Hmwk. 8 Help Chap. 7 Problems

- Basic Sampling Theory: all real-world signals are "band-limited"
  - $x_a(t) \xleftrightarrow{\mathcal{F}} X_a(\omega)$
  - prerequisite condition: $X_a(\omega) = 0$ for $|\omega| > \omega_M$
  - ω_M = max. frequency or bandwidth of signal
  - Nyquist rate: $\omega_s = 2\omega_M$ where: $\omega_s = 2\pi \frac{1}{T}$

    T = time between samples

    $1/T$ = samples per second (Hz)
  - As long as $\omega_s > 2\omega_M$

    ⇒ no aliasing

    ⇒ spectral replications at $m\omega_s$ (m = integer) don't overlap
- If $\omega_s < 2\omega_M$, aliasing starts at $\omega_s - \omega_M$
- There should be no energy above $\frac{\omega_s}{2}$, else aliasing will occur
- often first lowpass filter with cut-off at $\omega_s/2$ to avoid aliasing

• <u>Recall</u>: basic convolution result:

• If $x_1(t)$ is of duration T_1 secs
and $x_2(t)$ is of duration T_2 secs
$y(t) = x_1(t) * x_2(t)$ is (at most) of duration $T_1 + T_2$ secs

• <u>Recall</u>: FT property

$$z(t) = x_1(t)\,x_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega) = Z(\omega)$$

$$\left.\begin{array}{l} \text{If } X_1(\omega) = 0 \text{ for } |\omega| > \omega_{1_M} \\ \text{and } X_2(\omega) = 0 \text{ for } |\omega| > \omega_{2_M} \end{array}\right\} \begin{array}{l} \text{then } Z(\omega) = 0 \\ \text{for } |\omega| > \omega_{1_M} + \omega_{2_M} \end{array}$$

• Essentially, $X_1(\omega)$ of width $2\omega_{1_M}$ (centered at $\omega = 0$)
convolved with $X_2(\omega)$ of width $2\omega_{2_M}$ " "
is of width $2(\omega_{1_M} + \omega_{2_M})$ $\Rightarrow$ meaning $Z(\omega) \neq 0$
for $-(\omega_{1_M} + \omega_{2_M}) < \omega < \omega_{1_M} + \omega_{2_M}$

- Squaring is just a special case of $x_2(t) = x_1(t)$
- So, if $y(t) = x_1^2(t)$ with $X(\omega) = 0$ for $|\omega| > \omega_M$

then $Y(\omega) = 0$ for $|\omega| > 2\omega_M$

- squaring doubles the max. frequency

  $\Rightarrow$ doubles the bandwidth

- Similarly, cubing triples the bandwidth

- <u>In general</u>: nonlinear operations increase bandwidth
- What about $y(t) = x_1(t)\cos(\omega_0 t)$ $\Rightarrow x_2(t) = \cos(\omega_0 t)$
- in this case, ω_{2M} for $x_2(t) = \cos(\omega_0 t)$ is $\omega_{2M} = \omega_0$
- <u>Thus</u>: ω_M for $y(t)$ is $\omega_{1M} + \omega_0$ $\Rightarrow$ Nyquist rate $= 2(\omega_{1M} + \omega_0)$

• Running a signal through an LTI system $y(t) = x(t) * h(t)$ can <u>never</u> increase the max frequency (bandwidth)

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = X(\omega)\, H(\omega)$$

• If $X(\omega) = 0$ for $|\omega| > \omega_M$,

then $Y(\omega) = 0$ for $|\omega| > \omega_M$ $\Rightarrow$ guaranteed

• <u>However</u>, an LTI system can lower the bandwidth

$\Rightarrow$ a Lowpass Filter

• <u>Note</u>: differentiation is an LTI system

$$\frac{d\,x(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(\omega) \quad \Rightarrow H(\omega) = j\omega$$