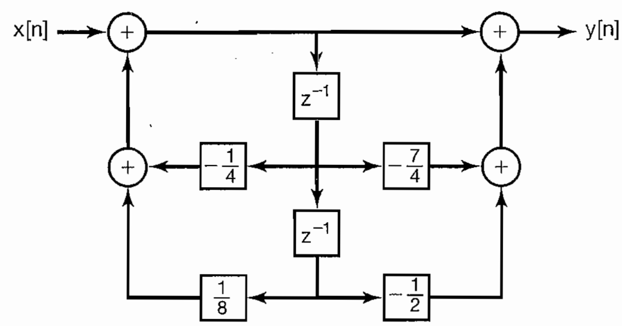

Figure 10.21 Direct-form representation for the system in Example 10.31.

and

$$H(z) = 4 + \frac{5/3}{1 + \frac{1}{2}z^{-1}} - \frac{14/3}{1 - \frac{1}{4}z^{-1}}. \tag{10.124}$$

Eq. (10.123) suggests a cascade-form representation, while eq. (10.124) leads to a parallel-form block diagram. These are also considered in Problem 10.38.

The concepts used in constructing block-diagram representations in the preceding examples can be applied directly to higher order systems, and several examples are considered in Problem 10.39. As in continuous time, there is typically considerable flexibility in doing this—e.g., in how numerator and denominator factors are paired in a product representation as in eq. (10.123), in the way in which each factor is implemented, and in the order in which the factors are cascaded. While all of these variations lead to representations of the same system, in practice there are differences in the behavior of the different block diagrams. Specifically, each block-diagram representation of a system can be translated directly into a computer algorithm for the implementation of the system. However, because the finite word length of a computer necessitates quantizing the coefficients in the block diagram and because there is numerical roundoff as the algorithm operates, each of these representations will lead to an algorithm that only approximates the behavior of the original system. Moreover, the errors in each of these approximations will be somewhat different. Because of these differences, considerable effort has been put into examining the relative merits of the various block-diagram representations in terms of their accuracy and sensitivity to quantization effects. For discussions of this subject, the reader may turn to the references on digital signal processing in the bibliography at the end of the book.

10.9 THE UNILATERAL z-TRANSFORM

The form of the z-transform considered thus far in this chapter is often referred to as the *bilateral z-transform.* As was the case with the Laplace transform, there is an alternative form, referred to as the *unilateral z-transform,* that is particularly useful in analyzing causal systems specified by linear constant-coefficient difference equations with nonzero initial conditions (i.e., systems that are not initially at rest). In this section, we introduce the unilateral z-transform and illustrate some of its properties and uses, paralleling our discussion of the unilateral Laplace transform in Section 9.9.