Recapping: Sampling Theory and Discrete-Time Fourier Transform (DTFT)

- Sampling a sinewave:

$$x(t) = \cos(\omega_a t)$$

$$x[n] = x_a(nT_s) = \cos(\omega_a n T_s)$$

$$= \cos([\omega_a T_s] n)$$

$$= \cos(\omega_d n)$$

- dictates relationship between DT frequency variable and CT frequency variable

$$\boxed{\omega_d = \omega_a T_s}$$

OR, since $F_s = \frac{1}{T_s}$, $\boxed{\omega_d = \frac{\omega_a}{F_s}}$

Side note: Definition of DTFT motivated by passing infinite duration sinewave thru LTI DT system:

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_0 (n-k)}$$

$$= H(\omega_0)\, e^{j\omega_0 n}$$

where:

$$H(\omega) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-jk\omega}$$

Since k is dummy variable of summation:

$$x[n] \xleftrightarrow{DTFT} X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- At the same time, consider the DTFT of a sampled signal:

$$x_s(t) = x_a(t) \left(\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right)$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{x_a(nT_s)}_{x[n]} \delta(t - nT_s)$$

The (Continuous-Time) Fourier Transform of this may be expressed as below using the time-shift property

$$X_s(\omega) = \mathcal{F}\{x_s(t)\} = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT_s) \right\}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega nT_s} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega T_s)n}$$

- Using our relationship between the DT freq. variable and the CT frequency variable, we obtain same definition of DTFT as before:

$$X_d(\omega) = X_s(\omega)\Big|_{\omega = \omega T_s} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\omega_d = \omega_a T_s = \frac{\omega_a}{F_s}$$

Now, remember the other way to determine the CT Fourier Transform of the sampled signal $x_s(t) = x_a(t) \left(\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right)$ is to use the product rule / property: (3)

Since:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{\mathcal{F}} \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

And since:

$$X_s(\omega) = \mathcal{F}\left\{ x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} = \frac{1}{2\pi} \mathcal{F}\{x_a(t)\} * \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\}$$

We obtain the classic result that sampling in time gives rise to replications of the spectrum (CTFT) of $x_a(t)$ at every integer multiple of $\frac{2\pi}{T_s}$

$$X_s(\omega) = \frac{1}{2\pi} X_a(\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega T_s}{T_s} - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{1}{T_s}(\omega T_s - k2\pi)\right)$$

Since $\omega_d = \omega T_s$:

$$x[n] = x_a(t)\Big|_{t=nT_s} \xleftrightarrow{DTFT} X(\omega)$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(\frac{1}{T_s}(\omega - k2\pi)\right)$$

Two ways to provide:

$$= F_s \sum_{k=-\infty}^{\infty} X_a\left(F_s(\omega - k2\pi)\right)$$

1. First, compress $X_a(\omega)$ by the sampling rate, F_s, then repeat/replicate result obtained every integer multiple of 2π

OR:

2. First, replicate $X_a(\omega)$ every integer multiple of $\frac{2\pi}{T_s}$ (to create $X_s(\omega)$)

   and then compress analog frequency axis

   by F_s (divide each and every analog frequency by the sampling rate according to $\omega_d = \frac{\omega_a}{F_s}$)

- I prefer 2nd option $\Rightarrow$ take into account aliasing first (overlapping terms) and then compress by F_s
- Either way, there is also an amplitude scaling by the sampling rate $F_s = \frac{1}{T_s}$

See old Exam 3's at course web site for many examples.

• Note: a third way to proceed is to sample the analog signal, and see if the DTFT of the resulting discrete-time signal can be determined from a <u>combination</u> of the:

DTFT pairs listed in Table 5.2 (pg. 392)

<u>AND</u>

DTFT properties listed in Table 5.1 (pg. 391)