Key properties of Laplace Transform

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = H(s)\,X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)}$$

$$\frac{d^n}{dt^n} x(t) \xleftrightarrow{\mathcal{L}} s^n X(s)$$

Key Laplace Transform Pair

$$e^{at} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}$$

Note: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt$

$= \mathcal{L}\{x(t)\}$

Take Laplace Transform of both sides of a Differential Equation. Example:

$$y''(t) + a_1 y'(t) + a_0 y(t) = b_0 x(t) + b_1 x'(t)$$

$$s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_0 X(s) + b_1 s X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s}{s^2 + a_1 s + a_0} = \mathcal{L}\{h(t)\}$$

roots of denominator: poles

$$s^2 + a_1 s + a_0 = (s - p_1)(s - p_2)$$

<u>Note:</u> Obtain Fourier Transform of $h(t)$ by substituting $s = j\omega$

$$H(\omega) = \frac{b_0 + b_1 j\omega}{(j\omega)^2 + a_1 \cdot j\omega + a_0} = \mathcal{F}\{x(t)\}$$

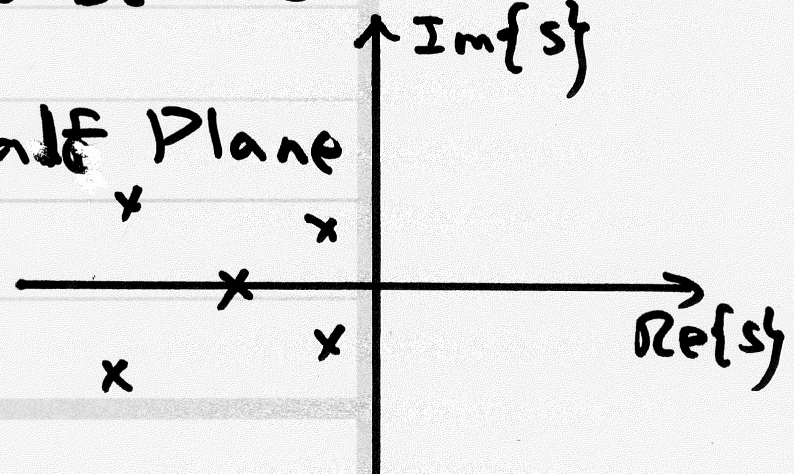
- Partial Fraction Expansion **IF** you want to determine impulse response $h(t)$

$$H(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} \qquad A_k = H(s)(s - p_k)\Big|_{s = p_k} \quad k = 1, 2$$

$$h(t) = A_1 e^{p_1 t} u(t) + A_2 e^{p_2 t} u(t)$$

- For stability: $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

- need terms to decay to zero
- need Real part of poles to be < 0
- need poles in the Left Half Plane



Key Properties of Z-Transform

$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{Z}} Y(z) = H(z)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$x[n+n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0} X(z)$$

$\Rightarrow$ DT Time-Shift plays same role in DT systems that differentian plays in CT systems!

Key Z-Transform Pair

$$\alpha^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha}$$

OR: $\alpha^{n-1} u[n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{z-\alpha} \left(= z^{-1} \frac{z}{z-\alpha}\right)$

Note: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

Take Z-Transform of both sides of a Difference Equation. 2^{nd} order Example

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1]$$

$$Y(z)\left(1 + a_1 z^{-1} + a_2 z^{-2}\right) = X(z)\left(b_0 + b_1 z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \cdot \frac{z^2}{z^2}$$

$$= \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2}$$

poles = roots of denominator polynomial

$$(z - p_1)(z - p_2) = z^2 + a_1 z + a_2$$

Note: Obtain Discrete-Time Fourier Transform of $h(n)$ by substituting $z = e^{jw}$

$$H(w) = b_0 e^{+j2w} + b_1 e^{+jw} \Big/ e^{j2w} + a_1 e^{jw} + a_2$$

Partial Fraction Expansion **IF** you want to determine impulse response $h[n]$

$$H(z) = \frac{A_1}{z-P_1} + \frac{A_2}{z-P_2} \qquad A_k = H(z)(z-P_k)\Big|_{z=P_k}$$

(if degree of numerator polynomial $\geq$ degree of denominator polynomial, first do long division)

$$h[n] = A_1 (P_1)^{n-1} u[n-1] + A_2 (P_2)^{n-1} u[n-1]$$

<u>For stability</u> $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

- Need terms to decay $\Rightarrow$ asymptotically approach zero
- Need poles to each have/satisfy $|P_k| < 1$
- Need poles inside unit circle