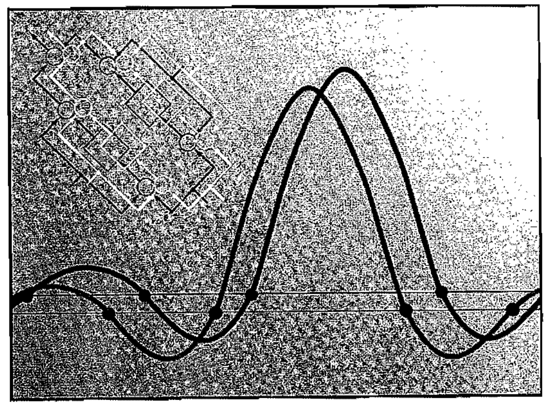
4

THE CONTINUOUS-TIME FOURIER TRANSFORM



4.0 INTRODUCTION

In Chapter 3, we developed a representation of periodic signals as linear combin complex exponentials. We also saw how this representation can be used in descr effect of LTI systems on signals.

In this and the following chapter, we extend these concepts to apply to signal not periodic. As we will see, a rather large class of signals, including all signals w energy, can also be represented through a linear combination of complex expc Whereas for periodic signals the complex exponential building blocks are harn related, for aperiodic signals they are infinitesimally close in frequency, and the r tation in terms of a linear combination takes the form of an integral rather than a resulting spectrum of coefficients in this representation is called the Fourier transl the synthesis integral itself, which uses these coefficients to represent the signal a combination of complex exponentials, is called the inverse Fourier transform.

The development of this representation for aperiodic signals in continuou one of Fourier's most important contributions, and our development of the Four form follows very closely the approach he used in his original work. In particula reasoned that an aperiodic signal can be viewed as a periodic signal with an in riod. More precisely, in the Fourier series representation of a periodic signal, as t increases the fundamental frequency decreases and the harmonically related co become closer in frequency. As the period becomes infinite, the frequency co form a continuum and the Fourier series sum becomes an integral. In the ne we develop the Fourier series representation for continuous-time periodic sig in the sections that follow we build on this foundation as we explore many of the i properties of the continuous-time Fourier transform that form the foundation of frequency-domain methods for continuous-time signals and systems. In Chapter 5, we parallel this development for discrete-time signals.

4.1 REPRESENTATION OF APERIODIC SIGNALS: THE CONTINUOUS-TIME FOURIER TRANSFORM

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

To gain some insight into the nature of the Fourier transform representation, we begin by revisiting the Fourier series representation for the continuous-time periodic square wave examined in Example 3.5. Specifically, over one period,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period T, as shown in Figure 4.1.

As determined in Example 3.5, the Fourier series coefficients a_k for this square wave are

$$\text{[eq. (3.44)]} \qquad a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, \tag{4.1}$$

where $\omega_0 = 2\pi/T$. In Figure 3.7, bar graphs of these coefficients were shown for a fixed value of T_1 and several different values of T.

An alternative way of interpreting eq. (4.1) is as samples of an envelope function, specifically,

$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}. \tag{4.2}$$

That is, with ω thought of as a continuous variable, the function $(2\sin\omega T_1)/\omega$ represents the envelope of Ta_k, and the coefficients a_k are simply equally spaced samples of this envelope. Also, for fixed T_1, the envelope of Ta_k is independent of T. In Figure 4.2, we again show the Fourier series coefficients for the periodic square wave, but this time as samples of the envelope of Ta_k, as specified in eq. (4.2). From the figure, we see that as

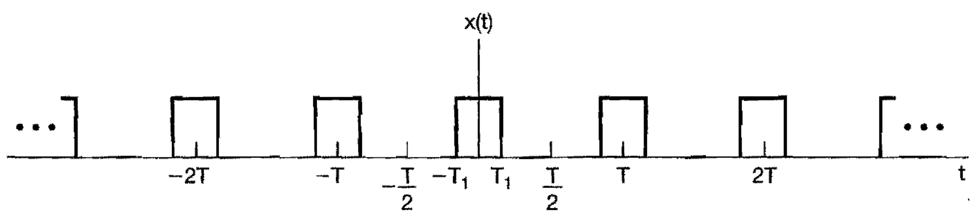


Figure 4.1 A continuous-time periodic square wave.

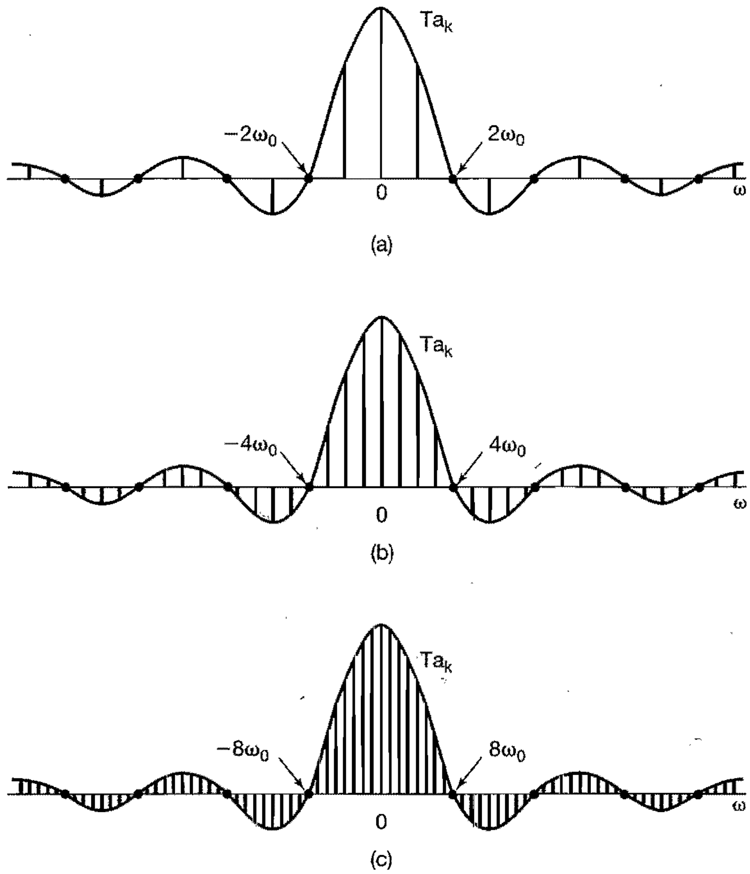


Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 fc several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi/T$ decreases, t envelope is sampled with a closer and closer spacing. As T becomes arbitrarily lar the original periodic square wave approaches a rectangular pulse (i.e., all that remains the time domain is an aperiodic signal corresponding to one period of the square wav Also, the Fourier series coefficients, multiplied by T, become more and more clos spaced samples of the envelope, so that in some sense (which we will specify short the set of Fourier series coefficients approaches the envelope function as $T \to \infty$.

This example illustrates the basic idea behind Fourier's development of a represe tation for aperiodic signals. Specifically, we think of an aperiodic signal as the limit o periodic signal as the period becomes arbitrarily large, and we examine the limiting b havior of the Fourier series representation for this signal. In particular, consider a sig $x(t)$ that is of finite duration. That is, for some number T_1, $x(t) = 0$ if $|t| > T_1$, as ill trated in Figure 4.3(a). From this aperiodic signal, we can construct a periodic signal $\tilde{x}$ for which $x(t)$ is one period, as indicated in Figure 4.3(b). As we choose the period T be larger, $\tilde{x}(t)$ is identical to $x(t)$ over a longer interval, and as $T \to \infty$, $\tilde{x}(t)$ is equal $x(t)$ for any finite value of t.

Let us now examine the effect of this on the Fourier series representation of $\tilde{x}$ Rewriting eqs. (3.38) and (3.39) here for convenience, with the integral in eq. (3.

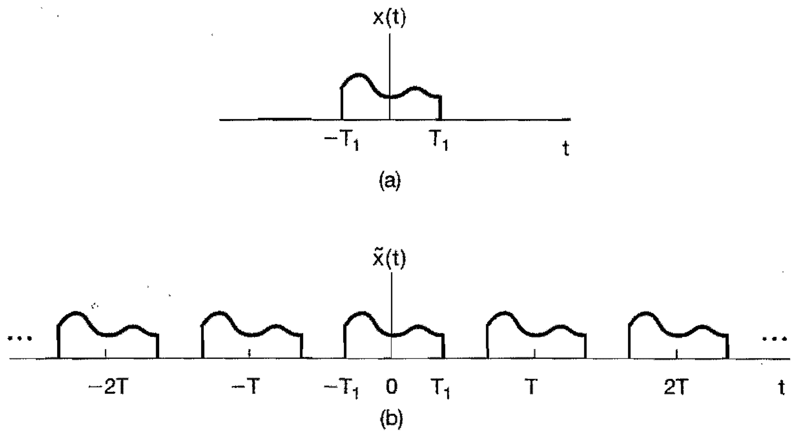


Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

carried out over the interval $-T/2 \leq t \leq T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{4.3}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt, \tag{4.4}$$

where $\omega_0 = 2\pi/T$. Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, eq. (4.4) can be rewritten as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt, \tag{4.5}$$

we have, for the coefficients a_k,

$$a_k = \frac{1}{T}X(jk\omega_0). \tag{4.6}$$

Combining eqs. (4.6) and (4.3), we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t},$$

or equivalently, since $2\pi/T = \omega_0$,

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \tag{4.7}$$

As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, in the limit eq. (4.7) becomes a representation of $x(t)$. Furthermore, $\omega_0 \to 0$ as $T \to \infty$, and the right-hand side of eq. (4.7) passes to an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure 4.4. Each term in the summation on the right-hand side is the area of a rectangle of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0. (Here, t is regarded as fixed.) As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$. Therefore, using the fact that $\tilde{x}(t) \to x(t)$ as $T \to \infty$, we see that eqs. (4.7) and (4.5) respectively become

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \tag{4.8}$$

and

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \tag{4.9}$$

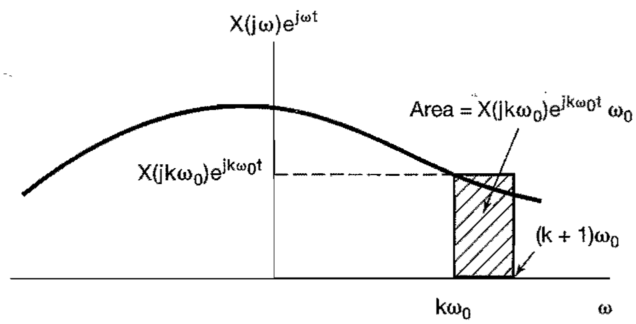


Figure 4.4 Graphical interpretation of eq. (4.7).

Equations (4.8) and (4.9) are referred to as the *Fourier transform pair*, with the function $X(j\omega)$ referred to as the *Fourier Transform* or *Fourier integral* of $x(t)$ and eq. (4.8) as the *inverse Fourier transform* equation. The *synthesis* equation (4.8) plays a role for aperiodic signals similar to that of eq. (3.38) for periodic signals, since both represent a signal as a linear combination of complex exponentials. For periodic signals, these complex exponentials have amplitudes $\{a_k\}$, as given by eq. (3.39), and occur at a discrete set of harmonically related frequencies $k\omega_0$, $k = 0, \pm 1, \pm 2, \ldots$. For aperiodic signals, the complex exponentials occur at a continuum of frequencies and, according to the synthesis equation (4.8), have "amplitude" $X(j\omega)(d\omega/2\pi)$. In analogy with the terminology used for the Fourier series coefficients of a periodic signal, the transform $X(j\omega)$ of an aperiodic signal $x(t)$ is commonly referred to as the *spectrum* of $x(t)$, as it provides us with the information needed for describing $x(t)$ as a linear combination (specifically, an integral) of sinusoidal signals at different frequencies.

Based on the above development, or equivalently on a comparison of eq. (4.9) and eq. (3.39), we also note that the Fourier coefficients a_k of a periodic signal $\tilde{x}(t)$ can be expressed in terms of equally spaced *samples* of the Fourier transform of one period of $\tilde{x}(t)$. Specifically, suppose that $\tilde{x}(t)$ is a periodic signal with period T and Fourier coefficients

a_k. Let $x(t)$ be a finite-duration signal that is equal to $\tilde{x}(t)$ over exactly one period—say, for $s \leq t \leq s+T$ for some value of s—and that is zero otherwise. Then, since eq. (3.39) allows us to compute the Fourier coefficients of $\tilde{x}(t)$ by integrating over any period, we can write

$$a_k = \frac{1}{T}\int_{s}^{s+T} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{s}^{s+T} x(t)e^{-jk\omega_0 t}dt.$$

Since $x(t)$ is zero outside the range $s \leq t \leq s+T$ we can equivalently write

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Comparing with eq. (4.9) we conclude that

$$a_k = \frac{1}{T}X(j\omega)\bigg|_{\omega = k\omega_0}, \tag{4.10}$$

where $X(j\omega)$ is the Fourier transform of $x(t)$. Equation 4.10 states that the Fourier coefficients of $\tilde{x}(t)$ are proportional to samples of the Fourier transform of one period of $\tilde{x}(t)$. This fact, which is often of use in practice, is examined further in Problem 4.37.

4.1.2 Convergence of Fourier Transforms

Although the argument we used in deriving the Fourier transform pair assumed that $x(t)$ was of arbitrary but finite duration, eqs. (4.8) and (4.9) remain valid for an extremely broad class of signals of infinite duration. In fact, our derivation of the Fourier transform suggests that a set of conditions like those required for the convergence of Fourier series should also apply here, and indeed, that can be shown to be the case.[1] Specifically, consider $X(j\omega)$ evaluated according to eq. (4.9), and let $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in the right-hand side of eq. (4.8). That is,

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega.$$

What we would like to know is when eq. (4.8) is valid [i.e., when is $\hat{x}(t)$ a valid representation of the original signal $x(t)$?]. If $x(t)$ has finite energy, i.e., if it is square integrable, so that

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty, \tag{4.11}$$

then we are guaranteed that $X(j\omega)$ is finite [i.e., eq. (4.9) converges] and that, with $e(t)$ denoting the error between $\hat{x}(t)$ and $x(t)$ [i.e., $e(t) = \hat{x}(t) - x(t)$],

[1]For a mathematically rigorous discussion of the Fourier transform and its properties and applications, see R. Bracewell, *The Fourier Transform and Its Applications,* 2nd ed. (New York: McGraw-Hill Book Company, 1986); A. Papoulis, *The Fourier Integral and Its Applications* (New York: McGraw-Hill Book Company, 1987); E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford: Clarendon Press, 1948); and the book by Dym and McKean referenced in footnote 2 of Chapter 3.

$$\int_{-\infty}^{+\infty} |e(t)|^2 dt = 0. \tag{4.12}$$

Equations (4.11) and (4.12) are the aperiodic counterparts of eqs. (3.51) and (3.54) for periodic signals. Thus, in a manner similar to that for periodic signals, if $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t, there is no energy in their difference.

Just as with periodic signals, there is an alternative set of conditions which are sufficient to ensure that $\hat{x}(t)$ is equal to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity. These conditions, again referred to as the Dirichlet conditions, require that:

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty. \tag{4.13}$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have a finite number of discontinuities have Fourier transforms.

Although the two alternative sets of conditions that we have given are sufficient to guarantee that a signal has a Fourier transform, we will see in the next section that periodic signals, which are neither absolutely integrable nor square integrable over an *infinite* interval, can be considered to have Fourier transforms if impulse functions are permitted in the transform. This has the advantage that the Fourier series and Fourier transform can be incorporated into a common framework, which we will find to be very convenient in subsequent chapters. Before examining the point further in Section 4.2, however, let us consider several examples of the Fourier transform.

4.1.3 Examples of Continuous-Time Fourier Transforms

Example 4.1

Consider the signal

$$x(t) = e^{-at}u(t) \quad a > 0.$$

From eq. (4.9),

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = -\frac{1}{a + j\omega} e^{-(a+j\omega)t} \bigg|_0^{\infty}.$$

That is,

$$X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0.$$

Since this Fourier transform is complex valued, to plot it as a function of ω, we express $X(j\omega)$ in terms of its magnitude and phase:

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Each of these components is sketched in Figure 4.5.

Note that if a is complex rather than real, then $x(t)$ is absolutely integrable as long as $\Re e\{a\} > 0$, and in this case the preceding calculation yields the same form for $X(j\omega)$. That is,

$$X(j\omega) = \frac{1}{a+j\omega}, \quad \Re e\{a\} > 0.$$

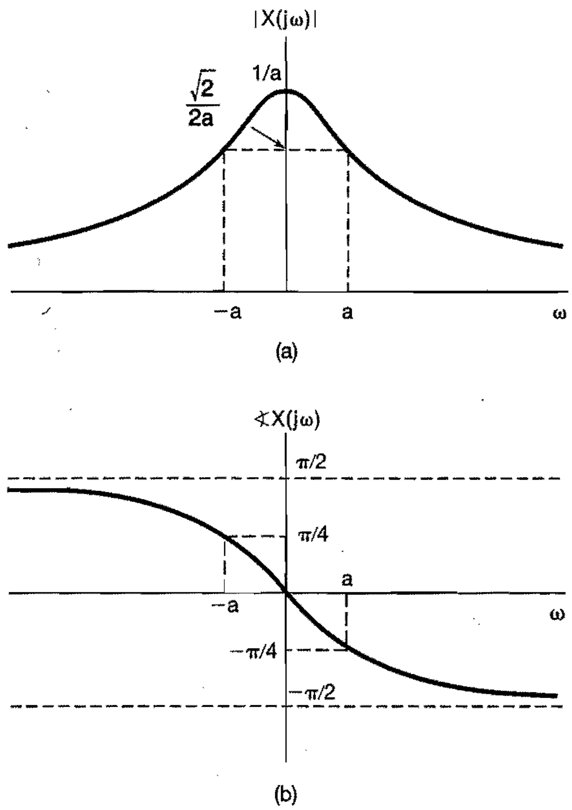


Figure 4.5 Fourier transform of the signal $x(t) = e^{-at}u(t), a > 0$, considered in Example 4.1.

Example 4.2

Let

$$x(t) = e^{-a|t|}, \quad a > 0.$$

This signal is sketched in Figure 4.6. The Fourier transform of the signal is

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|}e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$
$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$
$$= \frac{2a}{a^2 + \omega^2}.$$

In this case $X(j\omega)$ is real, and it is illustrated in Figure 4.7.

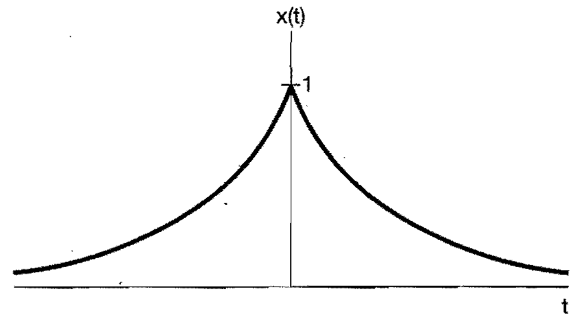


Figure 4.6 Signal $x(t) = e^{-a|t|}$ of Example 4.2.

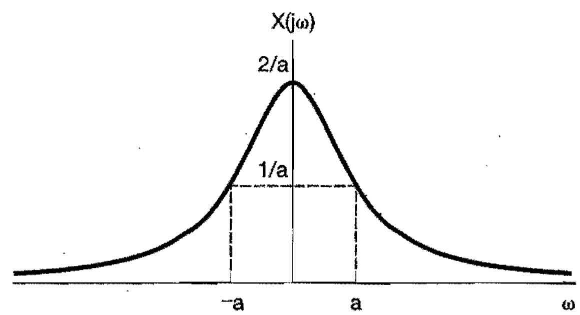


Figure 4.7 Fourier transform of the signal considered in Example 4.2 and depicted in Figure 4.6.

Example 4.3

Now let us determine the Fourier transform of the unit impulse

$$x(t) = \delta(t). \tag{4.14}$$

Substituting into eq. (4.9) yields

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = 1. \tag{4.15}$$

That is, the unit impulse has a Fourier transform consisting of equal contributions at *all* frequencies.

Example 4.4

Consider the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}, \tag{4.16}$$

as shown in Figure 4.8(a). Applying eq. (4.9), we find that the Fourier transform of this signal is

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin \omega T_1}{\omega}, \tag{4.17}$$

as sketched in Figure 4.8(b).

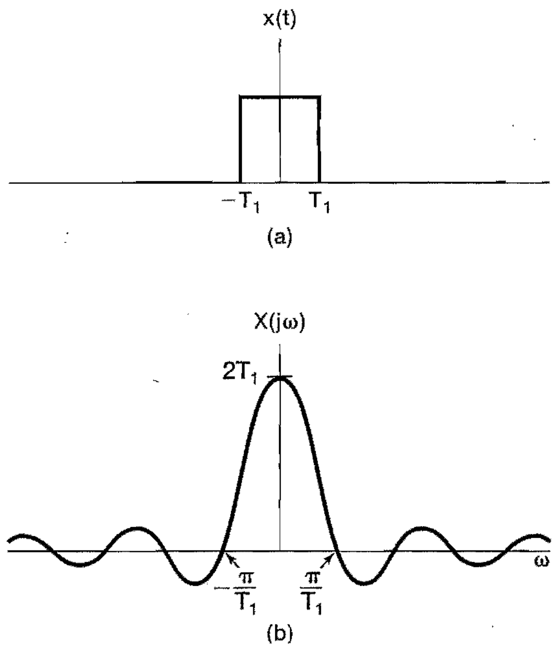


Figure 4.8 (a) The rectangular pulse signal of Example 4.4 and (b) its Fourier transform.

As we discussed at the beginning of this section, the signal given by eq. (4.16) can be thought of as the limiting form of a periodic square wave as the period becomes arbitrarily large. Therefore, we might expect that the convergence of the synthesis equation for this signal would behave in a manner similar to that observed in Example 3.5 for the square wave. This is, in fact, the case. Specifically, consider the inverse Fourier transform for the rectangular pulse signal:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\frac{\sin \omega T_1}{\omega} e^{j\omega t} d\omega.$$

Then, since $x(t)$ is square integrable,

$$\int_{-\infty}^{+\infty} |x(t) - \hat{x}(t)|^2 \, dt = 0.$$

Furthermore, because $x(t)$ satisfies the Dirichlet conditions, $\hat{x}(t) = x(t)$, except at the points of discontinuity, $t = \pm T_1$, where $\hat{x}(t)$ converges to 1/2, which is the average of the values of $x(t)$ on both sides of the discontinuity. In addition, the convergence of $\hat{x}(t)$ to $x(t)$ exhibits the Gibbs phenomenon, much as was illustrated for the periodic square wave in Figure 3.9. Specifically, in analogy with the finite Fourier series approximation, eq. (3.47), consider the following integral over a finite-length interval of frequencies:

$$\frac{1}{2\pi}\int_{-W}^{W} 2\frac{\sin \omega T_1}{\omega} e^{j\omega t} \, d\omega.$$

As $W \to \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities. Moreover, the signal exhibits ripples near the discontinuities. The peak amplitude of these ripples does not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

Example 4.5

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \tag{4.18}$$

This transform is illustrated in Figure 4.9(a). Using the synthesis equation (4.8), we can

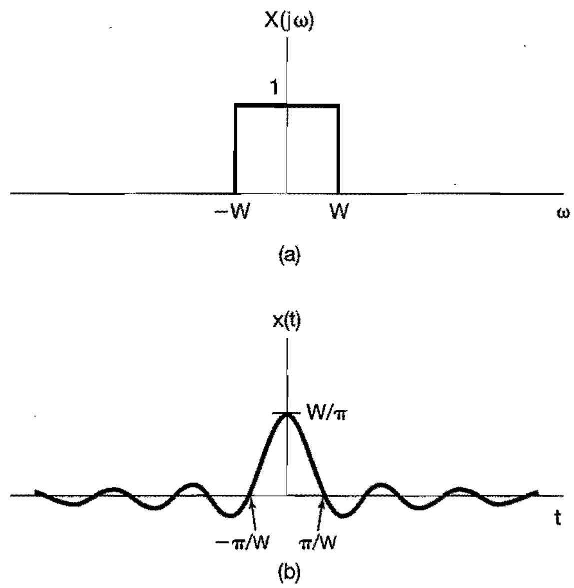


Figure 4.9 Fourier transform pair of Example 4.5: (a) Fourier transform for Example 4.5 and (b) the corresponding time function.

then determine

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\pi t}, \tag{4.19}$$

which is depicted in Figure 4.9(b).

Comparing Figures 4.8 and 4.9 or, equivalently, eqs. (4.16) and (4.17) with eqs. (4.18) and (4.19), we see an interesting relationship. In each case, the Fourier transform pair consists of a function of the form $(\sin a\theta)/b\theta$ and a rectangular pulse. However, in Example 4.4, it is the *signal* $x(t)$ that is a pulse, while in Example 4.5, it is the *transform* $X(j\omega)$. The special relationship that is apparent here is a direct consequence of the *duality property* for Fourier transforms, which we discuss in detail in Section 4.3.6.

Functions of the form given in eqs. (4.17) and (4.19) arise frequently in Fourier analysis and in the study of LTI systems and are referred to as *sinc functions*. A commonly used precise form for the sinc function is

$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{4.20}$$

The sinc function is plotted in Figure 4.10. Both of the signals in eqs. (4.17) and (4.19) can be expressed in terms of the sinc function:

$$\frac{2\sin\omega T_1}{\omega} = 2T_1 \,\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{\pi}\right).$$

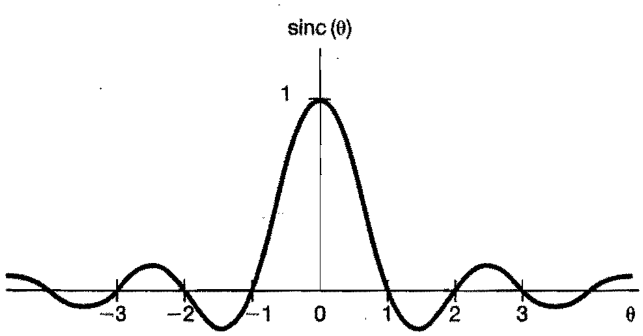


Figure 4.10 The sinc function.

Finally, we can gain insight into one other property of the Fourier transform by examining Figure 4.9, which we have redrawn as Figure 4.11 for several different values of W. From this figure, we see that as W increases, $X(j\omega)$ becomes broader, while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of this signal (i.e., the part of the signal for $|t| < \pi/W$) becomes narrower. In fact, in the limit as $W \to \infty$, $X(j\omega) = 1$ for all ω, and consequently, from Example 4.3, we see that $x(t)$ in eq. (4.19) converges to an impulse as $W \to \infty$. The behavior depicted in Figure 4.11 is an example of the inverse relationship that exists between the time and frequency domains,

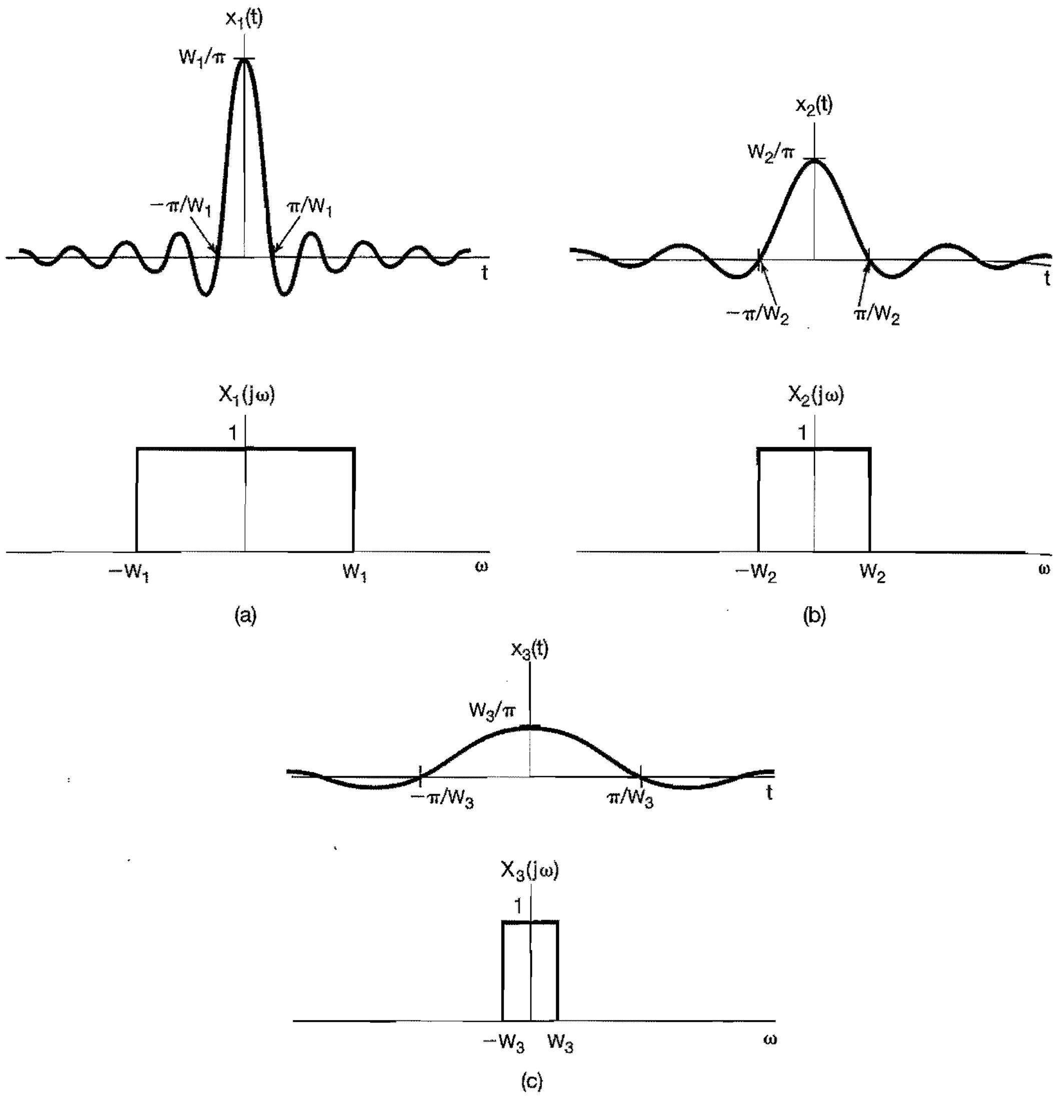


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W.

and we can see a similar effect in Figure 4.8, where an increase in T_1 broadens $x(t)$ but makes $X(j\omega)$ narrower. In Section 4.3.5, we provide an explanation of this behavior in the context of the scaling property of the Fourier transform.

4.2 THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

In the preceding section, we introduced the Fourier transform representation and gave several examples. While our attention in that section was focused on aperiodic signals, we can also develop Fourier transform representations for periodic signals, thus allowing us to consider both periodic and aperiodic signals within a unified context. In fact, as we will see, we can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of the impulses proportional to the Fourier series coefficients. This will turn out to be a very useful representation.

To suggest the general result, let us consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$; that is,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0). \tag{4.21}$$

To determine the signal $x(t)$ for which this is the Fourier transform, we can apply the inverse transform relation, eq. (4.8), to obtain

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega \\ &= e^{j\omega_0 t}. \end{aligned}$$

More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, that is,

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0), \tag{4.22}$$

then the application of eq. (4.8) yields

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \tag{4.23}$$

We see that eq. (4.23) corresponds exactly to the Fourier *series* representation of a periodic signal, as specified by eq. (3.38). Thus, the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

Example 4.6

Consider again the square wave illustrated in Figure 4.1. The Fourier series coefficients for this signal are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

and the Fourier transform of the signal is

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0),$$

which is sketched in Figure 4.12 for $T = 4T_1$. In comparison with Figure 3.7(a), the only differences are a proportionality factor of 2π and the use of impulses rather than a bar graph.

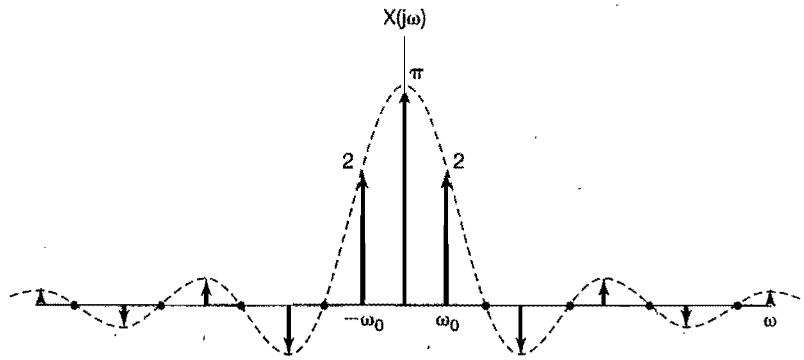


Figure 4.12 Fourier transform of a symmetric periodic square wave.

Example 4.7

Let

$$x(t) = \sin \omega_0 t.$$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2j},$$

$$a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

Thus, the Fourier transform is as shown in Figure 4.13(a). Similarly, for

$$x(t) = \cos \omega_0 t,$$

the Fourier series coefficients are

$$a_1 = a_{-1} = \frac{1}{2},$$

$$a_k = 0, \qquad k \neq 1 \quad \text{or} \quad -1.$$

The Fourier transform of this signal is depicted in Figure 4.13(b). These two transforms will be of considerable importance when we analyze sinusoidal modulation systems in Chapter 8.

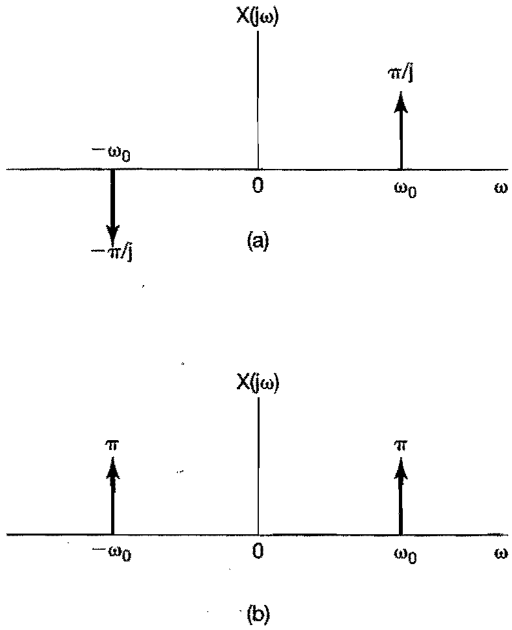


Figure 4.13 Fourier transforms of (a) $x(t) = \sin \omega_0 t$; (b) $x(t) = \cos \omega_0 t$.

Example 4.8

A signal that we will find extremely useful in our analysis of sampling systems in Chapter 7 is the impulse train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

which is periodic with period T, as indicated in Figure 4.14(a). The Fourier series coefficients for this signal were computed in Example 3.8 and are given by

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k in eq. (4.22) yields

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$, as sketched in Figure 4.14(b). Here again, we see an illustration of the inverse relationship between the time and the frequency domains. As the spacing between the impulses in the time domain (i.e., the period) gets longer, the spacing between the impulses in the frequency domain (namely, the fundamental frequency) gets smaller.

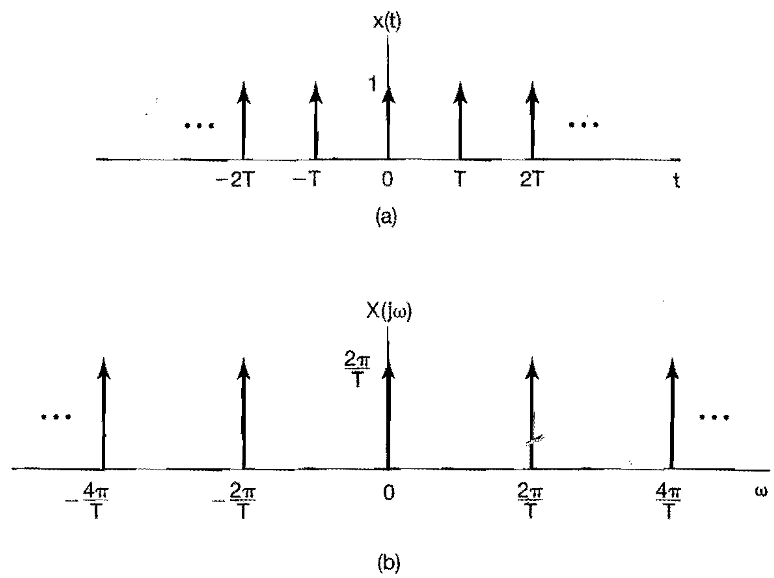


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

4.3 PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

In this and the following two sections, we consider a number of properties of the Fourier transform. A detailed listing of these properties is given in Table 4.1 in Section 4.6. As was the case for the Fourier series representation of periodic signals, these properties provide us with a significant amount of insight into the transform and into the relationship between the time-domain and frequency-domain descriptions of a signal. In addition, many of the properties are often useful in reducing the complexity of the evaluation of Fourier transforms or inverse transforms. Furthermore, as described in the preceding section, there is a close relationship between the Fourier series and Fourier transform representations of a periodic signal, and using this relationship, we can translate many of the Fourier transform properties into corresponding Fourier series properties, which we discussed independently in Chapter 3. (See, in particular, Section 3.5 and Table 3.1.)

Throughout the discussion in this section, we will be referring frequently to functions of time and their Fourier transforms, and we will find it convenient to use a shorthand notation to indicate the pairing of a signal and its transform. As developed in Section 4.1, a signal $x(t)$ and its Fourier transform $X(j\omega)$ are related by the Fourier transform synthesis and analysis equations,

[eq. (4.8)]
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \tag{4.24}$$

and

[eq. (4.9)]
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \tag{4.25}$$

We will sometimes find it convenient to refer to $X(j\omega)$ with the notation $\mathfrak{F}\{x(t)\}$ and to $x(t)$ with the notation $\mathfrak{F}^{-1}\{X(j\omega)\}$. We will also refer to $x(t)$ and $X(j\omega)$ as a Fourier transform pair with the notation

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega).$$

Thus, with reference to Example 4.1,

$$\frac{1}{a+j\omega} = \mathfrak{F}\{e^{-at}u(t)\},$$

$$e^{-at}u(t) = \mathfrak{F}^{-1}\left\{\frac{1}{a+j\omega}\right\},$$

and

$$e^{-at}u(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a+j\omega}.$$

4.3.1 Linearity

If

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega)$$

and

$$y(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} Y(j\omega),$$

then

$$\boxed{ax(t) + by(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega).} \tag{4.26}$$

The proof of eq. (4.26) follows directly by application of the analysis eq. (4.25) to $ax(t) + by(t)$. The linearity property is easily extended to a linear combination of an arbitrary number of signals.

4.3.2 Time Shifting

If

$$x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x(t - t_0) \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega).} \tag{4.27}$$

To establish this property, consider eq. (4.24):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Replacing t by $t - t_0$ in this equation, we obtain

$$\begin{aligned} x(t-t_0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}d\omega. \end{aligned}$$

Recognizing this as the synthesis equation for $x(t - t_0)$, we conclude that

$$\mathfrak{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega).$$

One consequence of the time-shift property is that a signal which is shifted in time does not have the *magnitude* of its Fourier transform altered. That is, if we express $X(j\omega)$ in polar form as

$$\mathfrak{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)},$$

then

$$\mathfrak{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j[\sphericalangle X(j\omega)-\omega t_0]}.$$

Thus, the effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-\omega t_0$, which is a linear function of ω.

Example 4.9

To illustrate the usefulness of the Fourier transform linearity and time-shift properties, let us consider the evaluation of the Fourier transform of the signal $x(t)$ shown in Figure 4.15(a).

First, we observe that $x(t)$ can be expressed as the linear combination

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5),$$

where the signals $x_1(t)$ and $x_2(t)$ are the rectangular pulse signals shown in Figure 4.15(b) and (c). Then, using the result from Example 4.4, we obtain

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \quad \text{and} \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}.$$

Finally, using the linearity and time-shift properties of the Fourier transform yields

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}.$$

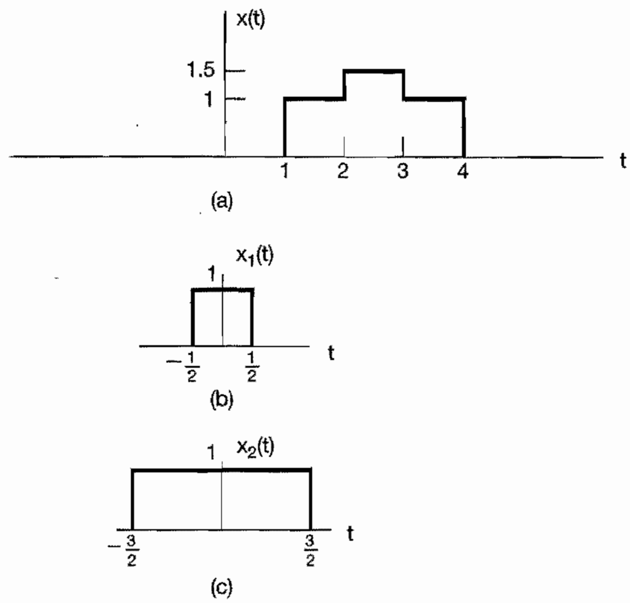


Figure 4.15 Decomposing a signal into the linear combination of two simpler signals. (a) The signal $x(t)$ for Example 4.9; (b) and (c) the two component signals used to represent $x(t)$.

4.3.3 Conjugation and Conjugate Symmetry

The conjugation property states that if

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega).} \tag{4.28}$$

This property follows from the evaluation of the complex conjugate of eq. (4.25):

$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* \\ &= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt. \end{aligned}$$

Replacing ω by $-\omega$, we see that

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}\,dt. \tag{4.29}$$

Recognizing that the right-hand side of eq. (4.29) is the Fourier transform analysis equation for $x^*(t)$, we obtain the relation given in eq. (4.28).

The conjugation property allows us to show that if $x(t)$ is real, then $X(j\omega)$ has *conjugate symmetry;* that is,

$$\boxed{X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}].} \tag{4.30}$$

Specifically, if $x(t)$ is real so that $x^*(t) = x(t)$, we have, from eq. (4.29),

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt = X(j\omega),$$

and eq. (4.30) follows by replacing ω with $-\omega$.

From Example 4.1, with $x(t) = e^{-at}u(t)$,

$$X(j\omega) = \frac{1}{a + j\omega}$$

and

$$X(-j\omega) = \frac{1}{a - j\omega} = X^*(j\omega).$$

As one consequence of eq. (4.30), if we express $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathcal{R}e\{X(j\omega)\} + j\mathcal{I}m\{X(j\omega)\},$$

then if $x(t)$ is real,

$$\mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\}$$

and

$$\mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\}.$$

That is, the real part of the Fourier transform is an *even* function of frequency, and the imaginary part is an *odd* function of frequency. Similarly, if we express $X(j\omega)$ in polar form as

$$X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)},$$

then it follows from eq. (4.30) that $|X(j\omega)|$ is an even function of ω and $\sphericalangle X(j\omega)$ is an odd function of ω. Thus, when computing or displaying the Fourier transform of a real-valued signal, the real and imaginary parts or magnitude and phase of the transform need only be specified for positive frequencies, as the values for negative frequencies can be determined directly from the values for $\omega > 0$ using the relationships just derived.

As a further consequence of eq. (4.30), if $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even. To see this, we write

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt,$$

or, with the substitution $\tau = -t$,

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}d\tau.$$

Since $x(-\tau) = x(\tau)$, we have

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega t}d\tau$$
$$= X(j\omega).$$

Thus, $X(j\omega)$ is an even function. This, together with eq. (4.30), also requires that $X^*(j\omega) = X(j\omega)$ [i.e., that $X(j\omega)$ is real]. Example 4.2 illustrates this property for the real, even signal $e^{-a|t|}$. In a similar manner, it can be shown that if $x(t)$ is a real and odd function of time, so that $x(t) = -x(-t)$, then $X(j\omega)$ is purely imaginary and odd.

Finally, as was discussed in Chapter 1, a real function $x(t)$ can always be expressed in terms of the sum of an even function $x_e(t) = \mathcal{E}v\{x(t)\}$ and an odd function $x_o(t) = \mathcal{O}d\{x(t)\}$; that is,

$$x(t) = x_e(t) + x_o(t).$$

From the linearity of the Fourier transform,

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},$$

and from the preceding discussion, $\mathfrak{F}\{x_e(t)\}$ is a real function and $\mathfrak{F}\{x_o(t)\}$ is purely imaginary. Thus, we can conclude that, with $x(t)$ real,

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j\omega),$$

$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\mathcal{I}m\{X(j\omega)\}.$$

One use of these symmetry properties is illustrated in the following example.

Example 4.10

Consider again the Fourier transform evaluation of Example 4.2 for the signal $x(t) = e^{-a|t|}$, where $a > 0$. This time we will utilize the symmetry properties of the Fourier transform to aid the evaluation process.

From Example 4.1, we have

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}.$$

Note that for $t > 0$, $x(t)$ equals $e^{-at}u(t)$, while for $t < 0$, $x(t)$ takes on mirror image values. That is,

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$
$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$
$$= 2\mathcal{E}v\{e^{-at}u(t)\}.$$

Since $e^{-at}u(t)$ is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathcal{E}v\{e^{-at}u(t)\} \stackrel{\mathfrak{F}}{\longleftrightarrow} \mathcal{R}e\left\{\frac{1}{a+j\omega}\right\}.$$

It follows that

$$X(j\omega) = 2\mathcal{R}e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2},$$

which is the same as the answer found in Example 4.2.

4.3.4 Differentiation and Integration

Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Then, by differentiating both sides of the Fourier transform synthesis equation (4.24), we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \stackrel{\mathfrak{F}}{\longleftrightarrow} j\omega X(j\omega). \tag{4.31}$$

This is a particularly important property, as it replaces the operation of differentiation in the time domain with that of multiplication by $j\omega$ in the frequency domain. We will find the substitution to be extremely useful in our discussion in Section 4.7 on the use of Fourier transforms for the analysis of LTI systems described by differential equations.

Since differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain, one might conclude that integration should involve division by $j\omega$ in the frequency domain. This is indeed the case, but it is only one part of the picture. The precise relationship is

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega). \tag{4.32}$$

The impulse term on the right-hand side of eq. (4.32) reflects the dc or average value that can result from integration.

The use of eqs. (4.31) and (4.32) is illustrated in the next two examples.

Example 4.11

Let us determine the Fourier transform $X(j\omega)$ of the unit step $x(t) = u(t)$, making use of eq. (4.32) and the knowledge that

$$g(t) = \delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} G(j\omega) = 1.$$

Noting that

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

and taking the Fourier transform of both sides, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

where we have used the integration property listed in Table 4.1. Since $G(j\omega) = 1$, we conclude that

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \tag{4.33}$$

Observe that we can apply the differentiation property of eq. (4.31) to recover the transform of the impulse. That is,

$$\delta(t) = \frac{du(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1,$$

where the last equality follows from the fact that $\omega\delta(\omega) = 0$.

Example 4.12

Suppose that we wish to calculate the Fourier transform $X(j\omega)$ for the signal $x(t)$ displayed in Figure 4.16(a). Rather than applying the Fourier integral directly to $x(t)$, we instead consider the signal

$$g(t) = \frac{d}{dt}x(t).$$

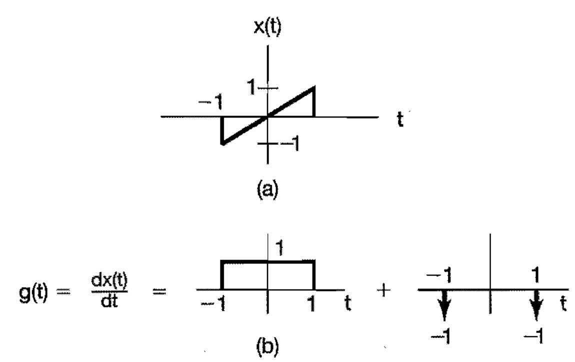


Figure 4.16 (a) A signal $x(t)$ for which the Fourier transform is to be evaluated; (b) representation of the derivative of $x(t)$ as the sum of two components.

As illustrated in Figure 4.16(b), $g(t)$ is the sum of a rectangular pulse and two impulses. The Fourier transforms of each of these component signals may be determined from Table 4.2:

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}.$$

Note that $G(0) = 0$. Using the integration property, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

With $G(0) = 0$ this becomes

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}.$$

The expression for $X(j\omega)$ is purely imaginary and odd, which is consistent with the fact that $x(t)$ is real and odd.

4.3.5 Time and Frequency Scaling

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega),$$

then

$$\boxed{x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right),} \tag{4.34}$$

where a is a nonzero real number. This property follows directly from the definition of the Fourier transform—specifically,

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt.$$

Using the substitution $\tau = at$, we obtain

$$\mathcal{F}\{x(at)\} = \begin{cases} \dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ -\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases},$$

which corresponds to eq. (4.34). Thus, aside from the amplitude factor $1/|a|$, a linear scaling in time by a factor of a corresponds to a linear scaling in frequency by a factor of $1/a$, and vice versa. Also, letting $a = -1$, we see from eq. (4.34) that

$$\boxed{x(-t) \overset{\mathfrak{F}}{\longleftrightarrow} X(-j\omega).} \tag{4.35}$$

That is, reversing a signal in time also reverses its Fourier transform.

A common illustration of eq. (4.34) is the effect on frequency content that results when an audiotape is recorded at one speed and played back at a different speed. If the playback speed is higher than the recording speed, corresponding to compression in time (i.e., $a > 1$), then the spectrum is expanded in frequency (i.e., the audible effect is that the playback frequencies are higher). Conversely, the signal played back will be scaled down in frequency if the playback speed is slower than the recording speed ($0 < a < 1$). For example, if a recording of the sound of a small bell ringing is played back at a reduced speed, the result will sound like the chiming of a larger and deeper sounding bell.

The scaling property is another example of the inverse relationship between time and frequency that we have already encountered on several occasions. For example, we have seen that as we increase the period of a sinusoidal signal, we decrease its frequency. Also, as we saw in Example 4.5 (see Figure 4.11), if we consider the transform

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases},$$

then as we increase W, the inverse transform of $X(j\omega)$ becomes narrower and taller and approaches an impulse as $W \to \infty$. Finally, in Example 4.8, we saw that the spacing in the frequency domain between impulses in the Fourier transform of a periodic impulse train is inversely proportional to the spacing in the time domain.

The inverse relationship between the time and frequency domains is of great importance in a variety of signal and systems contexts, including filtering and filter design, and we will encounter its consequences on numerous occasions in the remainder of the book. In addition, the reader may very well come across the implications of this property in studying a wide variety of other topics in science and engineering. One example is the uncertainty principle in physics; another is illustrated in Problem 4.49.

4.3.6 Duality

By comparing the transform and inverse transform relations given in eqs. (4.24) and (4.25), we observe that these equations are similar, but not quite identical, in form. This symmetry leads to a property of the Fourier transform referred to as *duality*. In Example 4.5, we alluded to duality when we noted the relationship that exists between the Fourier transform pairs of Examples 4.4 and 4.5. In the former example we derived the Fourier transform pair

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{\mathfrak{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}, \tag{4.36}$$

while in the latter we considered the pair

$$x_2(t) = \frac{\sin Wt}{\pi t} \overset{\mathfrak{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \tag{4.37}$$

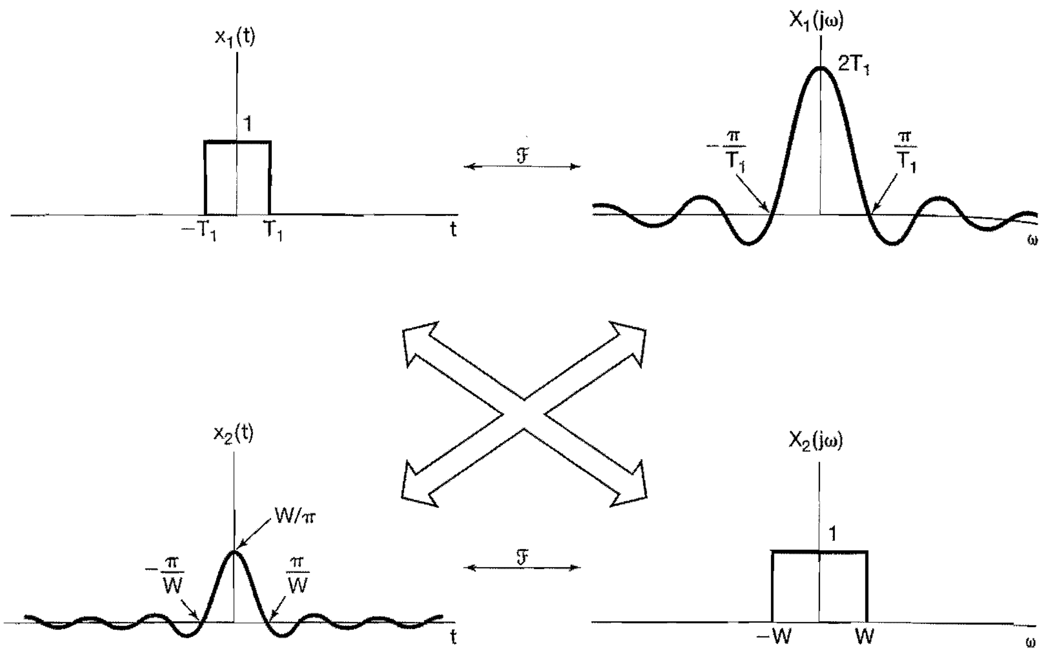


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

The two Fourier transform pairs and the relationship between them are depicted in Figure 4.17.

The symmetry exhibited by these two examples extends to Fourier transforms in general. Specifically, because of the symmetry between eqs. (4.24) and (4.25), for any transform pair, there is a dual pair with the time and frequency variables interchanged. This is best illustrated through an example.

Example 4.13

Let us consider using duality to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

In Example 4.2 we encountered a Fourier transform pair in which the Fourier transform, as a function of ω, had a form similar to that of the signal $x(t)$. Specifically, suppose we consider a signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

Then, from Example 4.2,

$$x(t) = e^{-|t|} \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega.$$

Now, interchanging the names of the variables t and ω, we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt. \tag{4.38}$$

The right-hand side of eq. (4.38) is the Fourier transform analysis equation for $2/(1+t^2)$, and thus, we conclude that

$$\mathfrak{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}.$$

The duality property can also be used to determine or to suggest other properties of Fourier transforms. Specifically, if there are characteristics of a function of time that have implications with regard to the Fourier transform, then the same characteristics associated with a function of frequency will have *dual* implications in the time domain. For example, in Section 4.3.4, we saw that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. From the preceding discussion, we might then suspect that multiplication by jt in the time domain corresponds roughly to differentiation in the frequency domain. To determine the precise form of this dual property, we can proceed in a fashion exactly analogous to that used in Section 4.3.4. Thus, if we differentiate the analysis equation (4.25) with respect to ω, we obtain

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jtx(t)e^{-j\omega t}dt. \tag{4.39}$$

That is,

$$\boxed{-jtx(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.} \tag{4.40}$$

Similarly, we can derive the dual properties of eqs. (4.27) and (4.32):

$$\boxed{e^{j\omega_0 t}x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j(\omega-\omega_0))} \tag{4.41}$$

and

$$\boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathfrak{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta.} \tag{4.42}$$

4.3.7 Parseval's Relation

If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then

$$\boxed{\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega.} \tag{4.43}$$

This expression, referred to as Parseval's relation, follows from direct application of the Fourier transform. Specifically,

$$\begin{aligned}\int_{-\infty}^{+\infty}|x(t)|^2dt &= \int_{-\infty}^{+\infty}x(t)x^*(t)dt \\ &= \int_{-\infty}^{+\infty}x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)e^{-j\omega t}d\omega\right]dt.\end{aligned}$$

Reversing the order of integration gives

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)\left[\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt\right]d\omega.$$

The bracketed term is simply the Fourier transform of $x(t)$; thus,

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega.$$

The term on the left-hand side of eq. (4.43) is the total energy in the signal $x(t)$. Parseval's relation says that this total energy may be determined either by computing the energy per unit time ($|x(t)|^2$) and integrating over all time or by computing the energy per unit frequency ($|X(j\omega)|^2/2\pi$) and integrating over all frequencies. For this reason, $|X(j\omega)|^2$ is often referred to as the *energy-density spectrum* of the signal $x(t)$. (See also Problem 4.45.) Note that Parseval's relation for finite-energy signals is the direct counterpart of Parseval's relation for periodic signals (eq. 3.67), which states that the average *power* of a periodic signal equals the sum of the average powers of its individual harmonic components, which in turn are equal to the squared magnitudes of the Fourier series coefficients.

Parseval's relation and other Fourier transform properties are often useful in determining some time domain characteristics of a signal directly from the Fourier transform. The next example is a simple illustration of this.

Example 4.14

For each of the Fourier transforms shown in Figure 4.18, we wish to evaluate the following time-domain expressions:

$$E = \int_{-\infty}^{\infty}|x(t)|^2dt$$

$$D = \left.\frac{d}{dt}x(t)\right|_{t=0}$$

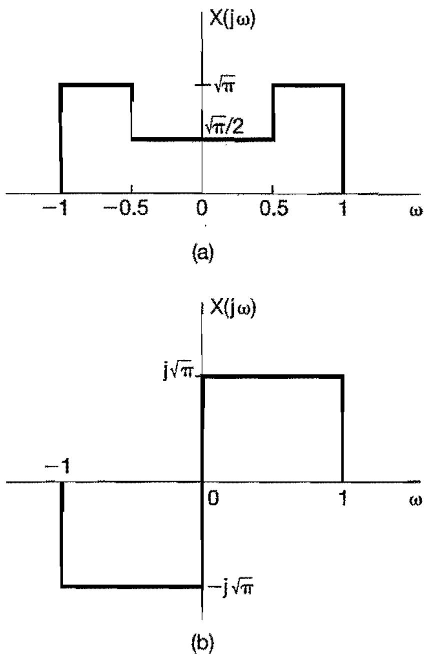


Figure 4.18 The Fourier transforms considered in Example 4.14.

To evaluate E in the frequency domain, we may use Parseval's relation. That is,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{4.44}$$

which evaluates to $\frac{5}{8}$ for Figure 4.18(a) and to 1 for Figure 4.18(b).

To evaluate D in the frequency domain, we first use the differentiation property to observe that

$$g(t) = \frac{d}{dt}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) = G(j\omega).$$

Noting that

$$D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega \tag{4.45}$$

we conclude:

$$D = \int_{-\infty}^{\infty} j\omega X(j\omega)\, d\omega \tag{4.46}$$

which evaluates to zero for figure 4.18(a) and to $\dfrac{-1}{(2\sqrt{\pi})}$ for Figure 4.18(b).

There are many other properties of the Fourier transform in addition to those we have already discussed. In the next two sections, we present two specific properties that play