Fourier Transform Examples. Hmwk.7 Help

Prob. 4.10 Find FT of: 1

$$x(t) = t\left(\frac{\sin(Wt)}{\pi t}\right)^2$$

- For Prob 4.10, $W=1$ => special case
- At least 2 ways to solve problem
- Method 1:

$$x(t) = t\,\frac{\sin(Wt)}{\pi t}\,\frac{\sin(Wt)}{\pi t}$$

$$= \frac{1}{\pi}\,\frac{\sin(Wt)}{\pi t}\,\sin(Wt)$$

recall:

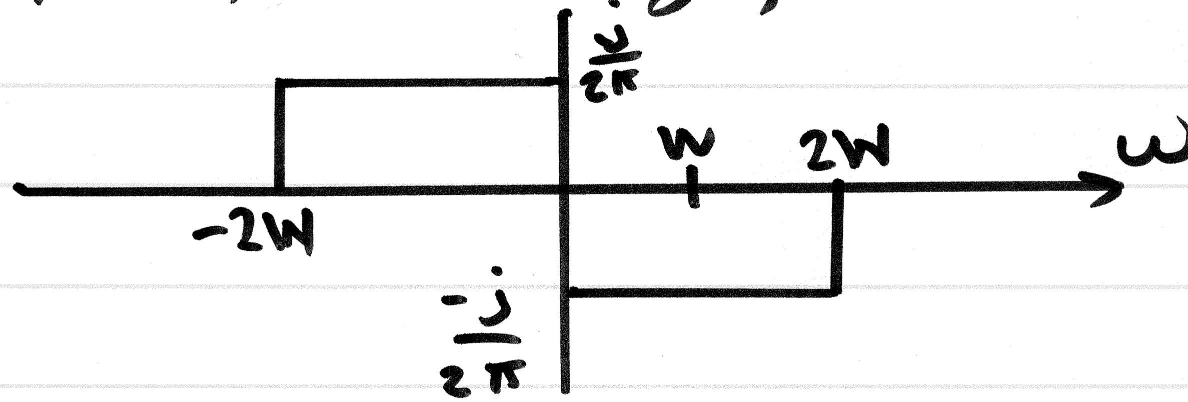
$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2W}\right)$$

$$x(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j}X(\omega-\omega_0) - \frac{1}{2j}X(\omega+\omega_0)$$

• Thus:

$$X(\omega) = \frac{1}{\pi}\left\{\frac{1}{2j}\,\text{rect}\left(\frac{\omega - W}{2W}\right) - \frac{1}{2j}\,\text{rect}\left(\frac{\omega + W}{2W}\right)\right\}$$

$\Rightarrow$ purely imaginary, since $x(t)$ is odd fn.

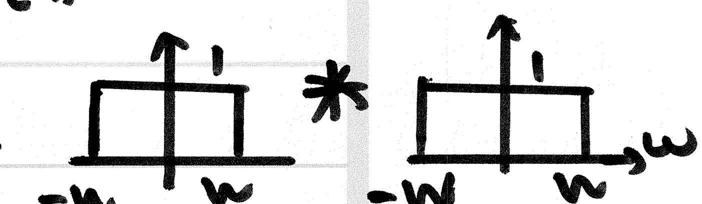


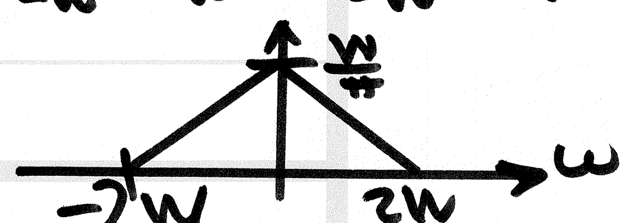
Method 2:

• recall: $x(t)\,y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega)$

• thus: $x^2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * X(\omega)$

$$\left\{\frac{\sin(Wt)}{\pi t}\right\}^2 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}$$





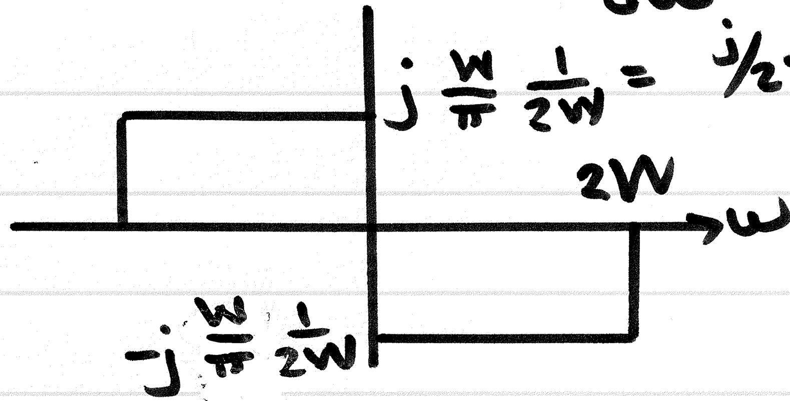
On crib sheet, have FT of $\frac{\sin(W_1 t)}{\pi t}\,\frac{\sin(W_2 t)}{\pi t}$

• recall: $t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\dfrac{dX(\omega)}{d\omega}$

③

• Thus:

slope $= \dfrac{\frac{W}{\pi}}{2W}$

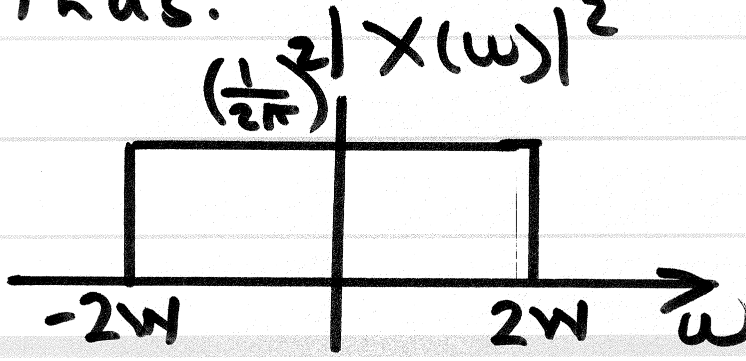


• same answer!

4.10 (b) Find: $A = \displaystyle\int_{-\infty}^{\infty} \left\{ t\left(\frac{\sin(Wt)}{\pi t}\right)^2 \right\}^2 dt$

Parseval's Theorem: $\displaystyle\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$

Thus:



Area under $= \left(\dfrac{1}{2\pi}\right)^2 (4W)$

$= \dfrac{W}{\pi^2}$

Prob. 4.12 (a) Find FT of $x(t) = t e^{-|t|}$

· again: $t\,x(t) \xleftrightarrow{\mathcal{F}} j \dfrac{dX(\omega)}{d\omega}$

Example 4.2 in text $\quad e^{-a|t|} \xleftrightarrow{\mathcal{F}} \dfrac{2a}{\omega^2 + a^2}$

Thus:
$$t e^{-|t|} \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega}\left\{\frac{2}{\omega^2+1}\right\} = j\,\frac{-2(2\omega)}{(\omega^2+1)^2}$$

(b) $\quad t e^{-|t|} \xleftrightarrow{\mathcal{F}} \dfrac{-4\omega j}{(1+\omega^2)^2}$

Recall Duality Property: $\quad x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$

$$X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$$

Thus:
$$\frac{-4jt}{(1+t^2)^2} \xleftrightarrow{\mathcal{F}} -\omega e^{-|\omega|}\ 2\pi \quad \Rightarrow \quad \frac{4t}{(1+t^2)^2} \xleftrightarrow{\mathcal{F}} -j\omega e^{-|\omega|}\ 2\pi$$

Prob. 4.27 $x(t) = u(t-1) - 2u(t-2) + u(t-3)$

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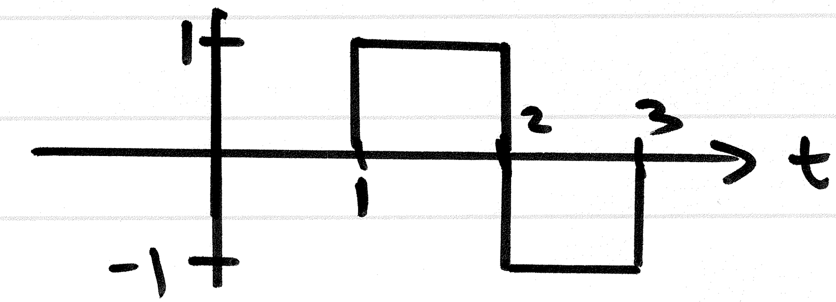


$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$ } replicates $x(t)$ above every T secs $\Rightarrow$ periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{k2\pi}{T}t}\,dt = \frac{1}{T}\int_{-\infty}^{\infty}\left\{x(t)\,\mathrm{rect}\left(\frac{t}{T}\right)e^{-j\omega t}\right\}dt$$

evaluated at $\omega = \frac{k2\pi}{T}$

Thus: $a_k = \frac{1}{T}\,\mathcal{F}\{\text{one period}\}\Big|_{\omega = k\frac{2\pi}{T}}$

• In this case: $x(t)$ is one period of $\tilde{x}(t)$

• $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow a_k = \frac{1}{T} X\left(k\frac{2\pi}{T}\right)$

• recall: $\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin\left(\frac{T}{2}\omega\right)}{\frac{\omega}{2}} = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$

$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$

• Thus: $X(\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} \left\{ e^{-j\frac{3}{2}\omega} - e^{-j\frac{5}{2}\omega} \right\}$

$= \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} e^{-j2\omega} \left\{ e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right\} \frac{2j}{2j}$

$= 4j \, \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega} e^{-j2\omega}$

$a_k = \frac{1}{T} 4j \, \frac{\sin^2\left(\frac{1}{2}k\frac{2\pi}{T}\right)}{k\,2\pi/T} e^{-j2k\frac{2\pi}{T}}$

Prob. 4.28.

For parts (i)-(iii), just use FT property

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$

- For parts (vi)-(viii), review Chap. 7 "Basic Sampling Theory" Handout

$$x(t)\,p(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}X(\omega) * P(\omega)$$

$$p(t)=\sum_{k=-\infty}^{\infty}\delta(t-nT) \xleftrightarrow{\mathcal{F}} P(\omega)=?$$

It's in Table 4.2 on page 329

$$=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\frac{2\pi}{T}t} \xleftrightarrow{\mathcal{F}} =\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\frac{2\pi}{T}\right)$$

periodic signal = FS expansion

FS coeffs $a_k=\frac{1}{T}$

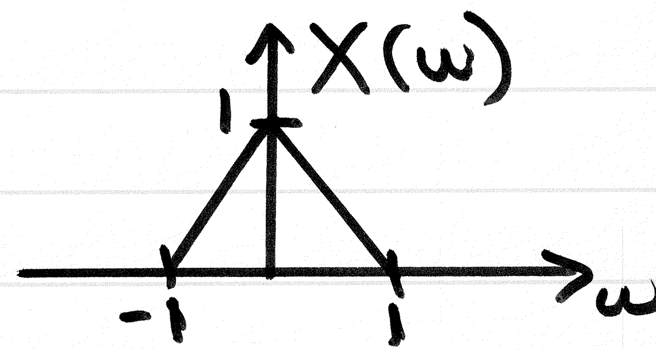
Thus:

$$x(t)\,p(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\cancel{2\pi}} X(\omega) * \frac{\cancel{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega) * \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$

For this problem,



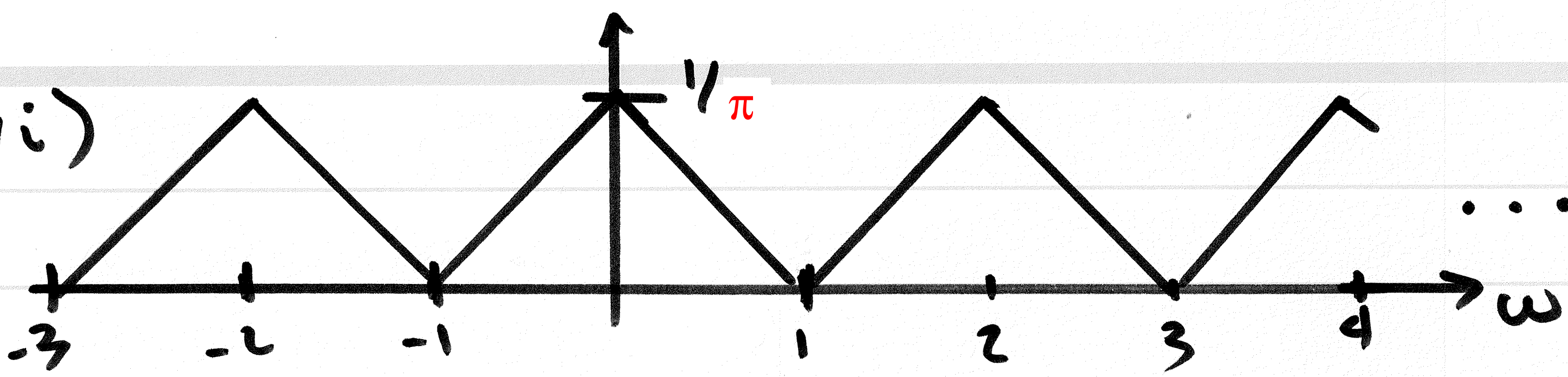
and

(vi) $T = \pi \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

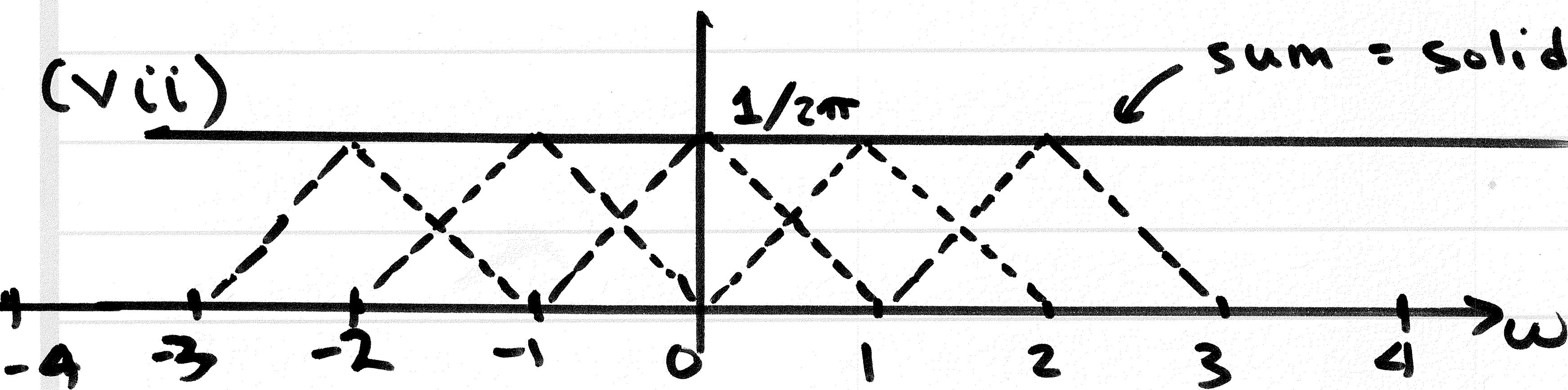
(vii) $T = 2\pi \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

(viii) $T = 4\pi \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{4\pi} = \frac{1}{2}$

(vi)



(vii)



(viii) same answer as (vii)

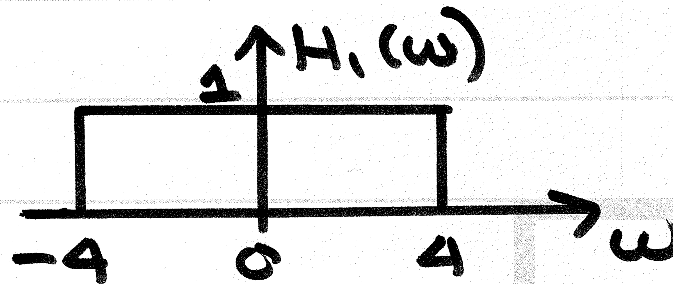
You get the double the number of triangles as occurring in in the answer to (vii), which can be viewed as the answer to (vii) plus the answer to (vi) shifted to the right by 1/2. That doubles the amplitude BUT the value of 1/T for (viii) is half of the value of 1/T for (vii) -- so you get the same amplitude.

Prob. 4.32 $\quad h(t) = \dfrac{\sin(4(t-1))}{\pi(t-1)}$

View as: $h(t) = \dfrac{\sin(4t)}{\pi t} * \delta(t-1)$

$$x(t) \longrightarrow \boxed{h_1(t) = \frac{\sin(4t)}{\pi t}} \xrightarrow{y_1(t)} \boxed{h_2(t) = \delta(t-1)} \longrightarrow y(t) = y_1(t-1)$$

$$h_1(t) = \frac{\sin(4t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$$



Recall for complex sinewave:

$$e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega_0)e^{j\omega_0 t}$$

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$$

- See Handout on Response of LTI Systems to Sinewaves. Generalizations

$$A\cos(\omega_0 t+\theta) \longrightarrow \underset{H(\omega)}{\boxed{h(t)}} \longrightarrow A\,|H(\omega_0)|\cos(\omega_0 t+\theta+\angle H(\omega_0))$$

$$H(\omega_0) = |H(\omega_0)|\, e^{j\angle H(\omega_0)}$$

polar form

$$A\sin(\omega_0 t+\theta) \longrightarrow \boxed{h(t)} \longrightarrow A\,|H(\omega_0)|\sin(\omega_0 t+\theta+\angle H(\omega_0))$$

For sum of sinewaves, invoke linearity

$\Rightarrow$ superposition

$\Rightarrow$ do one at a time and then sum

(a) $x_1(t) = \cos(6t + \frac{\pi}{2})$

since $H(6) = 0$, $y_1(t) = 0$

(b) $x_2(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \sin(k3t)$

$$= \underbrace{(1)\sin(0 \cdot t)}_{=0} + \underbrace{\frac{1}{2}\sin(3t)}_{\text{Passes}} + \underbrace{\frac{1}{4}\sin(6t) + \frac{1}{8}\sin(9t) + \ldots}_{\text{all rejected since } H(\omega)=0 \text{ for } \omega > 4}$$

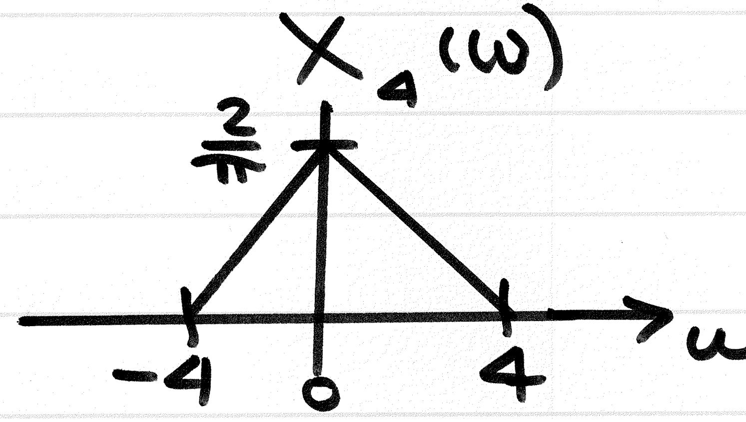
Final answer:

$$y_2(t) = \frac{1}{2}\sin(3(t-1))$$

(d) $x_4(t) = \left(\frac{\sin(2t)}{\pi t}\right)^2$

See handout on Fourier Transforms involving product of sinc functions

$$\left(\frac{\sin(2t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow}$$



Since $X_4(\omega) = 0$ for $\omega > 4$, passes right thru system $\Rightarrow$ ultimately delayed by 1

$$y_4(t) = \left(\frac{\sin(2(t-1))}{\pi(t-1)}\right)^2$$

See next page. This is a precursor -- not actually the problem

- For Hmwk. 7 (later), Prob. 4.12 uses Duality to find the FT of $x(t) = \dfrac{2a}{t^2+a^2}$ $a = \text{const.}$

$$X(\omega) = \int_{-\infty}^{\infty} \left(\frac{2a}{t^2+a^2}\right) e^{-j\omega t}\, dt = ?$$

hard integral to do!

- instead, employ duality:

Since: $$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$$

Then: $$\frac{2a}{t^2+a^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

Divide by 2 on both sides

$$\frac{a}{t^2+a^2} \longleftrightarrow \pi e^{-a|\omega|}$$

Prob. 4.12 $\quad e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{2}{1+\omega^2}$

(a) $t\, e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow}$?!

$$= j \frac{d}{d\omega}\left\{\frac{2}{1+\omega^2}\right\}$$

$$= j\, 2(-1)\, 2\omega \frac{1}{(1+\omega^2)^2}$$

(b) Duality Property:

$$-j4 \frac{t}{(1+t^2)^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi(-\omega)\, e^{-|-\omega|}$$

Multiply both sides by j:

$$\frac{4t}{(1+t^2)^2} \overset{\mathcal{F}}{\longleftrightarrow} -j 2\pi \omega e^{-|\omega|}$$