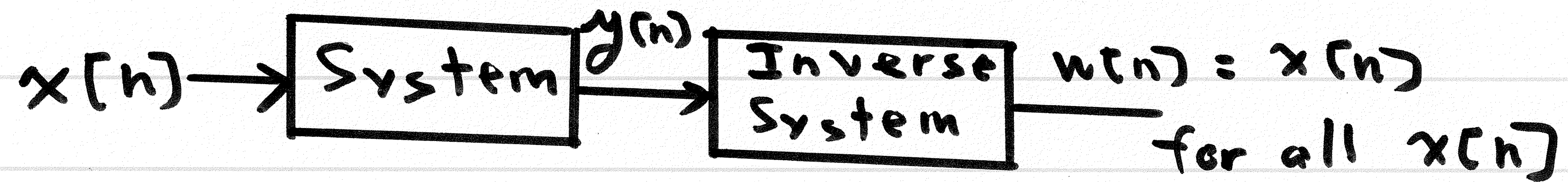
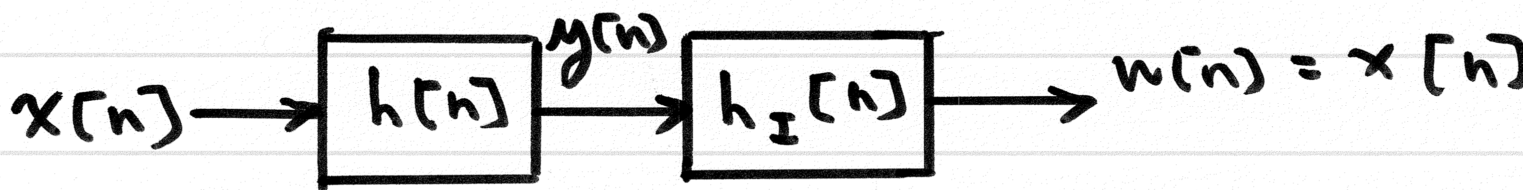
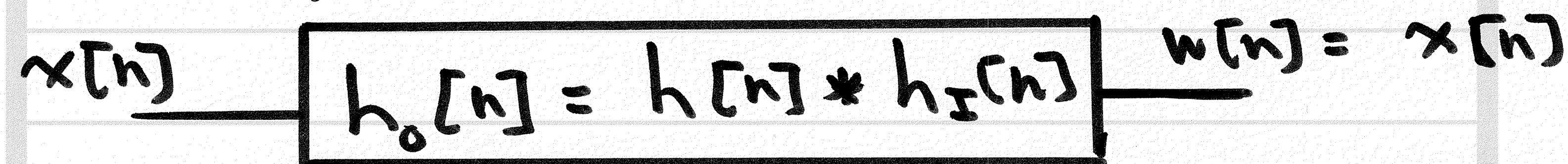
Invertibility and LTI Systems



- Above defines inverse system if it exists
- If the system is LTI, we have:



- Two LTI Systems in series can be replaced by a single LTI System with impulse response



- At the same time, we know $x[n] * \delta[n] = x[n]$
- Thus, if the inverse system exists for an LTI system, it must satisfy: $\boxed{h[n] * h_I[n] = \delta[n]}$

- **Example:**

$h[n] = u[n]$ and $h_I[n] = \delta[n] - \delta[n-1]$

form an inverse system pair

$$= \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \{\underset{\substack{\uparrow \\ n=0}}{1}, -1\}$$

$h[n] * h_I[n] =$

$u[n] * \{\delta[n] - \delta[n-1]\}$

$= u[n] - u[n-1]$

$= \delta[n]$ ✓ checks

- Note: $h[n] = u[n]$ is the impulse response of

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

and $h_I[n] = \delta[n] - \delta[n-1]$ is the impulse response of

$$y[n] = x[n] - x[n-1]$$

- Similarly for a CT LTI System with impulse response $h(t)$, if the inverse system exists, it must satisfy:

$$h(t) * h_I(t) = \delta(t)$$

where $h_I(t)$ is the impulse response of the inverse system

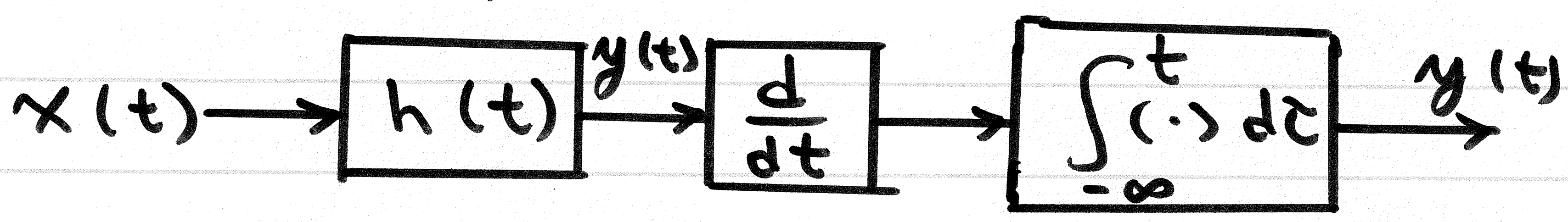
- Although it's peculiar to talk about the impulse response of a differentiator, one can use Leibniz's rule to show the following inverse system pair

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t} x(\tau)\,d\tau} \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow x(t)$$

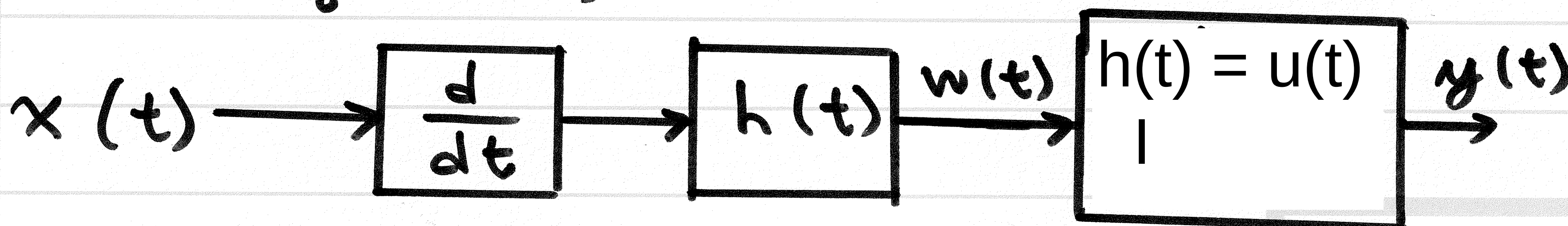
- since the order of LTI systems in series doesn't matter, we also have:

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \xrightarrow{y(t)} \boxed{\int_{-\infty}^{t} y(\tau)\,d\tau} \longrightarrow x(t)$$

• Consider an arbitrary LTI system whose impulse response is $h(t)$. We have:



• Since order of LTI Systems in series is inconsequential, we have:

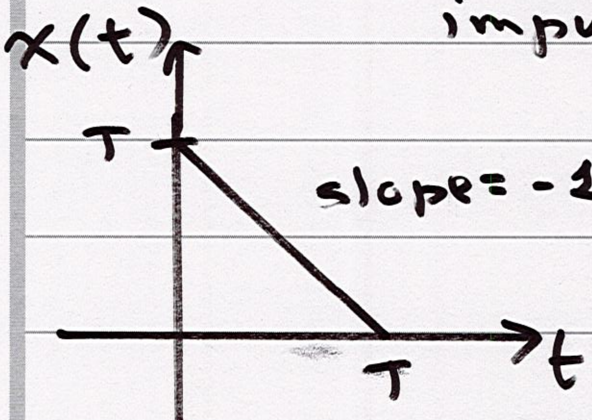


• This "trick" was sometimes used in ECE 202 to simplify the computation of $x(t) * h(t)$ if $x(t)$ was piecewise flat or piecewise linear

Example. Using "trick"

$$h_d(t) * u(t) = \delta(t)$$

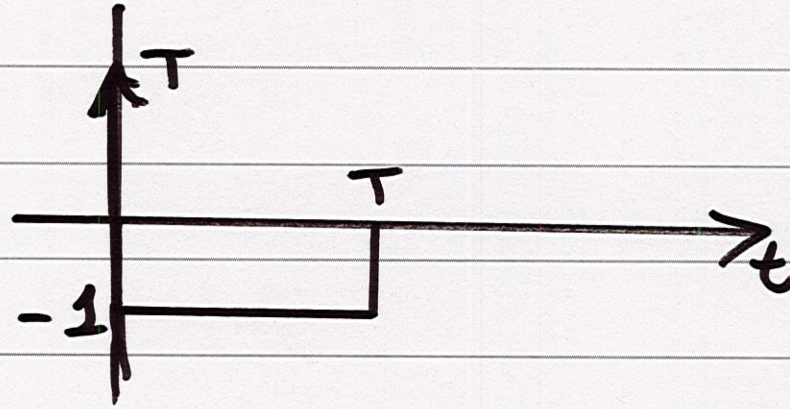
where: $\underbrace{h_d(t)}_{\text{impulse response of differentiator}} * x(t) = \frac{d}{dt} x(t)$



$$* \; h(t) = e^{at} u(t) = y(t) = ?$$

$$\begin{aligned}
y(t) &= x(t) * h(t) * \delta(t) \\
&= x(t) * \delta(t) * h(t) \\
&= x(t) * h_d(t) * u(t) * h(t) \\
&= \left\{ \frac{d}{dt} x(t) \right\} * h(t) * u(t)
\end{aligned}$$

$$\frac{d}{dt}x(t) = T\delta(t) - \{u(t) - u(t-T)\}$$



$$y(t) = \{T\delta(t) - (u(t) - u(t-T))\} * e^{at}u(t) * u(t$$

$$= T e^{at}u(t) * u(t) - e^{at}u(t) * u(t) * u(t)$$
$$+ e^{at}u(t) * u(t) * u(t-T)$$

$$\boxed{e^{at}u(t) * u(t) = \frac{1}{a}e^{at}u(t) - \frac{1}{a}u(t) = z(t)}$$

<u>Note:</u> $e^{at}u(t) * e^{at}u(t) = t e^{at}u(t)$

$a=0 \qquad u(t) * u(t) = t\,u(t)$

$$e^{at} u(t) * u(t) * u(t)$$

$$= \left(\frac{1}{a} e^{at} u(t) - \frac{1}{a} u(t) \right) * u(t)$$

$$= \frac{1}{a} \left\{ \frac{1}{a} e^{at} u(t) - \frac{1}{a} u(t) \right\} - \frac{1}{a} t\, u(t) = w(t)$$

Final answer:

$$y(t) = T z(t) - w(t) + w(t-T)$$

where: w(t) = (1/a) { z(t) - t u(t) }