investigation of harmonic time series in the 18th and 19th centuries by eminent scientists and mathematicians, including Gauss, and thus provided a second setting in which much of the initial work was done on discrete-time signals and systems.

In the mid-1960s an algorithm, now known as the fast Fourier transform, or FFT, was introduced. This algorithm, which was independently discovered by Cooley and Tukey in 1965, also has a considerable history and can, in fact, be found in Gauss' notebooks.[6] What made its modern discovery so important was the fact that the FFT proved to be perfectly suited for efficient digital implementation, and it reduced the time required to compute transforms by orders of magnitude. With this tool, many interesting but previously impractical ideas utilizing the discrete-time Fourier series and transform suddenly became practical, and the development of discrete-time signal and system analysis techniques moved forward at an accelerated pace.

What has emerged out of this long history is a powerful and cohesive framework for the analysis of continuous-time and discrete-time signals and systems and an extraordinarily broad array of existing and potential applications. In this and the following chapters, we will develop the basic tools of that framework and examine some of its important implications.

3.2 THE RESPONSE OF LTI SYSTEMS TO COMPLEX EXPONENTIALS

As we indicated in Section 3.0, it is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:

1. The set of basic signals can be used to construct a broad and useful class of signals.
2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

Much of the importance of Fourier analysis results from the fact that both of these properties are provided by the set of complex exponential signals in continuous and discrete time—i.e., signals of the form e^{st} in continuous time and z^n in discrete time, where s and z are complex numbers. In subsequent sections of this and the following two chapters, we will examine the first property in some detail. In this section, we focus on the second property and, in this way, provide motivation for the use of Fourier series and transforms in the analysis of LTI systems.

The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is,

$$\text{continuous time: } e^{st} \longrightarrow H(s)e^{st}, \tag{3.1}$$

$$\text{discrete time: } z^n \longrightarrow H(z)z^n, \tag{3.2}$$

where the complex amplitude factor $H(s)$ or $H(z)$ will in general be a function of the complex variable s or z. A signal for which the system output is a (possibly complex)

[6] M. T. Heideman, D. H. Johnson, and C. S. Burrus, "Gauss and the History of the Fast Fourier Transform," *The IEEE ASSP Magazine I* (1984), pp. 14–21.

constant times the input is referred to as an *eigenfunction* of the system, and the amplitude factor is referred to as the system's *eigenvalue*.

To show that complex exponentials are indeed eigenfunctions of LTI systems, let us consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t)$, we can determine the output through the use of the convolution integral, so that with $x(t) = e^{st}$

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau. \end{aligned} \tag{3.3}$$

Expressing $e^{s(t-\tau)}$ as $e^{st}e^{-s\tau}$, and noting that e^{st} can be moved outside the integral, we see that eq. (3.3) becomes

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau. \tag{3.4}$$

Assuming that the integral on the right-hand side of eq. (3.4) converges, the response to e^{st} is of the form

$$y(t) = H(s)e^{st}, \tag{3.5}$$

where $H(s)$ is a complex constant whose value depends on s and which is related to the system impulse response by

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau. \tag{3.6}$$

Hence, we have shown that complex exponentials are eigenfunctions of LTI systems. The constant $H(s)$ for a specific value of s is then the eigenvalue associated with the eigenfunction e^{st}.

In an exactly parallel manner, we can show that complex exponential sequences are eigenfunctions of discrete-time LTI systems. That is, suppose that an LTI system with impulse response $h[n]$ has as its input the sequence

$$x[n] = z^n, \tag{3.7}$$

where z is a complex number. Then the output of the system can be determined from the convolution sum as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}. \end{aligned} \tag{3.8}$$

From this expression, we see that if the input $x[n]$ is the complex exponential given by eq. (3.7), then, assuming that the summation on the right-hand side of eq. (3.8) converges, the output is the same complex exponential multiplied by a constant that depends on the value of z. That is,

$$y[n] = H(z)z^n, \tag{3.9}$$

where

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}. \tag{3.10}$$

Consequently, as in the continuous-time case, complex exponentials are eigenfunctions of discrete-time LTI systems. The constant $H(z)$ for a specified value of z is the eigenvalue associated with the eigenfunction z^n.

For the analysis of LTI systems, the usefulness of decomposing more general signals in terms of eigenfunctions can be seen from an example. Let $x(t)$ correspond to a linear combination of three complex exponentials; that is,

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}. \tag{3.11}$$

From the eigenfunction property, the response to each separately is

$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1) e^{s_1 t},$$
$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2) e^{s_2 t},$$
$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3) e^{s_3 t},$$

and from the superposition property the response to the sum is the sum of the responses, so that

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}. \tag{3.12}$$

More generally, in continuous time, eq. (3.5), together with the superposition property, implies that the representation of signals as a linear combination of complex exponentials leads to a convenient expression for the response of an LTI system. Specifically, if the input to a continuous-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x(t) = \sum_k a_k e^{s_k t}, \tag{3.13}$$

then the output will be

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{3.14}$$

In an exactly analogous manner, if the input to a discrete-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x[n] = \sum_k a_k z_k^n, \tag{3.15}$$

then the output will be

$$y[n] = \sum_k a_k H(z_k) z_k^n. \tag{3.16}$$

In other words, for both continuous time and discrete time, if the input to an LTI system is represented as a linear combination of complex exponentials, then the output can also be represented as a linear combination of the same complex exponential signals. Each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue $H(s_k)$ or $H(z_k)$ associated with the eigenfunction $e^{s_k t}$ or z_k^n, respectively. It was precisely this fact that Euler discovered for the problem of the vibrating string, that Gauss and others used in the analysis of time series, and that motivated Fourier and others after him to consider the question of how broad a class of signals could be represented as a linear combination of complex exponentials. In the next few sections we examine this question for periodic signals, first in continuous time and then in discrete time, and in Chapters 4 and 5 we consider the extension of these representations to aperiodic signals. Although in general, the variables s and z in eqs. (3.1)–(3.16) may be arbitrary complex numbers, Fourier analysis involves restricting our attention to particular forms for these variables. In particular, in continuous time we focus on purely imaginary values of s—i.e., $s = j\omega$—and thus, we consider only complex exponentials of the form $e^{j\omega t}$. Similarly, in discrete time we restrict the range of values of z to those of unit magnitude—i.e., $z = e^{j\omega}$—so that we focus on complex exponentials of the form $e^{j\omega n}$.

Example 3.1

As an illustration of eqs. (3.5) and (3.6), consider an LTI system for which the input $x(t)$ and output $y(t)$ are related by a time shift of 3, i.e.,

$$y(t) = x(t-3). \tag{3.17}$$

If the input to this system is the complex exponential signal $x(t) = e^{j2t}$, then, from eq. (3.17),

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}. \tag{3.18}$$

Equation (3.18) is in the form of eq. (3.5), as we would expect, since e^{j2t} is an eigenfunction. The associated eigenvalue is $H(j2) = e^{-j6}$. It is straightforward to confirm eq. (3.6) for this example. Specifically, from eq. (3.17), the impulse response of the system is $h(t) = \delta(t-3)$. Substituting into eq. (3.6), we obtain

$$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3)e^{-s\tau}\,d\tau = e^{-3s},$$

so that $H(j2) = e^{-j6}$.

As a second example, in this case illustrating eqs. (3.11) and (3.12), consider the input signal $x(t) = \cos(4t) + \cos(7t)$. From eq. (3.17), $y(t)$ will of course be

$$y(t) = \cos(4(t-3)) + \cos(7(t-3)). \tag{3.19}$$

To see that this will also result from eq. (3.12), we first expand $x(t)$ using Euler's relation:

$$x(t) = \tfrac{1}{2}e^{j4t} + \tfrac{1}{2}e^{-j4t} + \tfrac{1}{2}e^{j7t} + \tfrac{1}{2}e^{-j7t}. \tag{3.20}$$

From eqs. (3.11) and (3.12),

$$y(t) = \tfrac{1}{2}e^{-j12}e^{j4t} + \tfrac{1}{2}e^{j12}e^{-j4t} + \tfrac{1}{2}e^{-j21}e^{j7t} + \tfrac{1}{2}e^{j21}e^{-j7t},$$

or

$$\begin{aligned} y(t) &= \tfrac{1}{2}e^{j4(t-3)} + \tfrac{1}{2}e^{-j4(t-3)} + \tfrac{1}{2}e^{j7(t-3)} + \tfrac{1}{2}e^{-j7(t-3)} \\ &= \cos(4(t-3)) + \cos(7(t-3)). \end{aligned}$$

For this simple example, multiplication of each periodic exponential component of $x(t)$—for example, $\frac{1}{2}e^{j4t}$—by the corresponding eigenvalue—e.g., $H(j4) = e^{-j12}$—effectively causes the input component to shift in time by 3. Obviously, in this case we can determine $y(t)$ in eq. (3.19) by inspection rather than by employing eqs. (3.11) and (3.12). However, as we will see, the general property embodied in eqs. (3.11) and (3.12) not only allows us to calculate the responses of more complex LTI systems, but also provides the basis for the frequency domain representation and analysis of LTI systems.

3.3 FOURIER SERIES REPRESENTATION OF CONTINUOUS-TIME PERIODIC SIGNALS

3.3.1 Linear Combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a signal is periodic if, for some positive value of T,

$$x(t) = x(t+T) \quad \text{for all } t. \tag{3.21}$$

The fundamental period of $x(t)$ is the minimum positive, nonzero value of T for which eq. (3.21) is satisfied, and the value $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency.

In Chapter 1 we also introduced two basic periodic signals, the sinusoidal signal

$$x(t) = \cos\omega_0 t \tag{3.22}$$

and the periodic complex exponential

$$x(t) = e^{j\omega_0 t}. \tag{3.23}$$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Associated with the signal in eq. (3.23) is the set of *harmonically related* complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{3.24}$$

Each of these signals has a fundamental frequency that is a multiple of ω_0, and therefore, each is periodic with period T (although for $|k| \geq 2$, the fundamental period of $\phi_k(t)$ is a fraction of T). Thus, a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \tag{3.25}$$

is also periodic with period T. In eq. (3.25), the term for $k = 0$ is a constant. The terms for $k = +1$ and $k = -1$ both have fundamental frequency equal to ω_0 and are collectively referred to as the *fundamental components* or the *first harmonic components.* The two terms for $k = +2$ and $k = -2$ are periodic with half the period (or, equivalently, twice the frequency) of the fundamental components and are referred to as the *second harmonic components.* More generally, the components for $k = +N$ and $k = -N$ are referred to as the Nth harmonic components.

The representation of a periodic signal in the form of eq. (3.25) is referred to as the *Fourier series* representation. Before developing the properties of this representation, let us consider an example.

Example 3.2

Consider a periodic signal $x(t)$, with fundamental frequency 2π, that is expressed in the form of eq. (3.25) as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}, \tag{3.26}$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

Rewriting eq. (3.26) and collecting each of the harmonic components which have the same fundamental frequency, we obtain

$$\begin{aligned} x(t) = 1 &+ \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) \\ &+ \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}). \end{aligned} \tag{3.27}$$

Equivalently, using Euler's relation, we can write $x(t)$ in the form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t. \tag{3.28}$$

In Figure 3.4, we illustrate graphically how the signal $x(t)$ is built up from its harmonic components.

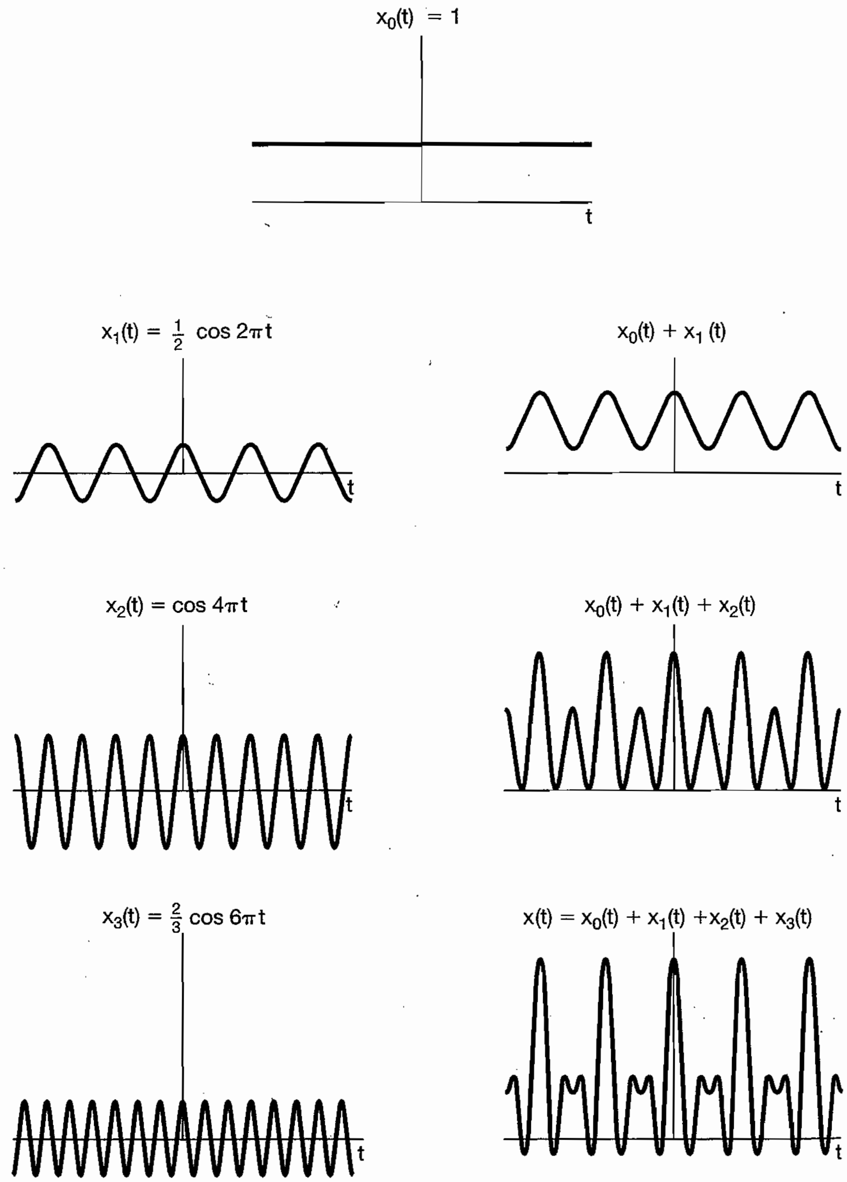

Figure 3.4 Construction of the signal $x(t)$ in Example 3.2 as a linear combination of harmonically related sinusoidal signals.

Equation (3.28) is an example of an alternative form for the Fourier series of real periodic signals. Specifically, suppose that $x(t)$ is real and can be represented in the form of eq. (3.25). Then, since $x^*(t) = x(t)$, we obtain

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}.$$

Replacing k by $-k$ in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t},$$

which, by comparison with eq. (3.25), requires that $a_k = a_{-k}^*$, or equivalently, that

$$a_k^* = a_{-k}. \tag{3.29}$$

Note that this is the case in Example 3.2, where the a_k's are in fact real and $a_k = a_{-k}$.

To derive the alternative forms of the Fourier series, we first rearrange the summation in eq. (3.25) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}].$$

Substituting a_k^* for a_{-k} from eq. (3.29), we obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}].$$

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{a_k e^{jk\omega_0 t}\}. \tag{3.30}$$

If a_k is expressed in polar form as

$$a_k = A_k e^{j\theta_k},$$

then eq. (3.30) becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{A_k e^{j(k\omega_0 t + \theta_k)}\}.$$

That is,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k). \tag{3.31}$$

Equation (3.31) is one commonly encountered form for the Fourier series of real periodic signals in continuous time. Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k,$$

where B_k and C_k are both real. With this expression for a_k, eq. (3.30) takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]. \tag{3.32}$$

In Example 3.2 the a_k's are all real, so that $a_k = A_k = B_k$, and therefore, both representations, eqs. (3.31) and (3.32), reduce to the same form, eq. (3.28).

Thus, for real periodic functions, the Fourier series in terms of complex exponentials, as given in eq. (3.25), is mathematically equivalent to either of the two forms in eqs. (3.31) and (3.32) that use trigonometric functions. Although the latter two are common forms for Fourier series,[7] the complex exponential form of eq. (3.25) is particularly convenient for our purposes, so we will use that form almost exclusively.

Equation (3.29) illustrates one of many properties associated with Fourier series. These properties are often quite useful in gaining insight and for computational purposes, and in Section 3.5 we collect together the most important of them. The derivation of several of them is considered in problems at the end of the chapter. In Section 4.3, we also will develop the majority of the properties within the broader context of the Fourier transform.

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Assuming that a given periodic signal can be represented with the series of eq. (3.25), we need a procedure for determining the coefficients a_k. Multiplying both sides of eq. (3.25) by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}. \tag{3.33}$$

Integrating both sides from 0 to $T = 2\pi/\omega_0$, we have

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt.$$

Here, T is the fundamental period of $x(t)$, and consequently, we are integrating over one period. Interchanging the order of integration and summation yields

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\,dt\right]. \tag{3.34}$$

The evaluation of the bracketed integral is straightforward. Rewriting this integral using Euler's formula, we obtain

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt. \tag{3.35}$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period $(T/|k-n|)$. Therefore, in eq. (3.35), we are integrating over an interval (of length T) that is an integral number of periods of these signals. Since the integral may be viewed as measuring the total area under the functions over the interval, we see that for $k \neq n$, both of the integrals on the right-hand side of eq. (3.35) are zero. For $k = n$, the integrand on the left-hand side of eq. (3.35) equals 1, and thus, the integral equals T. In sum, we then have

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

[7]In fact, in his original work, Fourier used the sine-cosine form of the Fourier series given in eq. (3.32).

and consequently, the right-hand side of eq. (3.34) reduces to Ta_n. Therefore,

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt, \tag{3.36}$$

which provides the equation for determining the coefficients. Furthermore, note that in evaluating eq. (3.35), the only fact that we used concerning the interval of integration was that we were integrating over an interval of length T, which is an integral number of periods of $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$. Therefore, we will obtain the same result if we integrate over any interval of length T. That is, if we denote integration over *any* interval of length T by $\int_T$, we have

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases},$$

and consequently,

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt. \tag{3.37}$$

To summarize, if $x(t)$ has a Fourier series representation [i.e., if it can be expressed as a linear combination of harmonically related complex exponentials in the form of eq. (3.25)], then the coefficients are given by eq. (3.37). This pair of equations, then, defines the Fourier series of a periodic continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \tag{3.38}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \tag{3.39}$$

Here, we have written equivalent expressions for the Fourier series in terms of the fundamnetal frequency ω_0 and the fundamental period T. Equation (3.38) is referred to as the *synthesis* equation and eq. (3.39) as the *analysis* equation. The set of coefficients $\{a_k\}$ are often called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$.[8] These complex coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component. The coefficient a_0 is the dc or constant component of $x(t)$ and is given by eq. (3.39) with $k = 0$. That is,

$$a_0 = \frac{1}{T}\int_T x(t)\,dt, \tag{3.40}$$

which is simply the average value of $x(t)$ over one period.

Equations (3.38) and (3.39) were known to both Euler and Lagrange in the middle of the 18th century. However, they discarded this line of analysis without having

[8]The term "spectral coefficient" is derived from problems such as the spectroscopic decomposition of light into spectral lines (i.e., into its elementary components at different frequencies). The intensity of any line in such a decomposition is a direct measure of the fraction of the total light energy at the frequency corresponding to the line.

examined the question of how large a class of periodic signals could, in fact, be represented in such a fashion. Before we turn to this question in the next section, let us illustrate the continuous-time Fourier series by means of a few examples.

Example 3.3

Consider the signal

$$x(t) = \sin \omega_0 t,$$

whose fundamental frequency is ω_0. One approach to determining the Fourier series coefficients for this signal is to apply eq. (3.39). For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express $\sin \omega_0 t$ as

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing the right-hand sides of this equation and eq. (3.38), we obtain

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$
$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example 3.4

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. As with Example 3.3, we can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j),$$
$$a_k = 0, \ |k| > 2.$$

In Figure 3.5, we show a bar graph of the magnitude and phase of a_k.

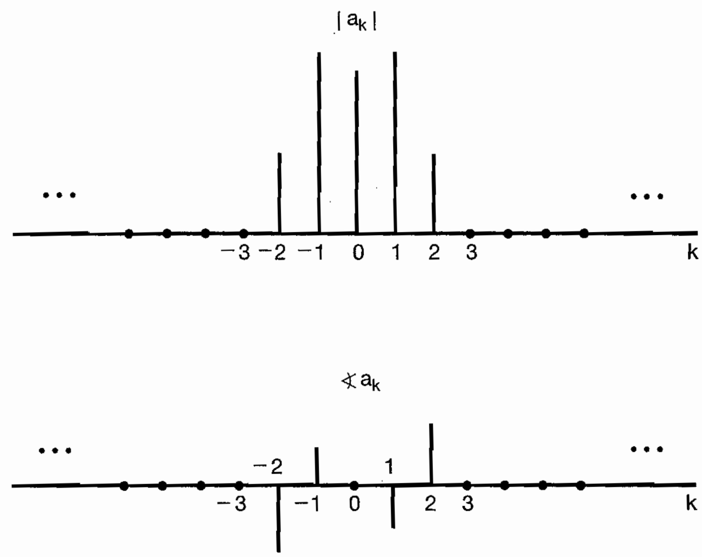

Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

Example 3.5

The periodic square wave, sketched in Figure 3.6 and defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \tag{3.41}$$

is a signal that we will encounter a number of times throughout this book. This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

To determine the Fourier series coefficients for $x(t)$, we use eq. (3.39). Because of the symmetry of $x(t)$ about $t = 0$, it is convenient to choose $-T/2 \le t < T/2$ as the

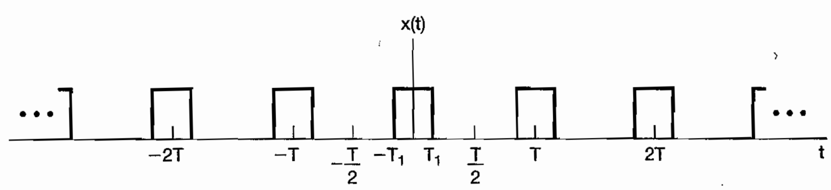

Figure 3.6 Periodic square wave.

interval over which the integration is performed, although any interval of length T is equally valid and thus will lead to the same result. Using these limits of integration and substituting from eq. (3.41), we have first, for $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}. \tag{3.42}$$

As mentioned previously, a_0 is interpreted to be the average value of $x(t)$, which in this case equals the fraction of each period during which $x(t) = 1$. For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

which we may rewrite as

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]. \tag{3.43}$$

Noting that the term in brackets is $\sin k\omega_0 T_1$, we can express the coefficients a_k as

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0, \tag{3.44}$$

where we have used the fact that $\omega_0 T = 2\pi$.

Figure 3.7 is a bar graph of the Fourier series coefficients for this example. In particular, the coefficients are plotted for a fixed value of T_1 and several values of T. For this specific example, the Fourier coefficients are real, and consequently, they can be depicted graphically with only a single graph. More generally, of course, the Fourier coefficients are complex, so that two graphs, corresponding to the real and imaginary parts, or magnitude and phase, of each coefficient, would be required. For $T = 4T_1$, $x(t)$ is a square wave that is unity for half the period and zero for half the period. In this case, $\omega_0 T_1 = \pi/2$, and from eq. (3.44),

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \quad k \neq 0, \tag{3.45}$$

while

$$a_0 = \frac{1}{2}. \tag{3.46}$$

From eq. (3.45), $a_k = 0$ for k even and nonzero. Also, $\sin(\pi k/2)$ alternates between ± 1 for successive odd values of k. Therefore,

$$\begin{aligned} a_1 &= a_{-1} = \frac{1}{\pi}, \\ a_3 &= a_{-3} = -\frac{1}{3\pi}, \\ a_5 &= a_{-5} = \frac{1}{5\pi}, \\ &\vdots \end{aligned}$$

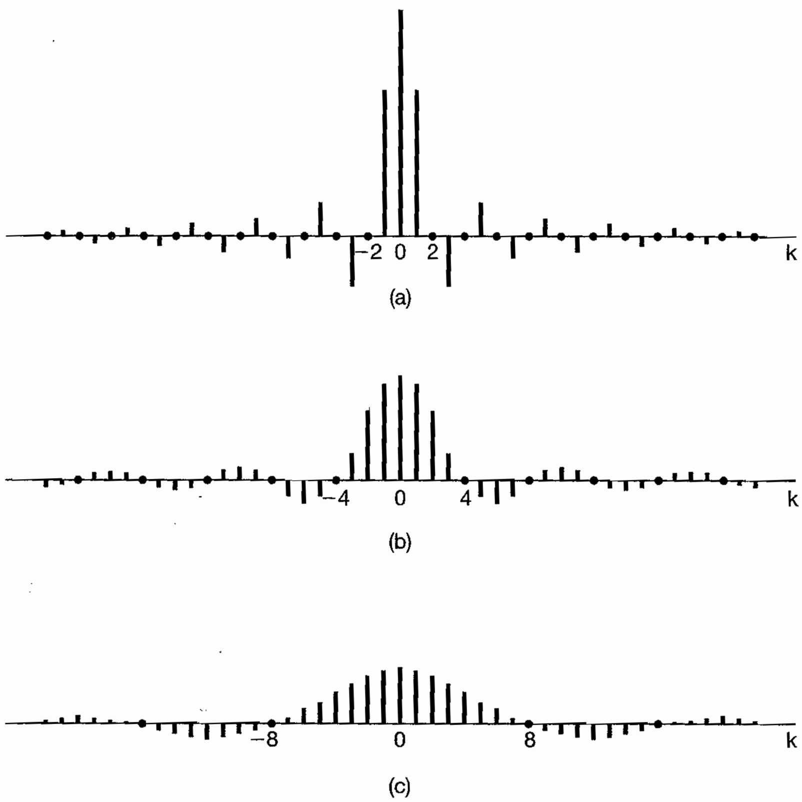

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

3.4 CONVERGENCE OF THE FOURIER SERIES

Although Euler and Lagrange would have been happy with the results of Examples 3.3 and 3.4, they would have objected to Example 3.5, since $x(t)$ is discontinuous while each of its harmonic components is continuous. Fourier, on the other hand, considered the same example and maintained that the Fourier series representation of the square wave is valid. In fact, Fourier maintained that *any* periodic signal could be represented by a Fourier series. Although this is not quite true, it *is* true that Fourier series can be used to represent an extremely large class of periodic signals, including the square wave and all other periodic signals with which we will be concerned in this book and which are of interest in practice.

To gain an understanding of the square-wave example and, more generally, of the question of the validity of Fourier series representations, let us examine the problem of approximating a given periodic signal $x(t)$ by a linear combination of a finite number of harmonically related complex exponentials—that is, by a finite series of the form poses. Specifically, since the signals differ only at isolated points, the integrals of both signals over any interval *are* identical. For this reason, the two signals behave identically under convolution and consequently are identical from the standpoint of the analysis of LTI systems.

To gain some additional understanding of *how* the Fourier series converges for a periodic signal with discontinuities, let us return to the example of a square wave. In particular, in 1898,[10] an American physicist, Albert Michelson, constructed a harmonic analyzer, a device that, for any periodic signal $x(t)$, would compute the truncated Fourier series approximation of eq. (3.52) for values of N up to 80. Michelson tested his device on many functions, with the expected result that $x_N(t)$ looked very much like $x(t)$. However, when he tried the square wave, he obtained an important and, to him, very surprising result. Michelson was concerned about the behavior he observed and thought that his device might have had a defect. He wrote about the problem to the famous mathematical physicist Josiah Gibbs, who investigated it and reported his explanation in 1899.

What Michelson had observed is illustrated in Figure 3.9, where we have shown $x_N(t)$ for several values of N for $x(t)$, a symmetric square wave ($T = 4T_1$). In each case, the partial sum is superimposed on the original square wave. Since the square wave satisfies the Dirichlet conditions, the limit as $N \to \infty$ of $x_N(t)$ at the discontinuities should be the average value of the discontinuity. We see from the figure that this is in fact the case, since for any N, $x_N(t)$ has exactly that value at the discontinuities. Furthermore, for any other value of t, say, $t = t_1$, we are guaranteed that

$$\lim_{N\to\infty} x_N(t_1) = x(t_1).$$

Therefore, the squared error in the Fourier series representation of the square wave has zero area, as in eqs. (3.53) and (3.54).

For this example, the interesting effect that Michelson observed is that the behavior of the partial sum in the vicinity of the discontinuity exhibits ripples and that the peak amplitude of these ripples does not seem to decrease with increasing N. Gibbs showed that these are in fact the case. Specifically, for a discontinuity of unity height, the partial sum exhibits a maximum value of 1.09 (i.e., an overshoot of 9% of the height of the discontinuity), no matter how large N becomes. One must be careful to interpret this correctly, however. As stated before, for any *fixed* value of t, say, $t = t_1$, the partial sums will converge to the correct value, and at the discontinuity they will converge to one-half the sum of the values of the signal on either side of the discontinuity. However, the closer t_1 is chosen to the point of discontinuity, the larger N must be in order to reduce the error below a specified amount. Thus, as N increases, the ripples in the partial sums become compressed toward the discontinuity, but for *any* finite value of N, the peak amplitude of the ripples remains constant. This behavior has come to be known as the *Gibbs phenomenon*. The implication is that the truncated Fourier series approximation $x_N(t)$ of a discontinuous signal $x(t)$ will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities. If such an approximation is used in practice, a large enough value of N should be chosen so as to guarantee that the total energy in these ripples is insignificant. In the limit, of course, we know that the energy in the approximation error vanishes and that the Fourier series representation of a discontinuous signal such as the square wave converges.

[10]The historical information used in this example is taken from the book by Lanczos referenced in footnote 1 of this chapter.

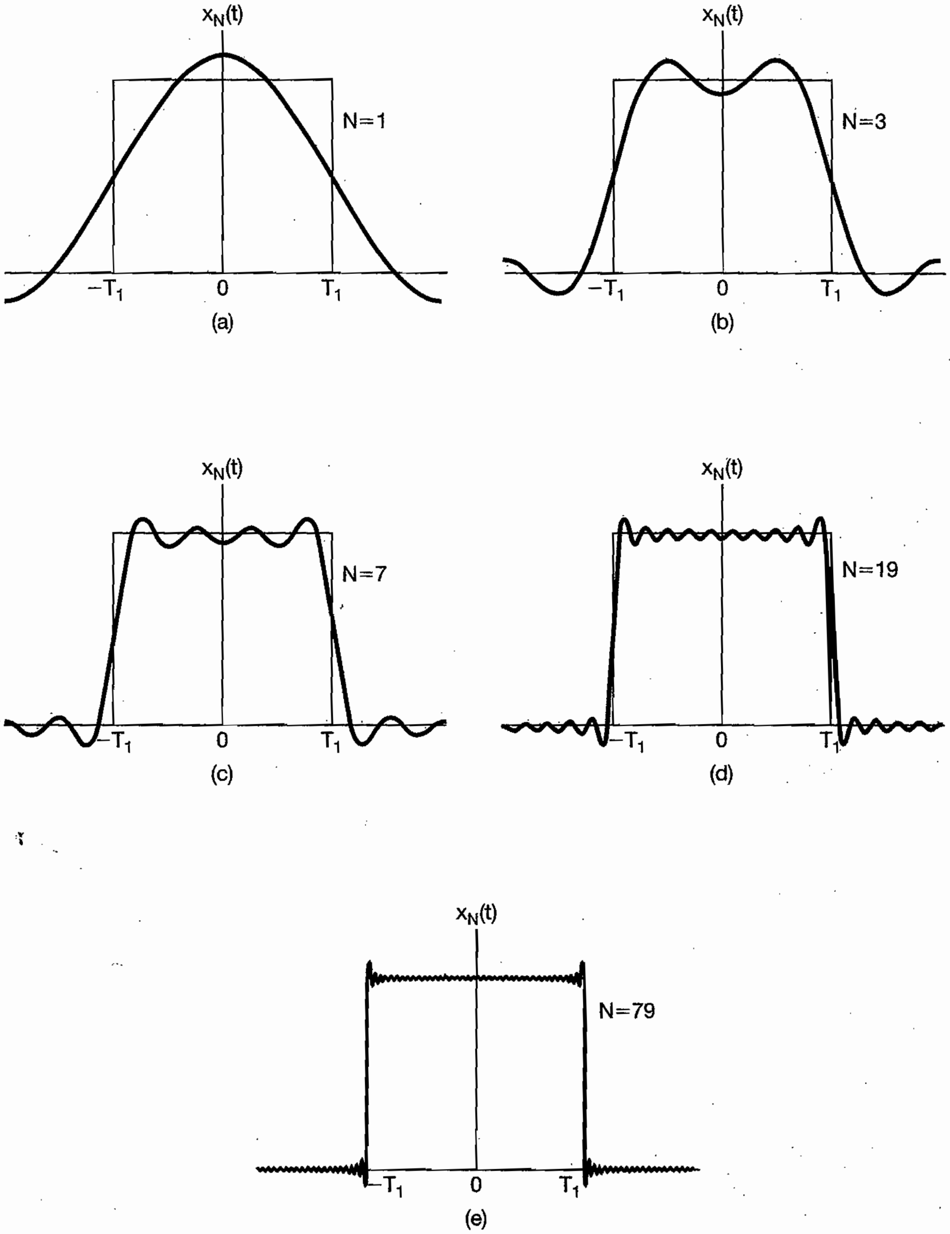

Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

3.5 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

As mentioned earlier, Fourier series representations possess a number of important properties that are useful for developing conceptual insights into such representations, and they can also help to reduce the complexity of the evaluation of the Fourier series of many signals. In Table 3.1 we have summarized these properties, several of which are considered in the problems at the end of this chapter. In Chapter 4, in which we develop the Fourier transform, we will see that most of these properties can be deduced from corresponding properties of the continuous-time Fourier transform. Consequently we limit ourselves here to the discussion of several of these properties to illustrate how they may be derived, interpreted, and used.

Throughout the following discussion of selected properties from Table 3.1, we will find it convenient to use a shorthand notation to indicate the relationship between a periodic signal and its Fourier series coefficients. Specifically, suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients of $x(t)$ are denoted by a_k, we will use the notation

$$x(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} a_k$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

3.5.1 Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and which have Fourier series coefficients denoted by a_k and b_k, respectively. That is,

$$x(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} b_k.$$

Since $x(t)$ and $y(t)$ have the same period T, it easily follows that any linear combination of the two signals will also be periodic with period T. Furthermore, the Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination of the Fourier series coefficients for $x(t)$ and $y(t)$. That is,

$$z(t) = Ax(t) + By(t) \stackrel{\mathfrak{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k. \tag{3.58}$$

The proof of this follows directly from the application of eq. (3.39). We also note that the linearity property is easily extended to a linear combination of an arbitrary number of signals with period T.

3.5.2 Time Shifting

When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved. The Fourier series coefficients b_k of the resulting signal $y(t) = x(t-t_0)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt. \tag{3.59}$$

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration T, we obtain

$$\begin{aligned}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau &= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau \\ &= e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k,\end{aligned} \tag{3.60}$$

where a_k is the kth Fourier series coefficient of $x(t)$. That is, if

$$x(t) \overset{\mathfrak{FS}}{\longleftrightarrow} a_k,$$

then

$$x(t - t_0) \overset{\mathfrak{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k.$$

One consequence of this property is that, when a periodic signal is shifted in time, the *magnitudes* of its Fourier series coefficients remain unaltered. That is, $|b_k| = |a_k|$.

3.5.3 Time Reversal

The period T of a periodic signal $x(t)$ also remains unchanged when the signal undergoes time reversal. To determine the Fourier series coefficients of $y(t) = x(-t)$, let us consider the effect of time reversal on the synthesis equation (3.38):

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{3.61}$$

Making the substitution $k = -m$, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}. \tag{3.62}$$

We observe that the right-hand side of this equation has the form of a Fourier series synthesis equation for $x(-t)$, where the Fourier series coefficients b_k are

$$b_k = a_{-k}. \tag{3.63}$$

That is, if

$$x(t) \overset{\mathfrak{FS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\mathfrak{FS}}{\longleftrightarrow} a_{-k}.$$

In other words time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients. An interesting consequence of the time-reversal property is that if $x(t)$ is even—that is, if $x(-t) = x(t)$—then its Fourier series coefficients are also even—i.e., $a_{-k} = a_k$. Similarly, if $x(t)$ is odd, so that $x(-t) = -x(t)$, then so are its Fourier series coefficients—i.e., $a_{-k} = -a_k$.

3.5.4 Time Scaling

Time scaling is an operation that in general changes the period of the underlying signal. Specifically, if $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$. Since the time-scaling operation applies directly to each of the harmonic components of $x(t)$, we may easily conclude that the Fourier coefficients for each of those components remain the same. That is, if $x(t)$ has the Fourier series representation in eq. (3.38), then

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

is the Fourier series representation of $x(\alpha t)$. We emphasize that, while the Fourier coefficients have not changed, the Fourier series representation *has* changed because of the change in the fundamental frequency.

3.5.5 Multiplication

Suppose that $x(t)$ and $y(t)$ are both periodic with period T and that

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for $x(t)$ and $y(t)$. The result is

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{3.64}$$

One way to derive this relationship (see Problem 3.46) is to multiply the Fourier series representations of $x(t)$ and $y(t)$ and to note that the kth harmonic component in the product will have a coefficient which is the sum of terms of the form $a_l b_{k-l}$. Observe that the sum on the right-hand side of eq. (3.64) may be interpreted as the discrete-time convolution of the sequence representing the Fourier coefficients of $x(t)$ and the sequence representing the Fourier coefficients of $y(t)$.

3.5.6 Conjugation and Conjugate Symmetry

Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation *and* time reversal on the corresponding Fourier series coefficients. That is, if

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}. \tag{3.65}$$

This property is easily proved by applying complex conjugation to both sides of eq. (3.38) and replacing the summation variable k by its negative.

Some interesting consequences of this property may be derived for $x(t)$ real—that is, when $x(t) = x^*(t)$. In particular, in this case, we see from eq. (3.65) that the Fourier series coefficients will be *conjugate symmetric*, i.e.,

$$a_{-k} = a_k^*, \tag{3.66}$$

as we previously saw in eq. (3.29). This in turn implies various symmetry properties (listed in Table 3.1) for the magnitudes, phases, real parts, and imaginary parts of the Fourier series coefficients of real signals. For example, from eq. (3.66), we see that if $x(t)$ is real, then a_0 is real and

$$|a_k| = |a_{-k}|.$$

Also, if $x(t)$ is real and even, then, from Section 3.5.3, $a_k = a_{-k}$. However, from eq. (3.66) we see that $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is, if $x(t)$ is real and even, then so are its Fourier series coefficients. Similarly, it can be shown that if $x(t)$ is real and odd, then its Fourier series coefficients are purely imaginary and odd. Thus, for example, $a_0 = 0$ if $x(t)$ is real and odd. This and the other symmetry properties of the Fourier series are examined further in Problem 3.42.

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

As shown in Problem 3.46, Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2, \tag{3.67}$$

where the a_k are the Fourier series coefficients of $x(t)$ and T is the period of the signal.

Note that the left-hand side of eq. (3.67) is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$. Also,

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2, \tag{3.68}$$

so that $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$. Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

3.5.8 Summary of Properties of the Continuous-Time Fourier Series

In Table 3.1, we summarize these and other important properties of continuous-time Fourier series.

3.5.9 Examples

Fourier series properties, such as those listed in Table 3.1, may be used to circumvent some of the algebra involved in determining the Fourier coefficients of a given signal. In the next

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \quad [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} \quad [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \tag{3.69}$$

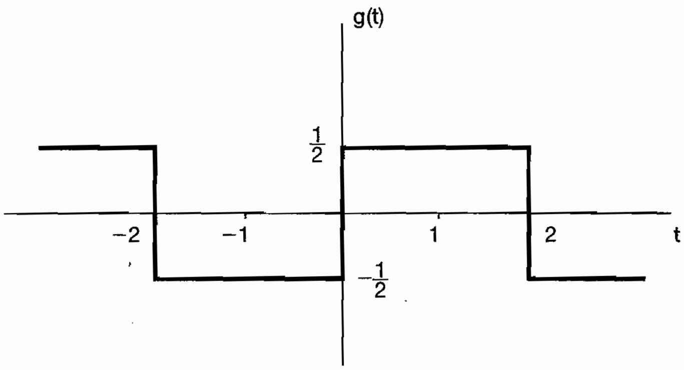

Figure 3.10 Periodic signal for Example 3.6.

The time-shift property in Table 3.1 indicates that, if the Fourier Series coefficients of $x(t)$ are denoted by a_k, the Fourier coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2}. \tag{3.70}$$

The Fourier coefficients of the *dc offset* in $g(t)$—i.e., the term $-1/2$ on the right-hand side of eq. (3.69)—are given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}. \tag{3.71}$$

Applying the linearity property in Table 3.1, we conclude that the coefficients for $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

where each a_k may now be replaced by the corresponding expression from eqs. (3.45) and (3.46), yielding

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}. \tag{3.72}$$

Example 3.7

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$ shown in Figure 3.11. The derivative of this signal is the signal $g(t)$ in Exam-

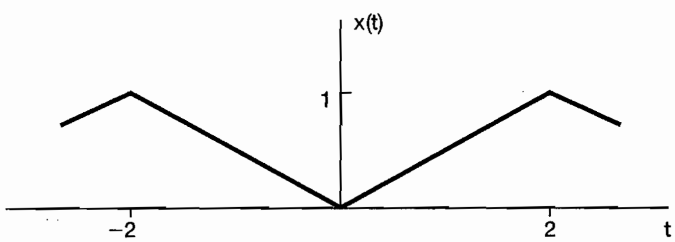

Figure 3.11 Triangular wave signal in Example 3.7.

ple 3.6. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k, we see that the differentiation property in Table 3.1 indicates that

$$d_k = jk(\pi/2)e_k. \tag{3.73}$$

This equation can be used to express e_k in terms of d_k, except when $k = 0$. Specifically, from eq. (3.72),

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0. \tag{3.74}$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

Example 3.8

Let us examine some properties of the Fourier series representation of a periodic train of impulses, or *impulse train.* This signal and its representation in terms of complex exponentials will play an important role when we discuss the topic of sampling in Chapter 7. The impulse train with period T may be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT); \tag{3.75}$$

it is illustrated in Figure 3.12(a). To determine the Fourier series coefficients a_k, we use eq. (3.39) and select the interval of integration to be $-T/2 \leq t \leq T/2$, avoiding the placement of impulses at the integration limits. Within this interval, $x(t)$ is the same as $\delta(t)$, and it follows that

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk2\pi t/T}dt = \frac{1}{T}. \tag{3.76}$$

In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index k). This is to be expected, since, according to Table 3.1, any real and even signal (such as our impulse train) should have real and even Fourier coefficients.

The impulse train also has a straightforward relationship to square-wave signals such as $g(t)$ in Figure 3.6, which we repeat in Figure 3.12(b). The derivative of $g(t)$ is the signal $q(t)$ illustrated in Figure 3.12(c). We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t + T_1) - x(t - T_1). \tag{3.77}$$

Using the properties of Fourier series, we can now compute the Fourier series coefficients of $q(t)$ and $g(t)$ without any further direct evaluation of the Fourier series analysis equation. First, from the time-shifting and linearity properties, we see from eq. (3.77) that the Fourier series coefficients b_k of $q(t)$ may be expressed in terms of the Fourier series coefficients a_k of $x(t)$; that is,

$$b_k = e^{jk\omega_0 T_1}a_k - e^{-jk\omega_0 T_1}a_k,$$

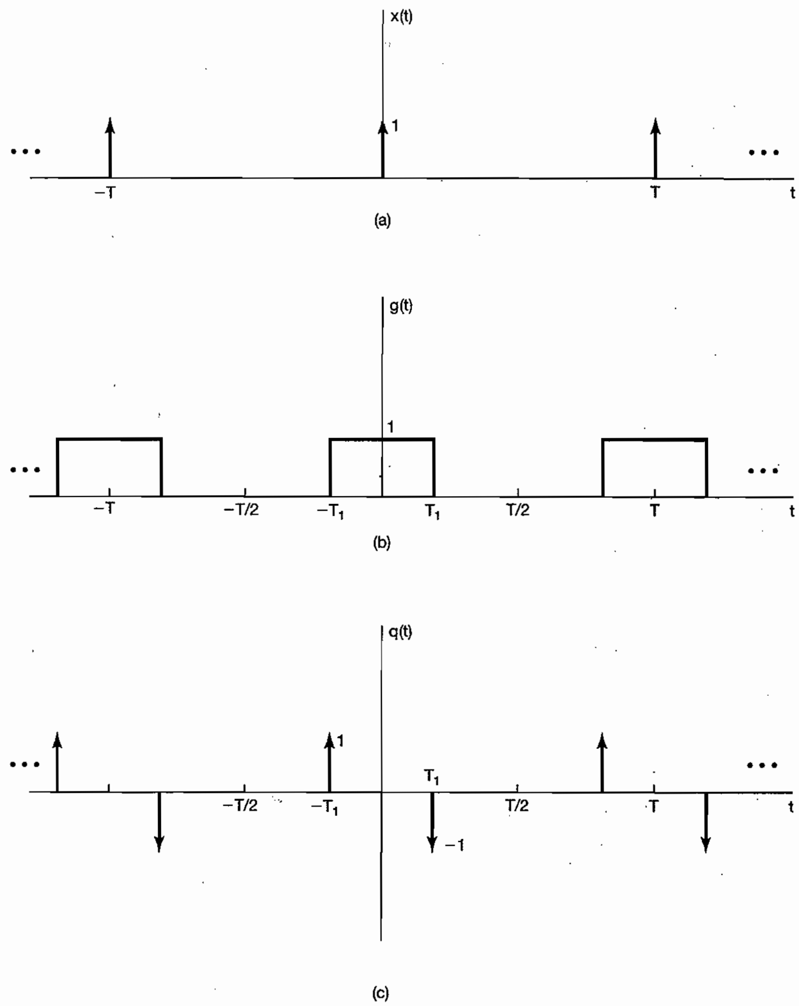

Figure 3.12 (a) Periodic train of impulses; (b) periodic square wave; (c) derivative of the periodic square wave in (b).

where $\omega_0 = 2\pi/T$. Using eq. (3.76), we then have

$$b_k = \frac{1}{T}[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] = \frac{2j\sin(k\omega_0 T_1)}{T}.$$

Finally, since $q(t)$ is the derivative of $g(t)$, we can use the differentiation property in Table 3.1 to write

$$b_k = jk\omega_0 c_k, \tag{3.78}$$

where the c_k are the Fourier series coefficients of $g(t)$. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0, \tag{3.79}$$

where we have used the fact that $\omega_0 T = 2\pi$. Note that eq. (3.79) is valid for $k \neq 0$, since we cannot solve for c_0 from eq. (3.78) with $k = 0$. However, since c_0 is just the average value of $g(t)$ over one period, we can determine it by inspection from Figure 3.12(b):

$$c_0 = \frac{2T_1}{T}. \tag{3.80}$$

Eqs. (3.80) and (3.79) are identical to eqs. (3.42) and (3.44), respectively, for the Fourier series coefficients of the square wave derived in Example 3.5.

The next example is chosen to illustrate the use of many of the properties in Table 3.1.

Example 3.9

Suppose we are given the following facts about a signal $x(t)$:

1. $x(t)$ is a real signal.
2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2}a_{-k}$ is odd.
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

Let us show that this information is sufficient to determine the signal $x(t)$ to within a sign factor. According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients a_k: a_0, a_1, and a_{-1}. Then, since $x(t)$ has fundamental frequency $\omega_0 = 2\pi/4 = \pi/2$, it follows that

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1}e^{-j\pi t/2}.$$

Since $x(t)$ is real (Fact 1), we can use the symmetry properties in Table 3.1 to conclude that a_0 is real and $a_1 = a*_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{j\pi t/2} + (a_1 e^{j\pi t/2})^* = a_0 + 2\Re e\{a_1 e^{j\pi t/2}\}. \tag{3.81}$$

Let us now determine the signal corresponding to the Fourier coefficients b_k given in Fact 4. Using the time-reversal property from Table 3.1, we note that a_{-k} corresponds to the signal $x(-t)$. Also, the time-shift property in the table indicates that multiplication of the kth Fourier coefficient by $e^{-jk\pi/2} = e^{-jk\omega_0}$ corresponds to the underlying signal being shifted by 1 to the right (i.e., having t replaced by $t-1$). We conclude that the coefficients b_k correspond to the signal $x(-(t-1)) = x(-t+1)$, which, according to Fact 4, must be odd. Since $x(t)$ is real, $x(-t+1)$ must also be real. From Table 3.1, it then follows that the Fourier coefficients of $x(-t+1)$ must be purely imaginary and odd. Thus, $b_0 = 0$ and $b_{-1} = -b_1$. Since time-reversal and time-shift operations cannot change the average power per period, Fact 5 holds even if $x(t)$ is replaced by $x(-t+1)$. That is,

$$\frac{1}{4}\int_4 |x(-t+1)|^2 dt = 1/2. \tag{3.82}$$