Example 1.1

Given the signal $x(t)$ shown in Figure 1.13(a), the signal $x(t + 1)$ corresponds to an advance (shift to the left) by one unit along the t axis as illustrated in Figure 1.13(b). Specifically, we note that the value of $x(t)$ at $t = t_0$ occurs in $x(t + 1)$ at $t = t_0 - 1$. For

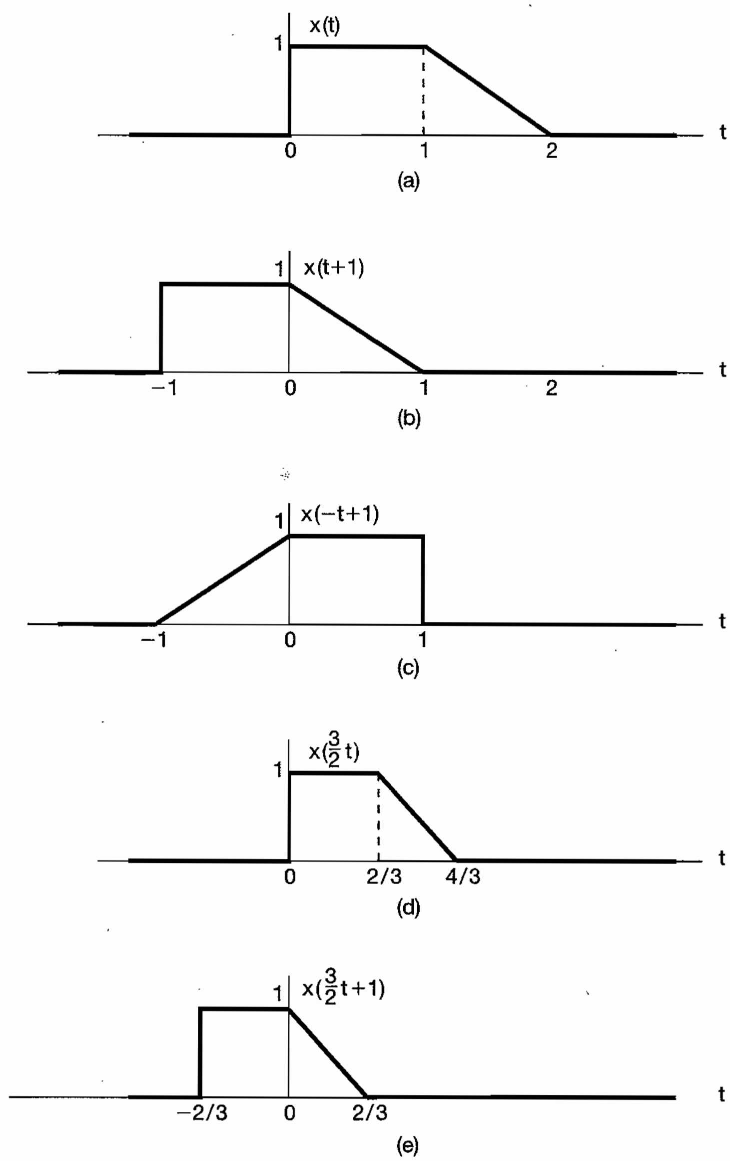


Figure 1.13 (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted signal $x(t + 1)$; (c) the signal $x(-t + 1)$ obtained by a time shift and a time reversal; (d) the time-scaled signal $x(\frac{3}{2}t)$; and (e) the signal $x(\frac{3}{2}t + 1)$ obtained by time-shifting and scaling.

example, the value of $x(t)$ at $t = 1$ is found in $x(t + 1)$ at $t = 1 - 1 = 0$. Also, since $x(t)$ is zero for $t < 0$, we have $x(t + 1)$ zero for $t < -1$. Similarly, since $x(t)$ is zero for $t > 2$, $x(t + 1)$ is zero for $t > 1$.

Let us also consider the signal $x(-t + 1)$, which may be obtained by replacing t with $-t$ in $x(t + 1)$. That is, $x(-t + 1)$ is the time reversed version of $x(t + 1)$. Thus, $x(-t + 1)$ may be obtained graphically by reflecting $x(t + 1)$ about the t axis as shown in Figure 1.13(c).

Example 1.2

Given the signal $x(t)$, shown in Figure 1.13(a), the signal $x(\frac{3}{2}t)$ corresponds to a linear compression of $x(t)$ by a factor of $\frac{2}{3}$ as illustrated in Figure 1.13(d). Specifically we note that the value of $x(t)$ at $t = t_0$ occurs in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}t_0$. For example, the value of $x(t)$ at $t = 1$ is found in $x(\frac{3}{2}t)$ at $t = \frac{2}{3}(1) = \frac{2}{3}$. Also, since $x(t)$ is zero for $t < 0$, we have $x(\frac{3}{2}t)$ zero for $t < 0$. Similarly, since $x(t)$ is zero for $t > 2$, $x(\frac{3}{2}t)$ is zero for $t > \frac{4}{3}$.

Example 1.3

Suppose that we would like to determine the effect of transforming the independent variable of a given signal, $x(t)$, to obtain a signal of the form $x(\alpha t + \beta)$, where α and β are given numbers. A systematic approach to doing this is to first delay or advance $x(t)$ in accordance with the value of β, and then to perform time scaling and/or time reversal on the resulting signal in accordance with the value of α. The delayed or advanced signal is linearly stretched if $|\alpha| < 1$, linearly compressed if $|\alpha| > 1$, and reversed in time if $\alpha < 0$.

To illustrate this approach, let us show how $x(\frac{3}{2}t + 1)$ may be determined for the signal $x(t)$ shown in Figure 1.13(a). Since $\beta = 1$, we first advance (shift to the left) $x(t)$ by 1 as shown in Figure 1.13(b). Since $|\alpha| = \frac{3}{2}$, we may linearly compress the shifted signal of Figure 1.13(b) by a factor of $\frac{2}{3}$ to obtain the signal shown in Figure 1.13(e).

In addition to their use in representing physical phenomena such as the time shift in a sonar signal and the speeding up or reversal of an audiotape, transformations of the independent variable are extremely useful in signal and system analysis. In Section 1.6 and in Chapter 2, we will use transformations of the independent variable to introduce and analyze the properties of systems. These transformations are also important in defining and examining some important properties of signals.

1.2.2 Periodic Signals

An important class of signals that we will encounter frequently throughout this book is the class of *periodic* signals. A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t + T) \tag{1.11}$$

for all values of t. In other words, a periodic signal has the property that it is unchanged by a time shift of T. In this case, we say that $x(t)$ is *periodic with period T*. Periodic continuous-time signals arise in a variety of contexts. For example, as illustrated in Problem 2.61, the natural response of systems in which energy is conserved, such as ideal LC circuits without resistive energy dissipation and ideal mechanical systems without frictional losses, are periodic and, in fact, are composed of some of the basic periodic signals that we will introduce in Section 1.3.

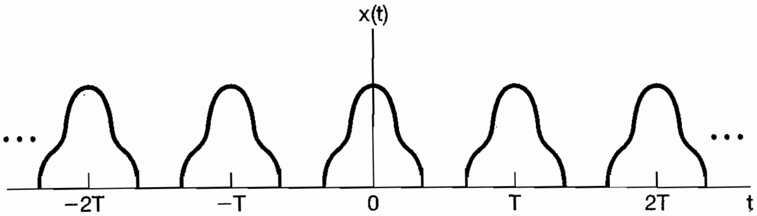


Figure 1.14 A continuous-time periodic signal.

An example of a periodic continuous-time signal is given in Figure 1.14. From the figure or from eq. (1.11), we can readily deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t + mT)$ for all t and for any integer m. Thus, $x(t)$ is also periodic with period $2T, 3T, 4T, \ldots$. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which eq. (1.11) holds. This definition of the fundamental period works, except if $x(t)$ is a constant. In this case the fundamental period is undefined, since $x(t)$ is periodic for *any* choice of T (so there is no smallest positive value). A signal $x(t)$ that is not periodic will be referred to as an *aperiodic* signal.

Periodic signals are defined analogously in discrete time. Specifically, a discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time shift of N, i.e., if

$$x[n] = x[n + N] \tag{1.12}$$

for all values of n. If eq. (1.12) holds, then $x[n]$ is also periodic with period $2N, 3N, \ldots$. The *fundamental period* N_0 is the smallest positive value of N for which eq. (1.12) holds. An example of a discrete-time periodic signal with fundamental period $N_0 = 3$ is shown in Figure 1.15.

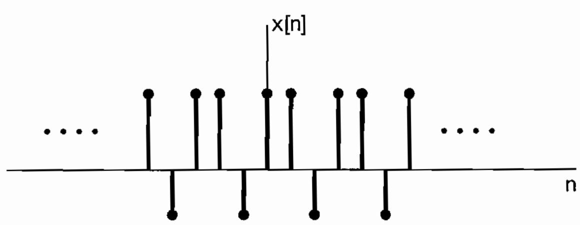


Figure 1.15 A discrete-time periodic signal with fundamental period $N_0 = 3$.

Example 1.4

Let us illustrate the type of problem solving that may be required in determining whether or not a given signal is periodic. The signal whose periodicity we wish to check is given by

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}. \tag{1.13}$$

From trigonometry, we know that $\cos(t + 2\pi) = \cos(t)$ and $\sin(t + 2\pi) = \sin(t)$. Thus, considering $t > 0$ and $t < 0$ separately, we see that $x(t)$ does repeat itself over every interval of length 2π. However, as illustrated in Figure 1.16, $x(t)$ also has a discontinuity at the time origin that does not recur at any other time. Since every feature in the shape of a periodic signal *must* recur periodically, we conclude that the signal $x(t)$ is not periodic.

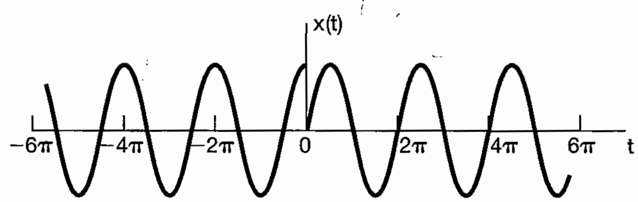


Figure 1.16 The signal $x(t)$ considered in Example 1.4.

1.2.3 Even and Odd Signals

Another set of useful properties of signals relates to their symmetry under time reversal. A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin. In continuous time a signal is even if

$$x(-t) = x(t), \tag{1.14}$$

while a discrete-time signal is even if

$$x[-n] = x[n]. \tag{1.15}$$

A signal is referred to as *odd* if

$$x(-t) = -x(t), \tag{1.16}$$

$$x[-n] = -x[n]. \tag{1.17}$$

An odd signal must necessarily be 0 at $t = 0$ or $n = 0$, since eqs. (1.16) and (1.17) require that $x(0) = -x(0)$ and $x[0] = -x[0]$. Examples of even and odd continuous-time signals are shown in Figure 1.17.

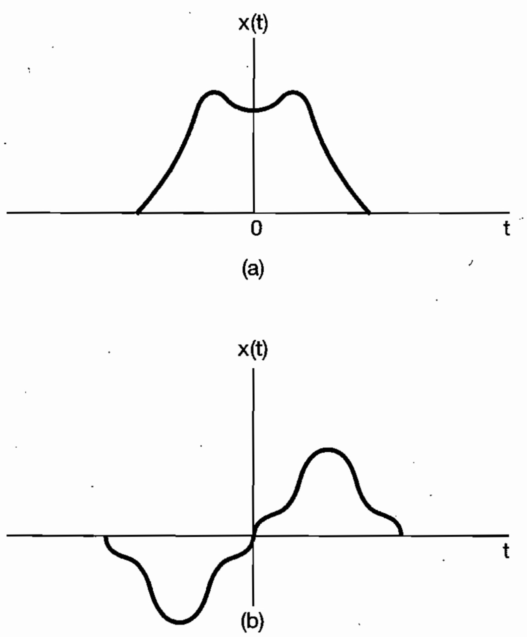


Figure 1.17 (a) An even continuous-time signal; (b) an odd continuous-time signal.

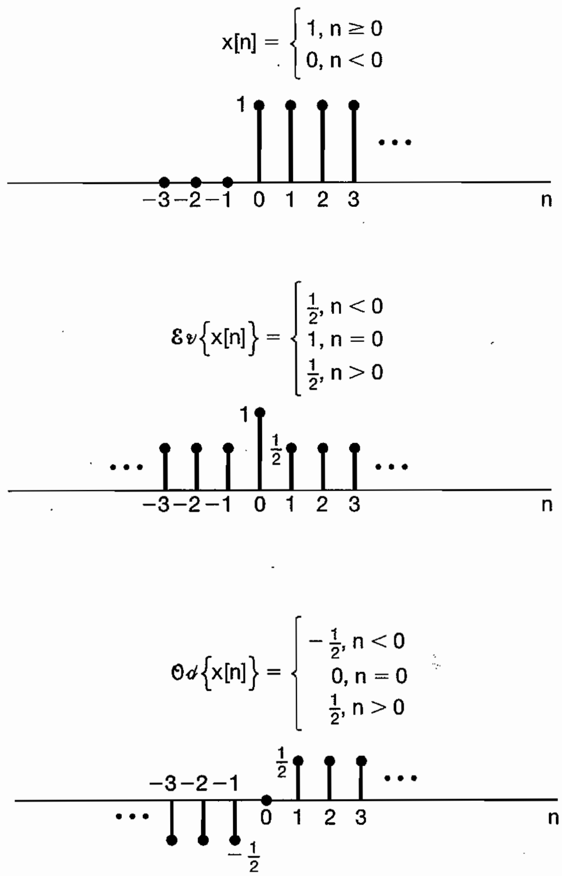


Figure 1.18 Example of the even-odd decomposition of a discrete-time signal.

An important fact is that any signal can be broken into a sum of two signals, one of which is even and one of which is odd. To see this, consider the signal

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)], \tag{1.18}$$

which is referred to as the *even part* of $x(t)$. Similarly, the *odd part* of $x(t)$ is given by

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]. \tag{1.19}$$

It is a simple exercise to check that the even part is in fact even, that the odd part is odd, and that $x(t)$ is the sum of the two. Exactly analogous definitions hold in the discrete-time case. An example of the even-odd decomposition of a discrete-time signal is given in Figure 1.18.

1.3 EXPONENTIAL AND SINUSOIDAL SIGNALS

In this section and the next, we introduce several basic continuous-time and discrete-time signals. Not only do these signals occur frequently, but they also serve as basic building blocks from which we can construct many other signals.

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time *complex exponential signal* is of the form

$$x(t) = Ce^{at}, \tag{1.20}$$

where C and a are, in general, complex numbers. Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

Real Exponential Signals

As illustrated in Figure 1.19, if C and a are real [in which case $x(t)$ is called a *real exponential*], there are basically two types of behavior. If a is positive, then as t increases $x(t)$ is a growing exponential, a form that is used in describing many different physical processes, including chain reactions in atomic explosions and complex chemical reactions. If a is negative, then $x(t)$ is a decaying exponential, a signal that is also used to describe a wide variety of phenomena, including the process of radioactive decay and the responses of RC circuits and damped mechanical systems. In particular, as shown in Problems 2.61 and 2.62, the natural responses of the circuit in Figure 1.1 and the automobile in Figure 1.2 are decaying exponentials. Also, we note that for $a = 0$, $x(t)$ is constant.

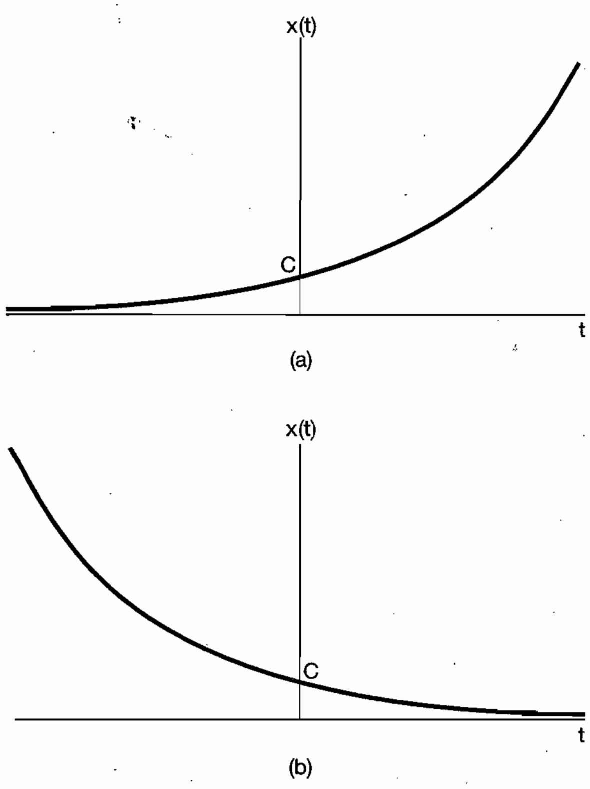


Figure 1.19 Continuous-time real exponential $x(t) = Ce^{at}$: (a) $a > 0$; (b) $a < 0$.

Periodic Complex Exponential and Sinusoidal Signals

A second important class of complex exponentials is obtained by constraining a to be purely imaginary. Specifically, consider

$$x(t) = e^{j\omega_0 t}. \tag{1.21}$$

An important property of this signal is that it is periodic. To verify this, we recall from eq. (1.11) that $x(t)$ will be periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}. \tag{1.22}$$

Or, since

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T},$$

it follows that for periodicity, we must have

$$e^{j\omega_0 T} = 1. \tag{1.23}$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$—that is, the smallest positive value of T for which eq. (1.23) holds—is

$$T_0 = \frac{2\pi}{|\omega_0|}. \tag{1.24}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \phi), \tag{1.25}$$

as illustrated in Figure 1.20. With seconds as the units of t, the units of ϕ and ω_0 are radians and radians per second, respectively. It is also common to write $\omega_0 = 2\pi f_0$, where f_0 has the units of cycles per second, or hertz (Hz). Like the complex exponential signal, the sinusoidal signal is periodic with fundamental period T_0 given by eq. (1.24). Sinusoidal and

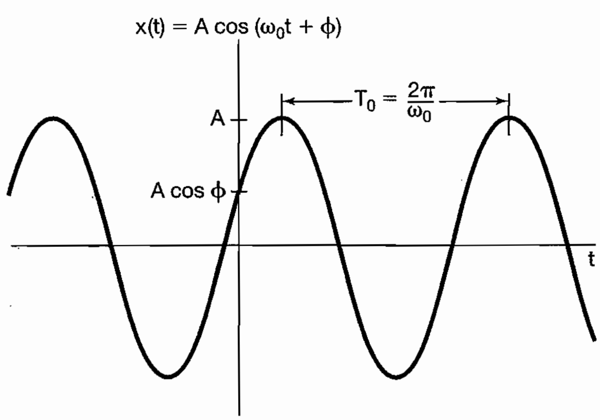


Figure 1.20 Continuous-time sinusoidal signal.

complex exponential signals are also used to describe the characteristics of many physical processes—in particular, physical systems in which energy is conserved. For example, as shown in Problem 2.61, the natural response of an LC circuit is sinusoidal, as is the simple harmonic motion of a mechanical system consisting of a mass connected by a spring to a stationary support. The acoustic pressure variations corresponding to a single musical tone are also sinusoidal.

By using Euler's relation,[2] the complex exponential in eq. (1.21) can be written in terms of sinusoidal signals with the same fundamental period:

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t. \tag{1.26}$$

Similarly, the sinusoidal signal of eq. (1.25) can be written in terms of periodic complex exponentials, again with the same fundamental period:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}. \tag{1.27}$$

Note that the two exponentials in eq. (1.27) have complex amplitudes. Alternatively, we can express a sinusoid in terms of a complex exponential signal as

$$A\cos(\omega_0 t + \phi) = A\,\mathcal{R}e\{e^{j(\omega_0 t+\phi)}\}, \tag{1.28}$$

where, if c is a complex number, $\mathcal{R}e\{c\}$ denotes its real part. We will also use the notation $\mathcal{I}m\{c\}$ for the imaginary part of c, so that, for example,

$$A\sin(\omega_0 t + \phi) = A\,\mathcal{I}m\{e^{j(\omega_0 t+\phi)}\}. \tag{1.29}$$

From eq. (1.24), we see that the fundamental period T_0 of a continuous-time sinusoidal signal or a periodic complex exponential is inversely proportional to $|\omega_0|$, which we will refer to as the *fundamental frequency*. From Figure 1.21, we see graphically what this means. If we decrease the magnitude of ω_0, we slow down the rate of oscillation and therefore increase the period. Exactly the opposite effects occur if we increase the magnitude of ω_0. Consider now the case $\omega_0 = 0$. In this case, as we mentioned earlier, $x(t)$ is constant and therefore is periodic with period T for any positive value of T. Thus, the fundamental period of a constant signal is undefined. On the other hand, there is no ambiguity in defining the fundamental frequency of a constant signal to be zero. That is, a constant signal has a zero rate of oscillation.

Periodic signals—and in particular, the complex periodic exponential signal in eq. (1.21) and the sinusoidal signal in eq. (1.25)—provide important examples of signals with infinite total energy but finite average power. For example, consider the periodic exponential signal of eq. (1.21), and suppose that we calculate the total energy and average power in this signal over one period:

$$\begin{aligned} E_{\text{period}} &= \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt \\ &= \int_0^{T_0} 1 \cdot dt = T_0, \end{aligned} \tag{1.30}$$

[2]Euler's relation and other basic ideas related to the manipulation of complex numbers and exponentials are considered in the mathematical review section of the problems at the end of the chapter.

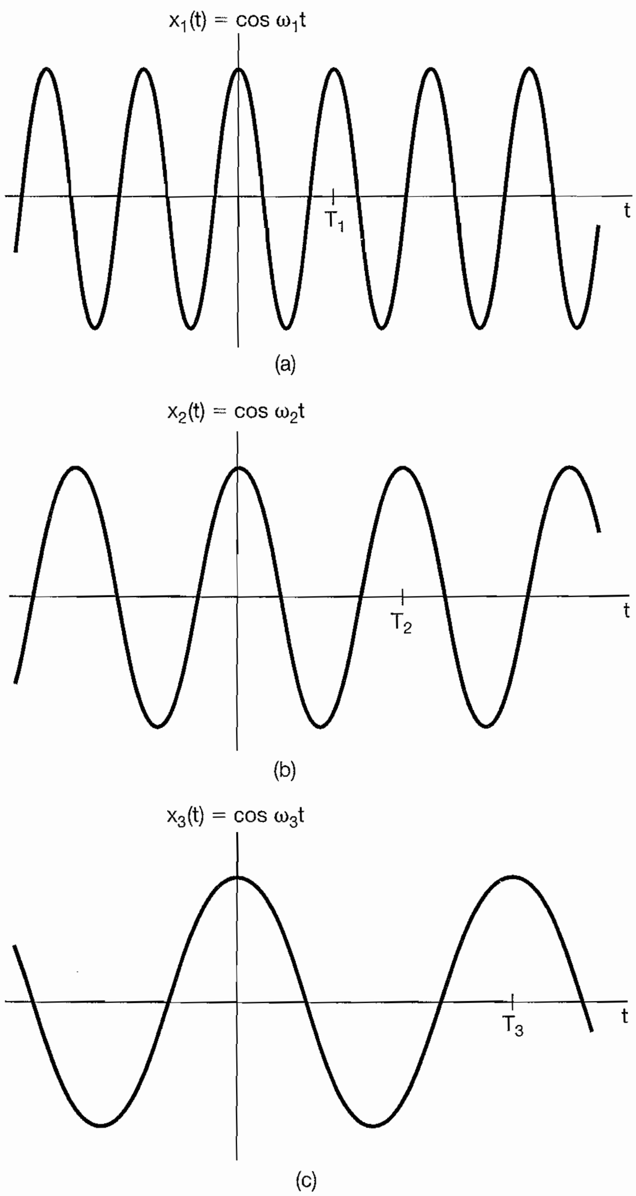


Figure 1.21 Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1. \tag{1.31}$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. However, each period of the signal looks exactly the same. Since the average power of the signal equals 1 over each period, averaging over multiple periods always yields an average power of 1. That is, the complex periodic exponential signal has finite average power equal to

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt = 1. \tag{1.32}$$

Problem 1.3 provides additional examples of energy and power calculations for periodic and aperiodic signals.

Periodic complex exponentials will play a central role in much of our treatment of signals and systems, in part because they serve as extremely useful building blocks for many other signals. We will often find it useful to consider sets of *harmonically related* complex exponentials—that is, sets of periodic exponentials, all of which are periodic with a common period T_0. Specifically, a necessary condition for a complex exponential $e^{j\omega t}$ to be periodic with period T_0 is that

$$e^{j\omega T_0} = 1, \tag{1.33}$$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.34}$$

Thus, if we define

$$\omega_0 = \frac{2\pi}{T_0}, \tag{1.35}$$

we see that, to satisfy eq. (1.34), ω must be an integer multiple of ω_0. That is, a harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.36}$$

For $k = 0$, $\phi_k(t)$ is a constant, while for any other value of k, $\phi_k(t)$ is periodic with fundamental frequency $|k|\omega_0$ and fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}. \tag{1.37}$$

The kth harmonic $\phi_k(t)$ is still periodic with period T_0 as well, as it goes through exactly $|k|$ of its fundamental periods during any time interval of length T_0.

Our use of the term "harmonic" is consistent with its use in music, where it refers to tones resulting from variations in acoustic pressure at frequencies that are integer multiples of a fundamental frequency. For example, the pattern of vibrations of a string on an instrument such as a violin can be described as a superposition—i.e., a weighted sum—of harmonically related periodic exponentials. In Chapter 3, we will see that we can build a very rich class of periodic signals using the harmonically related signals of eq. (1.36) as the building blocks.

Example 1.5

It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid. For example, suppose we wish to plot the magnitude of the signal

$$x(t) = e^{j2t} + e^{j3t}. \tag{1.38}$$

To do this, we first factor out a complex exponential from the right side of eq. (1.38), where the frequency of this exponential factor is taken as the average of the frequencies of the two exponentials in the sum. Doing this, we obtain

$$x(t) = e^{j2.5t}(e^{-j0.5} + e^{j0.5t}), \tag{1.39}$$

which, because of Euler's relation, can be rewritten as

$$x(t) = 2e^{j2.5t}\cos(0.5t). \tag{1.40}$$

From this, we can directly obtain an expression for the magnitude of $x(t)$:

$$|x(t)| = 2|\cos(0.5t)|. \tag{1.41}$$

Here, we have used the fact that the magnitude of the complex exponential $e^{j2.5t}$ is always unity. Thus, $|x(t)|$ is what is commonly referred to as a full-wave rectified sinusoid, as shown in Figure 1.22.

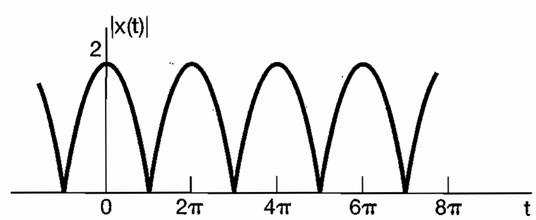


Figure 1.22 The full-wave rectified sinusoid of Example 1.5.

General Complex Exponential Signals

The most general case of a complex exponential can be expressed and interpreted in terms of the two cases we have examined so far: the real exponential and the periodic complex exponential. Specifically, consider a complex exponential Ce^{at}, where C is expressed in polar form and a in rectangular form. That is,

$$C = |C|e^{j\theta}$$

and

$$a = r + j\omega_0.$$

Then

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}. \tag{1.42}$$

Using Euler's relation, we can expand this further as

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta). \tag{1.43}$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal. For $r > 0$ they correspond to sinusoidal signals multiplied by a growing exponential, and for $r < 0$ they correspond to sinusoidal signals multiplied by a decaying exponential. These two cases are shown in Figure 1.23. The dashed lines in the figure correspond to the functions $\pm|C|e^{rt}$. From eq. (1.42), we see that $|C|e^{rt}$ is the magnitude of the complex exponential. Thus, the dashed curves act as an envelope for the oscillatory curve in the figure in that the peaks of the oscillations just reach these curves, and in this way the envelope provides us with a convenient way to visualize the general trend in the amplitude of the oscillations.

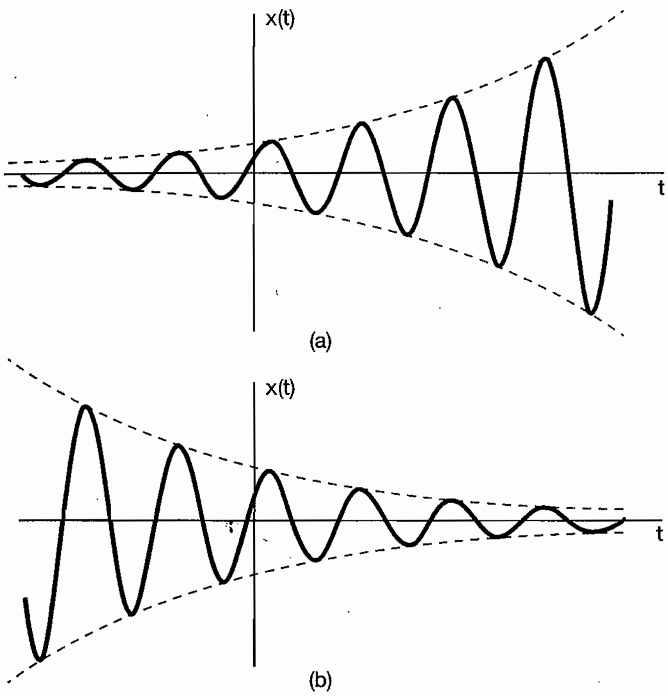


Figure 1.23 (a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

Sinusoidal signals multiplied by decaying exponentials are commonly referred to as *damped sinusoids.* Examples of damped sinusoids arise in the response of *RLC* circuits and in mechanical systems containing both damping and restoring forces, such as automotive suspension systems. These kinds of systems have mechanisms that dissipate energy (resistors, damping forces such as friction) with oscillations that decay in time. Examples illustrating such systems and their damped sinusoidal natural responses can be found in Problems 2.61 and 2.62.

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

As in continuous time, an important signal in discrete time is the *complex exponential signal* or *sequence,* defined by

$$x[n] = C\alpha^n, \tag{1.44}$$

where C and α are, in general, complex numbers. This could alternatively be expressed in the form

$$x[n] = Ce^{\beta n}, \tag{1.45}$$

where

$$\alpha = e^{\beta}.$$

Although the form of the discrete-time complex exponential sequence given in eq. (1.45) is more analogous to the form of the continuous-time exponential, it is often more convenient to express the discrete-time complex exponential sequence in the form of eq. (1.44).

Real Exponential Signals

If C and α are real, we can have one of several types of behavior, as illustrated in Figure 1.24. If $|\alpha| > 1$ the magnitude of the signal grows exponentially with n, while if $|\alpha| < 1$ we have a decaying exponential. Furthermore, if α is positive, all the values of $C\alpha^n$ are of the same sign, but if α is negative then the sign of $x[n]$ alternates. Note also that if $\alpha = 1$ then $x[n]$ is a constant, whereas if $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$. Real-valued discrete-time exponentials are often used to describe population growth as a function of generation and total return on investment as a function of day, month, or quarter.

Sinusoidal Signals

Another important complex exponential is obtained by using the form given in eq. (1.45) and by constraining β to be purely imaginary (so that $|\alpha| = 1$). Specifically, consider

$$x[n] = e^{j\omega_0 n}. \tag{1.46}$$

As in the continuous-time case, this signal is closely related to the sinusoidal signal

$$x[n] = A\cos(\omega_0 n + \phi). \tag{1.47}$$

If we take n to be dimensionless, then both ω_0 and ϕ have units of radians. Three examples of sinusoidal sequences are shown in Figure 1.25.

As before, Euler's relation allows us to relate complex exponentials and sinusoids:

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n \tag{1.48}$$

and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{1.49}$$

The signals in eqs. (1.46) and (1.47) are examples of discrete-time signals with infinite total energy but finite average power. For example, since $|e^{j\omega_0 n}|^2 = 1$, every sample of the signal in eq. (1.46) contributes 1 to the signal's energy. Thus, the total energy for $-\infty < n < \infty$ is infinite, while the average power per time point is obviously equal to 1. Other examples of energy and power calculations for discrete-time signals are given in Problem 1.3.

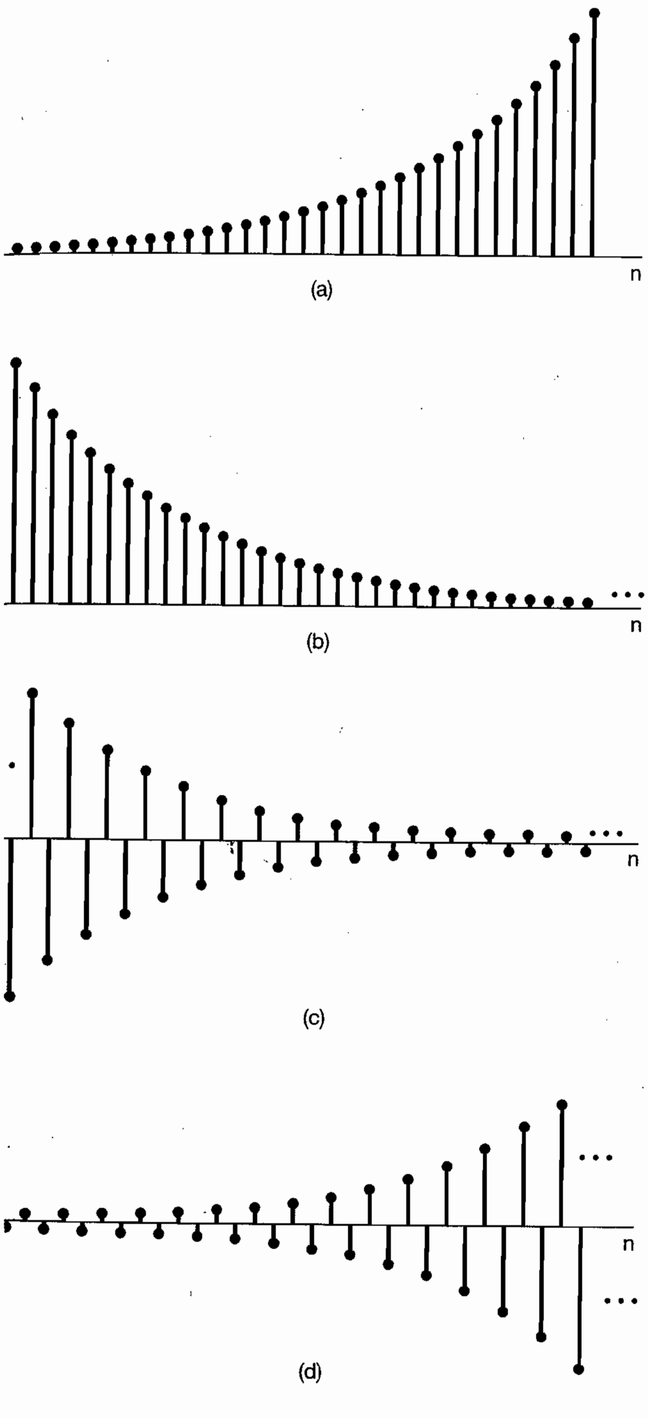


Figure 1.24 The real exponential signal $x[n] = C\alpha^n$:
(a) $\alpha > 1$; (b) $0 < \alpha < 1$;
(c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

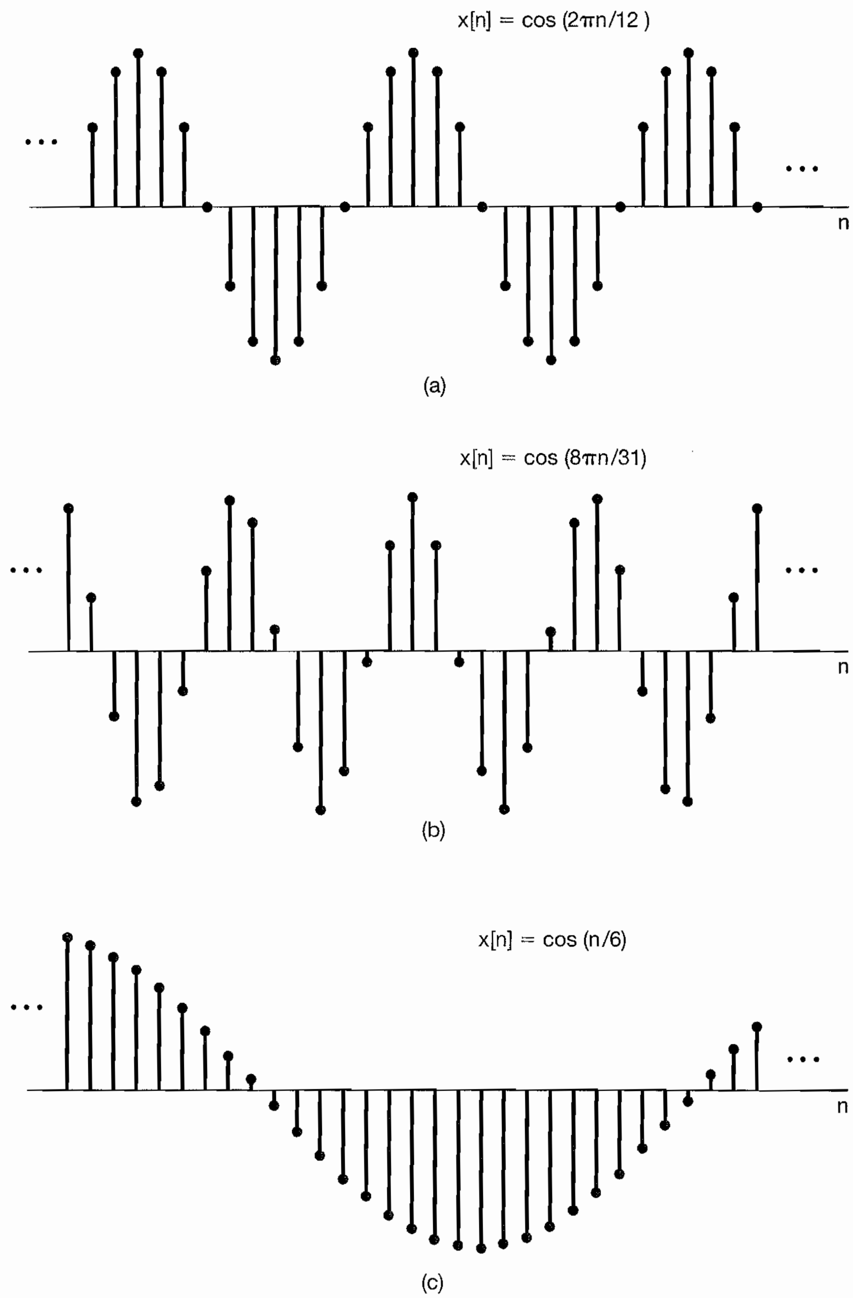


Figure 1.25 Discrete-time sinusoidal signals.

General Complex Exponential Signals

The general discrete-time complex exponential can be written and interpreted in terms of real exponentials and sinusoidal signals. Specifically, if we write C and α in polar form, viz.,

$$C = |C|e^{j\theta}$$

and

$$\alpha = |\alpha|e^{j\omega_0},$$

then

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta). \tag{1.50}$$

Thus, for $|\alpha| = 1$, the real and imaginary parts of a complex exponential sequence are sinusoidal. For $|\alpha| < 1$ they correspond to sinusoidal sequences multiplied by a decaying exponential, while for $|\alpha| > 1$ they correspond to sinusoidal sequences multiplied by a growing exponential. Examples of these signals are depicted in Figure 1.26.

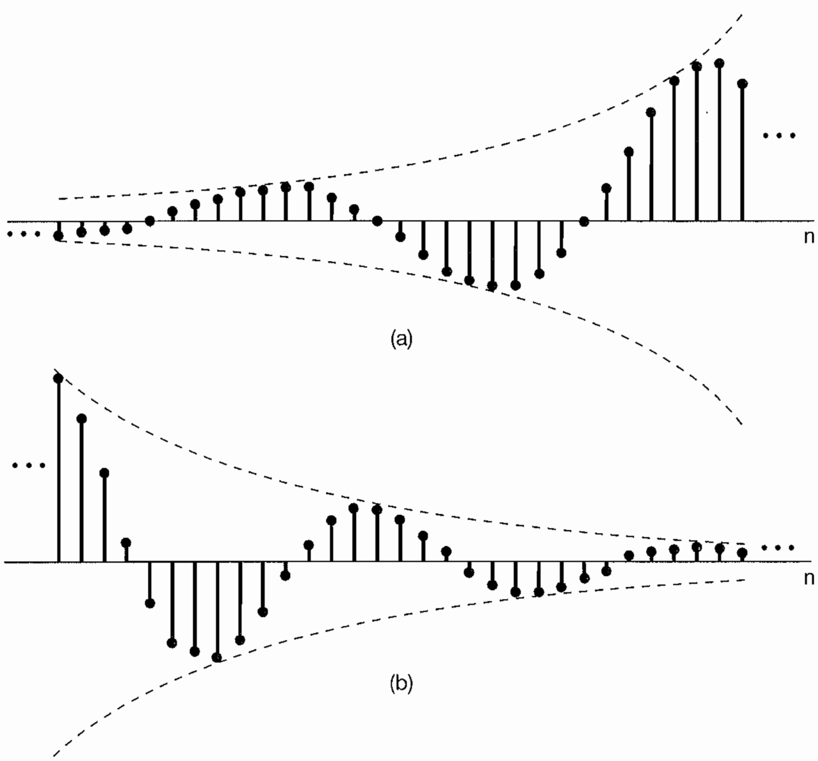


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

While there are many similarities between continuous-time and discrete-time signals, there are also a number of important differences. One of these concerns the discrete-time exponential signal $e^{j\omega_0 n}$. In Section 1.3.1, we identified the following two properties of its continuous-time counterpart $e^{j\omega_0 t}$: (1) the larger the magnitude of ω_0, the higher is the rate of oscillation in the signal; and (2) $e^{j\omega_0 t}$ is periodic for any value of ω_0. In this section we describe the discrete-time versions of both of these properties, and as we will see, there are definite differences between each of these and its continuous-time counterpart.

The fact that the first of these properties is different in discrete time is a direct consequence of another extremely important distinction between discrete-time and continuous-time complex exponentials. Specifically, consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}. \tag{1.51}$$

From eq. (1.51), we see that the exponential at frequency $\omega_0 + 2\pi$ is the *same* as that at frequency ω_0. Thus, we have a very different situation from the continuous-time case, in which the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In discrete time, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0. Although, according to eq. (1.51), any interval of length 2π will do, on most occasions we will use the interval $0 \le \omega_0 < 2\pi$ or the interval $-\pi \le \omega_0 < \pi$.

Because of the periodicity implied by eq. (1.51), the signal $e^{j\omega_0 n}$ does *not* have a continually increasing rate of oscillation as ω_0 is increased in magnitude. Rather, as illustrated in Figure 1.27, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we *decrease* the rate of oscillation until we reach $\omega_0 = 2\pi$, which produces the same constant sequence as $\omega_0 = 0$. Therefore, the low-frequency (that is, slowly varying) discrete-time exponentials have values of ω_0 near 0, 2π, and any other even multiple of π, while the high frequencies (corresponding to rapid variations) are located near $\omega_0 = \pm\pi$ and other odd multiples of π. Note in particular that for $\omega_0 = \pi$ or any other odd multiple of π,

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n, \tag{1.52}$$

so that this signal oscillates rapidly, changing sign at each point in time [as illustrated in Figure 1.27(e)].

The second property we wish to consider concerns the periodicity of the discrete-time complex exponential. In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{1.53}$$

or equivalently,

$$e^{j\omega_0 N} = 1. \tag{1.54}$$

For eq. (1.54) to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m, \tag{1.55}$$

or equivalently,

$$\frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{1.56}$$

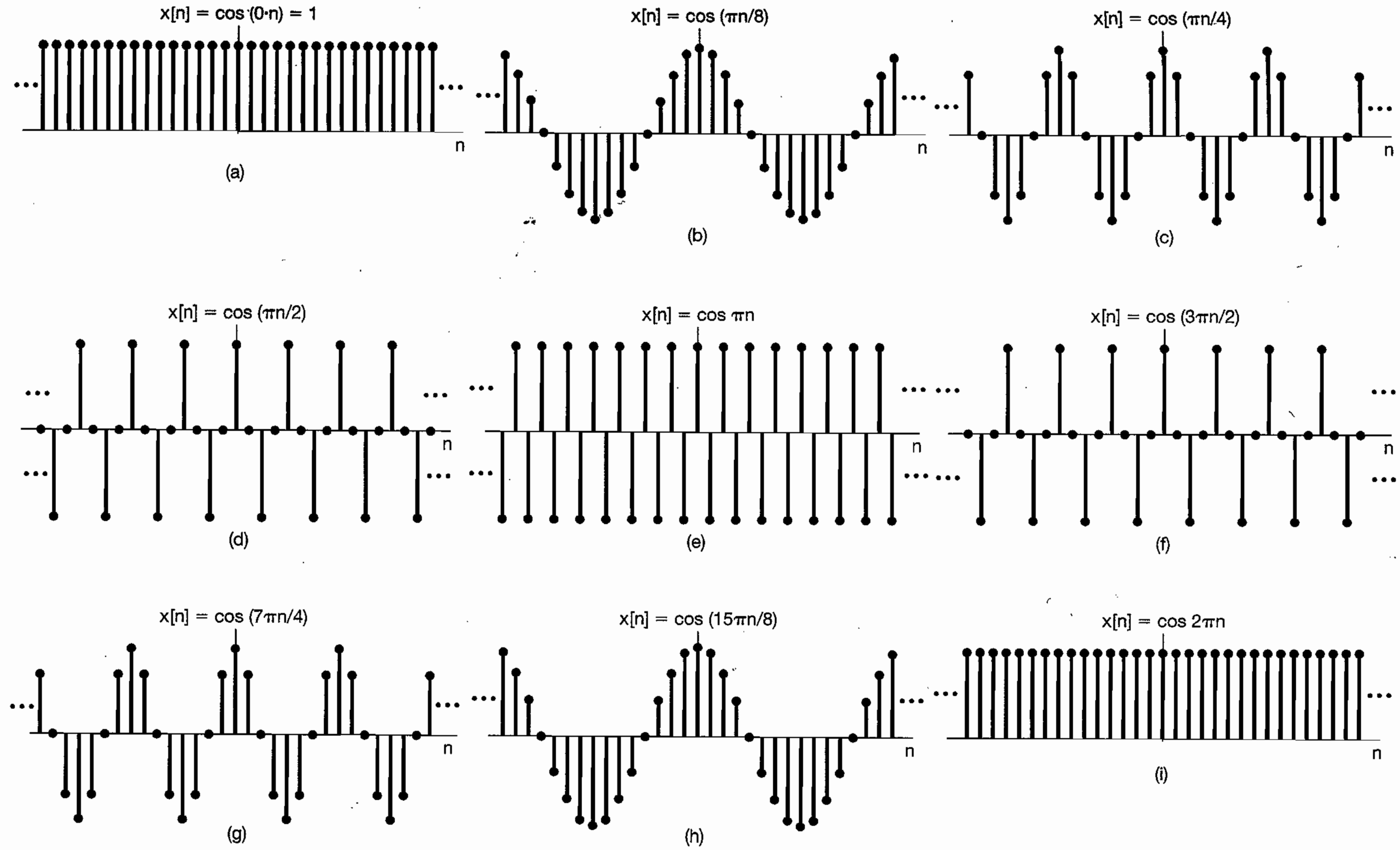


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

According to eq. (1.56), the signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is not periodic otherwise. These same observations also hold for discrete-time sinusoids. For example, the signals depicted in Figure 1.25(a) and (b) are periodic, while the signal in Figure 1.25(c) is not.

Using the calculations that we have just made, we can also determine the fundamental period and frequency of discrete-time complex exponentials, where we define the fundamental frequency of a discrete-time periodic signal as we did in continuous time. That is, if $x[n]$ is periodic with fundamental period N, its fundamental frequency is $2\pi/N$. Consider, then, a periodic complex exponential $x[n] = e^{j\omega_0 n}$ with $\omega_0 \neq 0$. As we have just seen, ω_0 must satisfy eq. (1.56) for some pair of integers m and N, with $N > 0$. In Problem 1.35, it is shown that if $\omega_0 \neq 0$ and if N and m have no factors in common, then the fundamental period of $x[n]$ is N. Using this fact together with eq. (1.56), we find that the fundamental frequency of the periodic signal $e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m}. \tag{1.57}$$

Note that the fundamental period can also be written as

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{1.58}$$

These last two expressions again differ from their continuous-time counterparts. In Table 1.1, we have summarized some of the differences between the continuous-time signal $e^{j\omega_0 t}$ and the discrete-time signal $e^{j\omega_0 n}$. Note that, as in the continuous-time case, the constant discrete-time signal resulting from setting $\omega_0 = 0$ has a fundamental frequency of zero, and its fundamental period is undefined.

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

*Assumes that m and N do not have any factors in common.

To gain some additional insight into these properties, let us examine again the signals depicted in Figure 1.25. First, consider the sequence $x[n] = \cos(2\pi n/12)$, depicted in Figure 1.25(a), which we can think of as the set of samples of the continuous-time sinusoid $x(t) = \cos(2\pi t/12)$ at integer time points. In this case, $x(t)$ is periodic with fundamental period 12 and $x[n]$ is also periodic with fundamental period 12. That is, the values of $x[n]$ repeat every 12 points, exactly in step with the fundamental period of $x(t)$.

In contrast, consider the signal $x[n] = \cos(8\pi n/31)$, depicted in Figure 1.25(b), which we can view as the set of samples of $x(t) = \cos(8\pi t/31)$ at integer points in time. In this case, $x(t)$ is periodic with fundamental period 31/4. On the other hand, $x[n]$ is periodic with fundamental period 31. The reason for this difference is that the discrete-time signal is defined only for integer values of the independent variable. Thus, there is no sample at time $t = 31/4$, when $x(t)$ completes one period (starting from $t = 0$). Similarly, there is no sample at $t = 2 \cdot 31/4$ or $t = 3 \cdot 31/4$, when $x(t)$ has completed two or three periods, but there is a sample at $t = 4 \cdot 31/4 = 31$, when $x(t)$ has completed *four* periods. This can be seen in Figure 1.25(b), where the pattern of $x[n]$ values does *not* repeat with each single cycle of positive and negative values. Rather, the pattern repeats after four such cycles, namely, every 31 points.

Similarly, the signal $x[n] = \cos(n/6)$ can be viewed as the set of samples of the signal $x(t) = \cos(t/6)$ at integer time points. In this case, the values of $x(t)$ at integer sample points never repeat, as these sample points never span an interval that is an exact multiple of the period, 12π, of $x(t)$. Thus, $x[n]$ is not periodic, although the eye visually interpolates between the sample points, suggesting the envelope $x(t)$, which *is* periodic. The use of the concept of sampling to gain insight into the periodicity of discrete-time sinusoidal sequences is explored further in Problem 1.36.

Example 1.6

Suppose that we wish to determine the fundamental period of the discrete-time signal

$$x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}. \qquad (1.59)$$

The first exponential on the right-hand side of eq. (1.59) has a fundamental period of 3. While this can be verified from eq. (1.58), there is a simpler way to obtain that answer. In particular, note that the angle $(2\pi/3)n$ of the first term must be incremented by a multiple of 2π for the values of this exponential to begin repeating. We then immediately see that if n is incremented by 3, the angle will be incremented by a single multiple of 2π. With regard to the second term, we see that incrementing the angle $(3\pi/4)n$ by 2π would require n to be incremented by 8/3, which is impossible, since n is restricted to being an integer. Similarly, incrementing the angle by 4π would require a noninteger increment of 16/3 to n. However, incrementing the angle by 6π requires an increment of 8 to n, and thus the fundamental period of the second term is 8.

Now, for the entire signal $x[n]$ to repeat, each of the terms in eq. (1.59) must go through an integer number of its own fundamental period. The smallest increment of n that accomplishes this is 24. That is, over an interval of 24 points, the first term on the right-hand side of eq. (1.59) will have gone through eight of its fundamental periods, the second term through three of its fundamental periods, and the overall signal $x[n]$ through exactly one of its fundamental periods.

As in continuous time, it is also of considerable value in discrete-time signal and system analysis to consider sets of harmonically related periodic exponentials—that is, periodic exponentials with a common period N. From eq. (1.56), we know that these are precisely the signals which are at frequencies which are multiples of $2\pi/N$. That is,

$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots. \qquad (1.60)$$

In the continuous-time case, all of the harmonically related complex exponentials $e^{jk(2\pi/T)t}$, $k = 0, \pm 1, \pm 2, \ldots$, are distinct. However, because of eq. (1.51), this is *not* the case in discrete time. Specifically,

$$\begin{aligned}\phi_{k+N}[n] &= e^{j(k+N)(2\pi/N)n} \\ &= e^{jk(2\pi/N)n}e^{j2\pi n} = \phi_k[n].\end{aligned} \tag{1.61}$$

This implies that there are only N distinct periodic exponentials in the set given in eq. (1.60). For example,

$$\phi_0[n] = 1, \phi_1[n] = e^{j2\pi n/N}, \phi_2[n] = e^{j4\pi n/N}, \ldots, \phi_{N-1}[n] = e^{j2\pi(N-1)n/N} \tag{1.62}$$

are all distinct, and any other $\phi_k[n]$ is identical to one of these (e.g., $\phi_N[n] = \phi_0[n]$ and $\phi_{-1}[n] = \phi_{N-1}[n]$).

1.4 THE UNIT IMPULSE AND UNIT STEP FUNCTIONS

In this section, we introduce several other basic signals—specifically, the unit impulse and step functions in continuous and discrete time—that are also of considerable importance in signal and system analysis. In Chapter 2, we will see how we can use unit impulse signals as basic building blocks for the construction and representation of other signals. We begin with the discrete-time case.

1.4.1 The Discrete-Time Unit Impulse and Unit Step Sequences

One of the simplest discrete-time signals is the *unit impulse* (or *unit sample*), which is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{1.63}$$

and which is shown in Figure 1.28. Throughout the book, we will refer to $\delta[n]$ interchangeably as the unit impulse or unit sample.

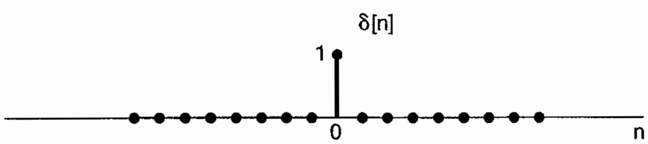


Figure 1.28 Discrete-time unit impulse (sample).

A second basic discrete-time signal is the discrete-time *unit step,* denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}. \tag{1.64}$$

The unit step sequence is shown in Figure 1.29.

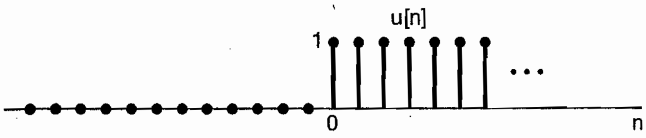


Figure 1.29 Discrete-time unit step sequence.

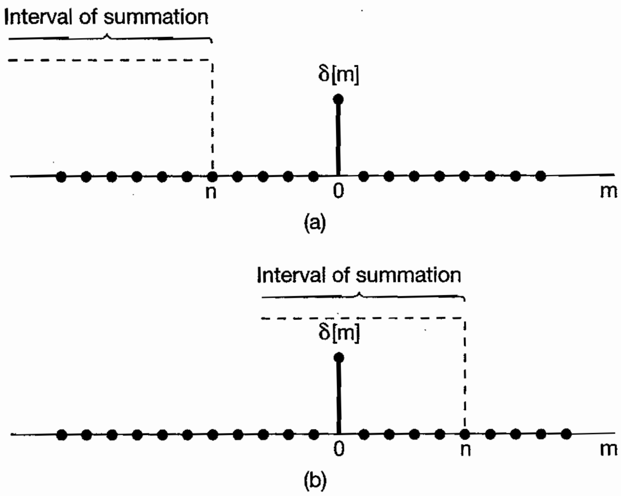


Figure 1.30 Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

There is a close relationship between the discrete-time unit impulse and unit step. In particular, the discrete-time unit impulse is the *first difference* of the discrete-time step

$$\delta[n] = u[n] - u[n-1]. \tag{1.65}$$

Conversely, the discrete-time unit step is the *running sum* of the unit sample. That is,

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{1.66}$$

Equation (1.66) is illustrated graphically in Figure 1.30. Since the only nonzero value of the unit sample is at the point at which its argument is zero, we see from the figure that the running sum in eq. (1.66) is 0 for $n < 0$ and 1 for $n \geq 0$. Furthermore, by changing the variable of summation from m to $k = n - m$ in eq. (1.66), we find that the discrete-time unit step can also be written in terms of the unit sample as

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k],$$

or equivalently,

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{1.67}$$

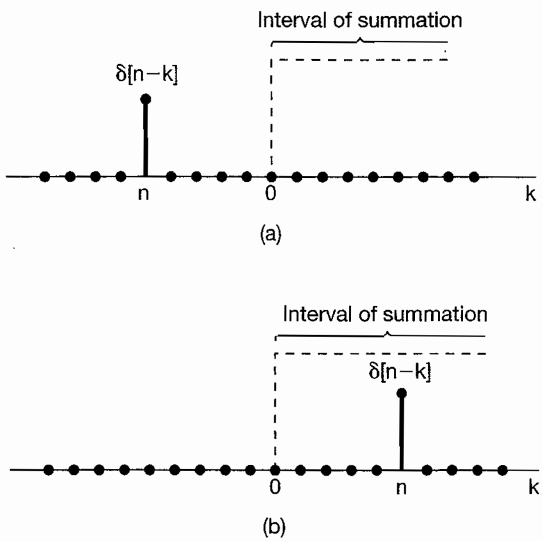


Figure 1.31 Relationship given in eq. (1.67): (a) $n < 0$; (b) $n > 0$.

Equation (1.67) is illustrated in Figure 1.31. In this case the nonzero value of $\delta[n-k]$ is at the value of k equal to n, so that again we see that the summation in eq. (1.67) is 0 for $n < 0$ and 1 for $n \geq 0$.

An interpretation of eq. (1.67) is as a superposition of delayed impulses; i.e., we can view the equation as the sum of a unit impulse $\delta[n]$ at $n = 0$, a unit impulse $\delta[n-1]$ at $n = 1$, another, $\delta[n-2]$, at $n = 2$, etc. We will make explicit use of this interpretation in Chapter 2.

The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n]. \tag{1.68}$$

More generally, if we consider a unit impulse $\delta[n-n_0]$ at $n = n_0$, then

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]. \tag{1.69}$$

This sampling property of the unit impulse will play an important role in Chapters 2 and 7.

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time *unit step function* $u(t)$ is defined in a manner similar to its discrete-time counterpart. Specifically,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}, \tag{1.70}$$

as is shown in Figure 1.32. Note that the unit step is discontinuous at $t = 0$. The continuous-time *unit impulse function* $\delta(t)$ is related to the unit step in a manner analogous

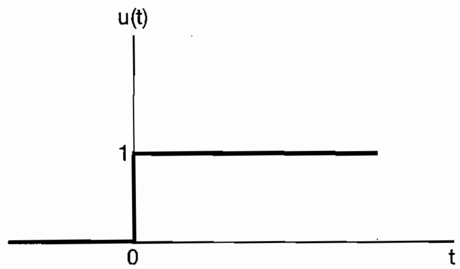


Figure 1.32 Continuous-time unit step function.

to the relationship between the discrete-time unit impulse and step functions. In particular, the continuous-time unit step is the *running integral* of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \tag{1.71}$$

This also suggests a relationship between $\delta(t)$ and $u(t)$ analogous to the expression for $\delta[n]$ in eq. (1.65). In particular, it follows from eq. (1.71) that the continuous-time unit impulse can be thought of as the *first derivative* of the continuous-time unit step:

$$\delta(t) = \frac{du(t)}{dt}. \tag{1.72}$$

In contrast to the discrete-time case, there is some formal difficulty with this equation as a representation of the unit impulse function, since $u(t)$ is discontinuous at $t = 0$ and consequently is formally not differentiable. We can, however, interpret eq. (1.72) by considering an approximation to the unit step $u_\Delta(t)$, as illustrated in Figure 1.33, which rises from the value 0 to the value 1 in a short time interval of length Δ. The unit step, of course, changes values instantaneously and thus can be thought of as an idealization of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. Formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \to 0$. Let us now consider the derivative

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}, \tag{1.73}$$

as shown in Figure 1.34.

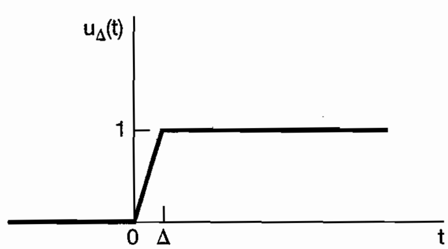


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

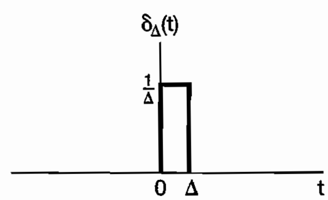


Figure 1.34 Derivative of $u_\Delta(t)$.

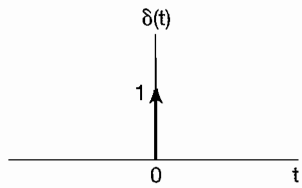


Figure 1.35 Continuous-time unit impulse.

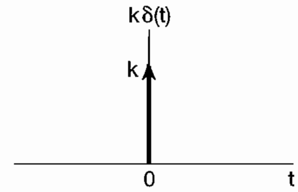


Figure 1.36 Scaled impulse.

Note that $\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintaining its unit area. Its limiting form,

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t), \tag{1.74}$$

can then be thought of as an idealization of the short pulse $\delta_\Delta(t)$ as the duration Δ becomes insignificant. Since $\delta(t)$ has, in effect, no duration but unit area, we adopt the graphical notation for it shown in Figure 1.35, where the arrow at $t = 0$ indicates that the area of the pulse is concentrated at $t = 0$ and the height of the arrow and the "1" next to the arrow are used to represent the *area* of the impulse. More generally, a scaled impulse $k\delta(t)$ will have an area k, and thus,

$$\int_{-\infty}^{t} k\delta(\tau)\,d\tau = ku(t).$$

A scaled impulse with area k is shown in Figure 1.36, where the height of the arrow used to depict the scaled impulse is chosen to be proportional to the area of the impulse.

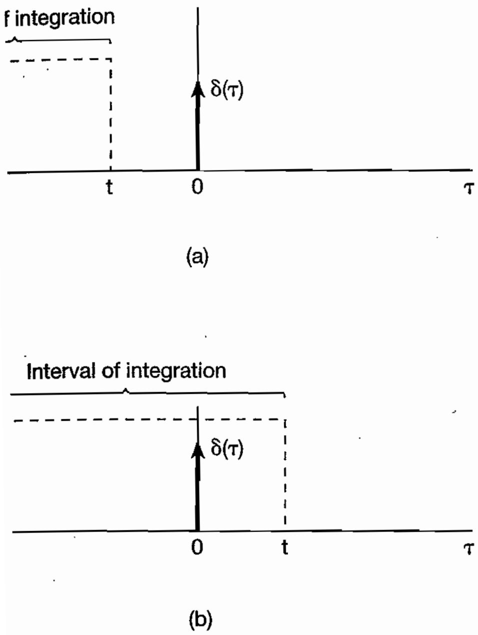
As with discrete time, we can provide a simple graphical interpretation of the running integral of eq. (1.71); this is shown in Figure 1.37. Since the area of the continuous-time unit impulse $\delta(\tau)$ is concentrated at $\tau = 0$, we see that the running integral is 0 for $t < 0$ and 1 for $t > 0$. Also, we note that the relationship in eq. (1.71) between the continuous-time unit step and impulse can be rewritten in a different form, analogous to the discrete-time form in eq. (1.67), by changing the variable of integration from τ to $\sigma = t - \tau$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma),$$

or equivalently,

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\,d\sigma. \tag{1.75}$$

The graphical interpretation of this form of the relationship between $u(t)$ and $\delta(t)$ is given in Figure 1.38. Since in this case the area of $\delta(t-\sigma)$ is concentrated at the point $\sigma = t$, we again see that the integral in eq. (1.75) is 0 for $t < 0$ and 1 for $t > 0$. This type of graphical interpretation of the behavior of the unit impulse under integration will be extremely useful in Chapter 2.



37 Running integral given in eq. (1.71): (b) $t > 0$.

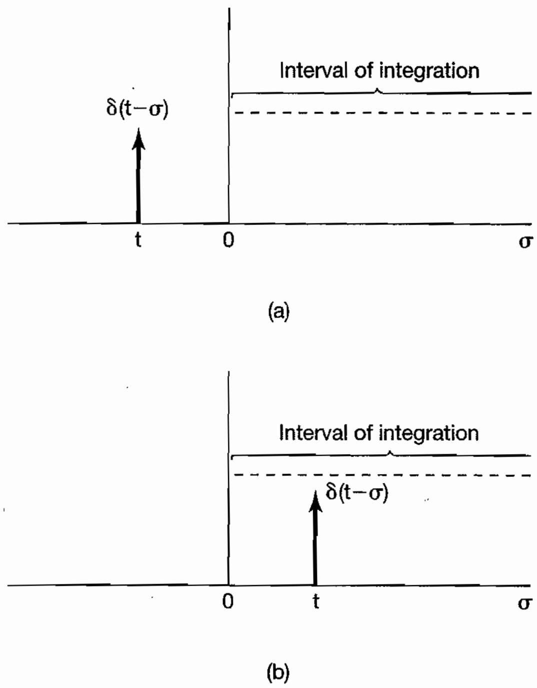


Figure 1.38 Relationship given in eq. (1.75): (a) $t < 0$; (b) $t > 0$.

As with the discrete-time impulse, the continuous-time impulse has a very important sampling property. In particular, for a number of reasons it will be important to consider the product of an impulse and more well-behaved continuous-time functions $x(t)$. The interpretation of this quantity is most readily developed using the definition of $\delta(t)$ according to eq. (1.74). Specifically, consider

$$x_1(t) = x(t)\delta_\Delta(t).$$

In Figure 1.39(a) we have depicted the two time functions $x(t)$ and $\delta_\Delta(t)$, and in Figure 1.39(b) we see an enlarged view of the nonzero portion of their product. By construction, $x_1(t)$ is zero outside the interval $0 \le t \le \Delta$. For Δ sufficiently small so that $x(t)$ is approximately constant over this interval,

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t).$$

Since $\delta(t)$ is the limit as $\Delta \to 0$ of $\delta_\Delta(t)$, it follows that

$$x(t)\delta(t) = x(0)\delta(t). \tag{1.76}$$

By the same argument, we have an analogous expression for an impulse concentrated at an arbitrary point, say, t_0. That is,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

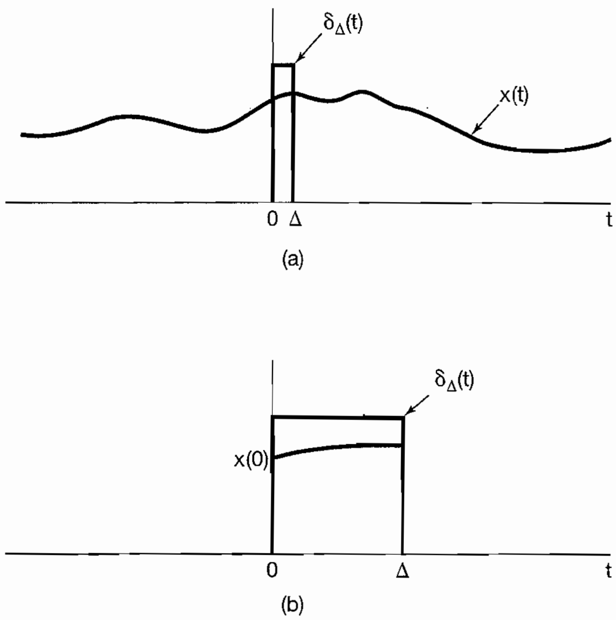


Figure 1.39 The product $x(t)\delta_\Delta(t)$: (a) graphs of both functions; (b) enlarged view of the nonzero portion of their product.

Although our discussion of the unit impulse in this section has been somewhat informal, it does provide us with some important intuition about this signal that will be useful throughout the book. As we have stated, the unit impulse should be viewed as an idealization. As we illustrate and discuss in more detail in Section 2.5, any real physical system has some inertia associated with it and thus does not respond instantaneously to inputs. Consequently, if a pulse of sufficiently short duration is applied to such a system, the system response will not be noticeably influenced by the pulse's duration or by the details of the shape of the pulse, for that matter. Instead, the primary characteristic of the pulse that will matter is the net, integrated effect of the pulse—i.e., its area. For systems that respond much more quickly than others, the pulse will have to be of much shorter duration before the details of the pulse shape or its duration no longer matter. Nevertheless, for any physical system, we can always find a pulse that is "short enough." The unit impulse then is an idealization of this concept—the pulse that is short enough for *any* system. As we will see in Chapter 2, the response of a system to this idealized pulse plays a crucial role in signal and system analysis, and in the process of developing and understanding this role, we will develop additional insight into the idealized signal.[3]

[3]The unit impulse and other related functions (which are often collectively referred to as *singularity functions*) have been thoroughly studied in the field of mathematics under the alternative names of *generalized functions* and the *theory of distributions.* For more detailed discussions of this subject, see *Distribution Theory and Transform Analysis,* by A. H. Zemanian (New York: McGraw-Hill Book Company, 1965), *Generalised Functions,* by R.F. Hoskins (New York: Halsted Press, 1979), or the more advanced text, *Fourier Analysis and Generalized Functions,* by M. J. Lighthill (New York: Cambridge University Press, 1958). Our discussion of singularity functions in Section 2.5 is closely related in spirit to the mathematical theory described in these texts and thus provides an informal introduction to concepts that underlie this topic in mathematics.

Example 1.7

Consider the discontinuous signal $x(t)$ depicted in Figure 1.40(a). Because of the relationship between the continuous-time unit impulse and unit step, we can readily calculate and graph the derivative of this signal. Specifically, the derivative of $x(t)$ is clearly 0, except at the discontinuities. In the case of the unit step, we have seen [eq. (1.72)] that differentiation gives rise to a unit impulse located at the point of discontinuity. Furthermore, by multiplying both sides of eq. (1.72) by any number k, we see that the derivative of a unit step with a discontinuity of size k gives rise to an impulse of area k at the point of discontinuity. This rule also holds for any other signal with a jump discontinuity, such as $x(t)$ in Figure 1.40(a). Consequently, we can sketch its derivative $\dot{x}(t)$, as in Figure 1.40(b), where an impulse is placed at each discontinuity of $x(t)$, with area equal to the size of the discontinuity. Note, for example, that the discontinuity in $x(t)$ at $t = 2$ has a value of -3, so that an impulse scaled by -3 is located at $t = 2$ in the signal $\dot{x}(t)$.

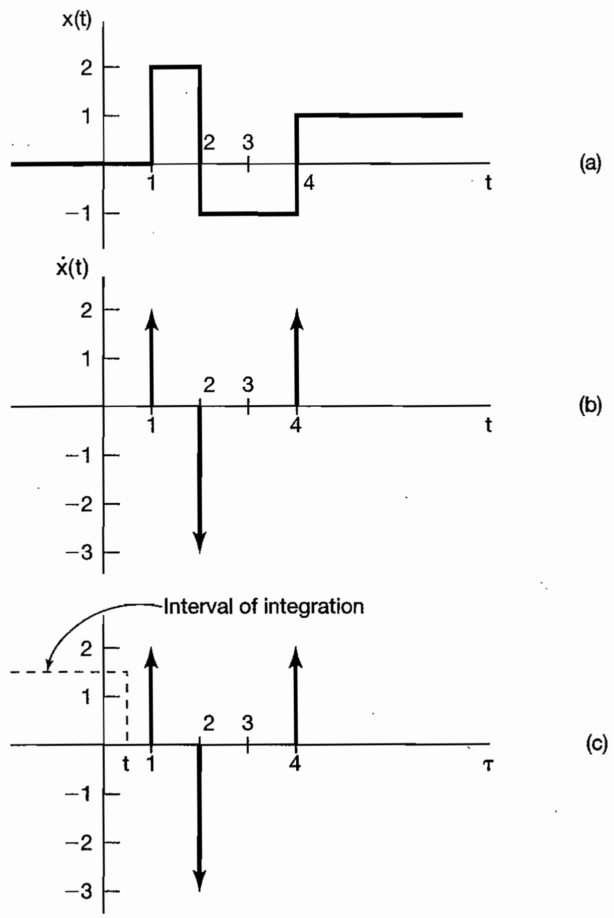


Figure 1.40 (a) The discontinuous signal $x(t)$ analyzed in Example 1.7; (b) its derivative $\dot{x}(t)$; (c) depiction of the recovery of $x(t)$ as the running integral of $\dot{x}(t)$, illustrated for a value of t between 0 and 1.