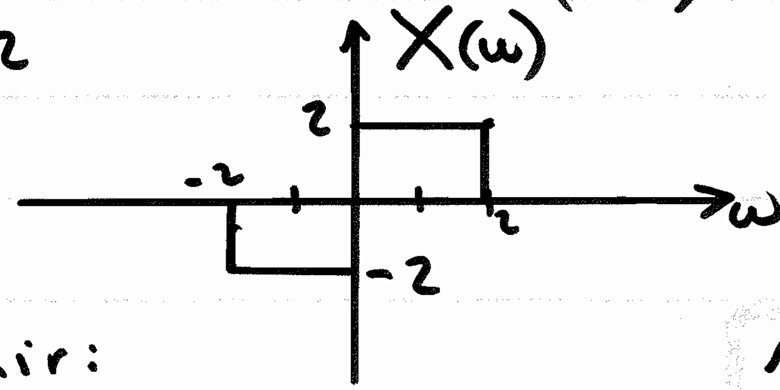
# Example Problems on Fourier Transform (FT) Using FT Pairs and FT Properties

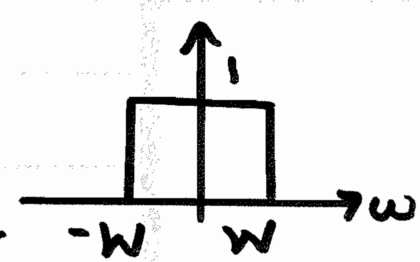
Prob. 4.4 (Hmwk 6) Part (b)

$$X(\omega)=\begin{cases}2, & 0\le\omega\le 2\\ -2, & -2\le\omega<0\\ 0, & |\omega|>2\end{cases} \quad = -2\,\text{rect}\left(\frac{\omega+1}{2}\right)+2\,\text{rect}\left(\frac{\omega-1}{2}\right)$$




Recall basic FT pair:

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \text{rect}\left(\frac{\omega}{2W}\right)$$




$\boxed{\text{Consider } W=1}$

- Also, recall FT property: $e^{j\omega_0 t}x(t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$

consider $\omega_0=1$, $\omega_0=-1$

• also, using linearity property of FT, we have:

$$x(t) = -2\,\frac{\sin(t)}{\pi t}e^{-jt} + 2\,\frac{\sin(t)}{\pi t}e^{jt}$$

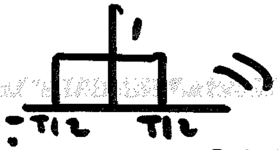
$$= \frac{\sin(t)}{\pi t}\,4j\left\{\frac{e^{jt}-e^{-jt}}{2j}\right\}$$

$$= 4j\,\frac{\sin(t)}{\pi t}\sin(t) = 4j\,\frac{\sin^2(t)}{\pi t}$$

<u>Prob. 4.21</u> Hmwk 6: a, b, c, e, g

Do part (c) first:

$$x(t) = \begin{cases} 1+\cos(\pi t), & |t|<1 \\ 0, & |t|>1 \end{cases} = \big(1+\cos(\pi t)\big)\,\mathrm{rect}\left(\frac{t}{2}\right)$$




- Basic FT pair: $\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$

- Basic FT property: $e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$

leads to:

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(\omega+\omega_0)+\frac{1}{2}X(\omega-\omega_0)$$

since $\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$ and linearity prop. of FT

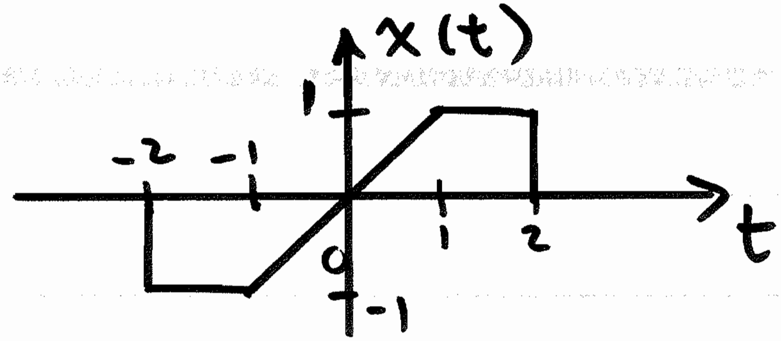
Consider $T=2$ and $\omega_0=\pi$ and $\omega_0=0$:

$$X(\omega) = \frac{\sin(\omega)}{\omega/2} + \frac{1}{2}\,\frac{\sin(\omega+\pi)}{\frac{(\omega+\pi)}{2}} + \frac{1}{2}\,\frac{\sin(\omega-\pi)}{\frac{(\omega-\pi)}{2}}$$

$$= \sin(\omega)\left\{\frac{2}{\omega} - \frac{1}{\omega+\pi} - \frac{1}{\omega-\pi}\right\}$$

$$= \sin(\omega)\left\{\frac{2}{\omega} - \frac{2\omega}{\omega^2-\pi^2}\right\} = 2\sin(\omega)\left\{\frac{1}{\omega} - \frac{\omega}{\omega^2-\pi^2}\right\}$$

Prob. 4.21 (g)




$$x(t) = -\,\mathrm{rect}\left(\frac{t+\frac{3}{2}}{1}\right) + t\,\mathrm{rect}\left(\frac{t}{2}\right) + \mathrm{rect}\left(\frac{t-\frac{3}{2}}{1}\right)$$

FT pair: $\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{\sin\left(T\frac{\omega}{2}\right)}{\omega/2}$

FT property: $x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$

$$t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega)$$

$$X(\omega) = -\frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} e^{j\frac{3}{2}\omega} + j\frac{d}{d\omega}\left\{\frac{\sin(\omega)}{\omega/2}\right\} + \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} e^{-j\frac{3}{2}\omega}$$

$$= -4j\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}\sin\left(\frac{3}{2}\omega\right) + j2\left\{\frac{\cos(\omega)\,\omega - \sin(\omega)}{\omega^2}\right\}$$

- Other variation on modulation property:

$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}X(\omega-\omega_0) - \frac{1}{2j}X(\omega+\omega_0)$$

Use for Probs. 4.21 (b) and (e)

4.21 (b) $\quad e^{-3|t|}\sin(2t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$

FT pair from Ex. 4.2 in text on pg. 292:

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$$

THUS:

$$X(\omega) = \frac{1}{2j}\frac{2(3)}{(\omega-2)^2+3^2} - \frac{1}{2j}\frac{2(3)}{(\omega+2)^2+3^2}$$

$$= 3j\left\{\frac{1}{(\omega+2)^2+9} - \frac{1}{(\omega-2)^2+9}\right\}$$

Prob. 4.21 (e)

$$x(t) = \left(t e^{-2t} u(t)\right) \sin(4t)$$

$$t e^{-at} u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{(a+j\omega)^2} \qquad a = 2$$

$$\omega_0 = 4$$

$$x(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j} X(\omega - \omega_0) - \frac{1}{2j} X(\omega + \omega_0)$$

$$X(\omega) = \frac{1}{2j} \frac{1}{\left(2 + j(\omega - 4)\right)^2} - \frac{1}{2j} \frac{1}{\left(2 + j(\omega + 4)\right)^2}$$

**Prob. 4.38** We previously proved modulation property as requested in part (a):

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$X(\omega-\omega_0) = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left( x(t) e^{j\omega_0 t} \right) e^{-j\omega t}\, dt$$

Since FT is unique, this proves:

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$$

Part (b): prove same property using multiplication property: $x(t)\, y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$

So, let: $y(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega-\omega_0)$

Thus:

$$x(t)\,e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * 2\pi\delta(\omega-\omega_0)$$

$$= X(\omega) * \delta(\omega-\omega_0)$$

$$= X(\omega-\omega_0)$$

Alternative proof of modulation property.

- On an exam, simply use modulation property

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} X(\omega+\omega_0) + \frac{1}{2} X(\omega-\omega_0)$$

- no need to use multiplication property if multiplying by a sinewave

Prob. 4.39 See proof of Duality Property posted at course web site under "Chapter 4"

If: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

Then $X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$

For example: $\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$

Thus: $\dfrac{\sin\left(T\frac{t}{2}\right)}{\frac{t}{2}} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \mathrm{rect}\left(\frac{-\omega}{T}\right)$

We previously showed with $T=2W$:

$$\frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2W}\right)$$

- For part (b), alternative derivation of the FT of a sinewave $e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$ using Duality Property and basic FT pair

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

Time-Shift Property dictates: $\boxed{x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)}$

$$\delta(t+B) \overset{\mathcal{F}}{\longleftrightarrow} e^{jB\omega}(1) = e^{jB\omega}$$

- Applying Duality Property:

$$e^{jBt} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(-\omega+B)$$

- note: Dirac Delta fn. is a symmetric fn.: $\delta(-t) = \delta(t)$, $\delta(-\omega) = \delta(\omega)$
- Thus:

$$e^{jBt} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega - B)$$

- which we had proved previously $(\omega_0 = B)$

- For Hmwk. 7 (later), Prob. 4.12 uses Duality to find the FT of $x(t) = \frac{2a}{t^2+a^2}$ $a = \text{const.}$

$$X(\omega) = \int_{-\infty}^{\infty} \left(\frac{2a}{t^2+a^2}\right) e^{-j\omega t}\, dt = ?$$

hard integral to do!

- instead, employ duality:

Since: $e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$

Then: $\frac{2a}{t^2+a^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$

Divide by 2 on both sides

$$\frac{a}{t^2+a^2} \longleftrightarrow \pi e^{-a|\omega|}$$

**Prob. 4.23** exercises on even and odd part properties of FT which are proved in a separate handout posted at course web site

- For $x(t)$ real-valued, if $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$

then:

even part of $x(t)$

$$x_e(t) = \frac{x(t) + x(-t)}{2} \xleftrightarrow{\mathcal{F}} \mathrm{Re}\{X(\omega)\} = \text{real part of } X(\omega)$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} \xleftrightarrow{\mathcal{F}} j\,\mathrm{Im}\{X(\omega)\} = \text{imaginary part of } X(\omega)$$

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Note: for $x(t)$ real-valued: $X(-\omega) = X^*(\omega)$

$\Rightarrow |X(-\omega)| = |X(\omega)| \Rightarrow$ magnitude is even fn. of $\omega$

$\Rightarrow \angle X(-\omega) = -\angle X(\omega) \Rightarrow$ phase is odd fn. of $\omega$

$$\boxed{X(\omega) = |X(\omega)| e^{j\angle X(\omega)}}$$

polar form

Prob. 4.23 (cont.) $x_o(t) = \begin{cases} e^{-t}, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$

$$x_o(t) = e^{-t}\{u(t) - u(t-1)\}$$

$$= e^{-t}u(t) - e^{-1}e^{-(t-1)}u(t-1)$$

Since: $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{a+j\omega}$ $\boxed{x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(\omega)}$

We have: $X_o(\omega) = \dfrac{1}{1+j\omega}\left(1 - e^{-1}e^{-j\omega}\right)$

$$= \frac{1 - e^{-1}(\cos\omega - j\sin\omega)}{1+j\omega} \cdot \frac{(1-j\omega)}{(1-j\omega)}$$

Note:
x_1 (t) is two times the even part of x_o (t) and
x_2 (t) is two times the imaginary part of x_o (t)

$$= \underbrace{\frac{2 - 2e^{-1}\cos(\omega) - 2e^{-1}\omega\sin(\omega)}{1+\omega^2}}_{\mathrm{Re}\{X_o(\omega)\}} + j\underbrace{\left\{\frac{-2\omega + 2e^{-1}\sin(\omega) + 2e^{-1}\omega\cos(\omega)}{1+\omega^2}\right\}}_{\mathrm{Im}\{X_o(\omega)\}}$$

## Example. Prob. 4.21 (d)

$$x(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - kT) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$$

Basic FT Pair: $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$

Basic FT Property: $x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) e^{-j\omega t_0}$

Thus: $\delta(t - kT) \overset{\mathcal{F}}{\longleftrightarrow} 1 \cdot e^{-j\omega kT}$

$$= \left(e^{-j\omega T}\right)^k$$

Since FT satisfies linearity property

$$X(\omega) = \sum_{k=0}^{\infty} \alpha^k e^{-j\omega kT} = \sum_{k=0}^{\infty} \left(\alpha e^{-j\omega T}\right)^k$$

$$= \frac{1}{1 - \alpha e^{-j\omega T}} \quad \text{since } |\alpha| < 1$$

## Example. Prob. 4.21 (i)

$$x(t) = \begin{cases} 1 - t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} = (1 - t^2)\, \mathrm{rect}\left(\frac{t - \frac{1}{2}}{1}\right)$$

$$= \mathrm{rect}\left(\frac{t - \frac{1}{2}}{1}\right) - t^2\, \mathrm{rect}\left(\frac{t - \frac{1}{2}}{1}\right)$$

Basic FT pair: $\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \dfrac{\sin\left(\frac{\omega}{2}T\right)}{\omega/2}$ $\left(\begin{matrix}T = 1\\ \text{here}\end{matrix}\right)$

Two FT properties:

$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(\omega)\, e^{-j\omega t_0}$ and $t\, x(t) \xleftrightarrow{\mathcal{F}} j\, \dfrac{d\, X(\omega)}{d\omega}$

Generalization: $t^n x(t) \xleftrightarrow{\mathcal{F}} (j)^n \dfrac{d^n X(\omega)}{d\omega^n}$

Thus:

$$X(\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} e^{-j\omega\frac{1}{2}} - (-1)\frac{d^2}{d\omega^2}\left\{\frac{\sin\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}}}{\omega/2}\right\}$$

Look at term to be differentiated:

$$\sin\left(\frac{\omega}{2}\right) = \frac{1}{2j} e^{j\frac{\omega}{2}} - \frac{1}{2j} e^{-j\frac{\omega}{2}}$$

Thus, we have:

$$+j \frac{d^2}{d\omega^2}\left\{\frac{1 - e^{-j\omega}}{\omega}\right\} = -j \frac{d^2}{d\omega^2}\left(\omega^{-1}\right) + j \frac{d^2}{d\omega^2}\left\{\frac{e^{-j\omega}}{\omega}\right\}$$

$$= -j \frac{2}{\omega^3} + j \frac{d}{d\omega}\left\{\frac{-j\omega e^{-j\omega} - e^{-j\omega}}{\omega^2}\right\}$$

$$= -j \frac{2}{\omega^3} + \frac{d}{d\omega}\left\{\frac{e^{-j\omega}}{\omega}\right\} - j \frac{d}{d\omega}\left\{\frac{e^{-j\omega}}{\omega^2}\right\}$$

$$= -j \frac{2}{\omega^3} + \left\{\frac{\omega e^{-j\omega} - e^{-j\omega}}{\omega}\right\} - j \left\{\frac{\omega^2 e^{-j\omega} - 2\omega e^{-j\omega}}{\omega^4}\right\}$$

Still some sign errors and j's missing in last line - you can fix :)