Spring 2024

ECE 301 HW9 Solutions

5.1) $X(\omega) = \sum_n x[n] e^{-j\omega n}$

a: $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$

$$X(\omega) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} e^{-j\omega n} \qquad \text{Let } l = n-1 \qquad n = l+1$$

$$= \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)^{l} e^{-j\omega(l+1)}$$

$$= e^{-j\omega} \sum_{l=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^{l} = e^{-j\omega} \frac{1}{1 - \frac{1}{2} e^{-j\omega}} = \frac{e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}}$$

b: $x[n] = \left(\frac{1}{2}\right)^{|n-1|}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n-1|} e^{-j\omega n} = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|m|} e^{-j\omega(m+1)}$$

Let $m = n-1$

$n = m+1$

$$= e^{-j\omega}\left[\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m} e^{-j\omega m} + \sum_{m=-\infty}^{-1} \left(\frac{1}{2}\right)^{-m} e^{-j\omega m} \right]$$

Let $l = -m$

$$= e^{-j\omega}\left[\sum_{m=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^{m} + \sum_{l=1}^{\infty} \left(\frac{1}{2} e^{j\omega}\right)^{l} \right]$$

$$= e^{-j\omega}\left[\frac{1}{1 - \frac{1}{2} e^{-j\omega}} + \sum_{l=0}^{\infty} \left(\frac{1}{2} e^{j\omega}\right)^{l} - \underbrace{1}_{\left(\frac{1}{2} e^{j\omega}\right)^{l}\big|_{l=0}} \right]$$

$$= e^{-j\omega}\left[\frac{1}{1 - \frac{1}{2} e^{-j\omega}} + \frac{1}{1 - \frac{1}{2} e^{j\omega}} - 1 \right]$$

$$= e^{-j\omega}\left[\frac{1 - \frac{1}{2} e^{j\omega} + 1 - \frac{1}{2} e^{-j\omega} - \left(1 - \frac{1}{2} e^{j\omega}\right)\left(1 - \frac{1}{2} e^{-j\omega}\right)}{\left(1 - \frac{1}{2} e^{-j\omega}\right)\left(1 - \frac{1}{2} e^{j\omega}\right)} \right]$$

$$= e^{-j\omega}\left[\frac{2 - \cos(\omega) - \left(1 - \frac{1}{2} e^{-j\omega} - \frac{1}{2} e^{j\omega} + \frac{1}{4}\right)}{1 - \frac{1}{2} e^{-j\omega} - \frac{1}{2} e^{j\omega} + \frac{1}{4}} \right]$$

$$= e^{-j\omega}\left[\frac{\frac{3}{4} - \cos(\omega) + \cos(\omega)}{\frac{5}{4} - \cos(\omega)} \right] = \frac{3 e^{-j\omega}}{5 - 4\cos(\omega)}$$

5.2) $X(\omega) = \sum_n x[n] e^{-j\omega n}$

a: $x[n] = \delta[n-1] + \delta[n+1]$

$$X(\omega) = \sum_n \left(\delta[n-1] + \delta[n+1]\right) e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=1} + e^{-j\omega n}\Big|_{n=-1} \quad \text{from the two } \delta\text{'s}$$

$$= e^{-j\omega} + e^{j\omega} = 2\cos(\omega)$$

b: $x[n] = \delta[n+2] + \delta[n-2]$

$$X(\omega) = \sum_n \left(\delta[n+2] + \delta[n-2]\right) e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=-2} + e^{-j\omega n}\Big|_{n=2} \quad \text{from the two } \delta\text{'s}$$

$$= e^{j2\omega} + e^{-j2\omega} = 2\cos(2\omega)$$

5.21) a: $x[n] = u[n-2] - u[n-6]$

$$= \begin{cases} 1 & 2 \le n \le 5 \\ 0 & ow \end{cases}$$

Let $x_0[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & ow \end{cases}$ $\quad x[n] = x_0[n-2] \quad N = 4$

$$X_0(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$X(\omega) = e^{-j2\omega} X_0(\omega)$$

$$= \frac{e^{-j2\omega} - e^{-j6\omega}}{1-e^{-j\omega}} = e^{-j4\omega}\frac{e^{j2\omega}-e^{-j2\omega}}{1-e^{-j\omega}} = e^{-j4\omega}\frac{2j\sin(2\omega)}{1-e^{-j\omega}}$$

b: $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$

$$X(\omega) = \sum_n \left(\tfrac{1}{2}\right)^{-n} u[n-1]\, e^{-j\omega n}$$

Let $l = -n$

$$= \sum_l \left(\tfrac{1}{2}\right)^l u[l-1]\, e^{+j\omega l} = \sum_{l=1}^{\infty} \left(\tfrac{1}{2}\right)^l e^{j\omega l}$$

$$= \sum_{l=0}^{\infty} \left(\tfrac{1}{2} e^{j\omega}\right)^l - 1 = \frac{1}{1-\frac{1}{2}e^{j\omega}} - 1$$

$$= \frac{1-\left(1-\frac{1}{2}e^{j\omega}\right)}{1-\frac{1}{2}e^{j\omega}} = \frac{\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}$$

c: $x[n] = \left(\frac{1}{3}\right)^{|n|} u[-n-2]$

$$X(\omega) = \sum_{n=-\infty}^{-2} \left(\tfrac{1}{3}\right)^{|n|} e^{-j\omega n} = \sum_{l=2}^{\infty} \left(\tfrac{1}{3}\right)^{|l|} e^{j\omega l}$$

Let $l = -n$

$$= \sum_{l=0}^{\infty} \left(\tfrac{1}{3}\right)^l e^{j\omega l} - \tfrac{1}{3}e^{j\omega} - 1$$

$$= \frac{1}{1-\frac{1}{3}e^{j\omega}} - \frac{1}{3}e^{j\omega} - 1 = \frac{1-\frac{1}{3}e^{j\omega}+\frac{1}{9}e^{j2\omega}-1+\frac{1}{3}e^{j\omega}}{1-\frac{1}{3}e^{j\omega}}$$

$$= \frac{\frac{1}{9}e^{j2\omega}}{1-\frac{1}{3}e^{j\omega}}$$

d: $x[n] = 2^n \sin(\frac{\pi}{4}n)\, u[-n]$

$= (\frac{1}{2})^{-n} \sin(\frac{\pi}{4}n)\, u[-n]$

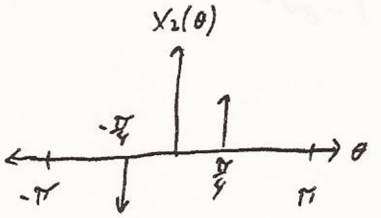
Let $x_1[n] = (\frac{1}{2})^{-n} u[-n]$ & $x_2[n] = \sin(\frac{\pi}{4}n)$

$$X_1(\omega) = \sum_{n=-\infty}^{0} (\tfrac{1}{2})^{-n} e^{-j\omega n} = \sum_{l=0}^{\infty} (\tfrac{1}{2} e^{j\omega})^{l} = \frac{1}{1-\frac{1}{2}e^{j\omega}}$$

Let $l = -n$

$$X_2(\omega) = \frac{\pi}{j} \sum_{l} \left[\delta(\omega - \tfrac{\pi}{4} - 2\pi l) - \delta(\omega + \tfrac{\pi}{4} - 2\pi l)\right]$$ from Table 5.2

$$X(\omega) = \frac{1}{2\pi} X_1(\omega) \circledast X_2(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(\theta) X_1(\omega-\theta)\, d\theta$$



$$= \frac{1}{2\pi}\left[\frac{\pi}{j} X_1(\omega - \tfrac{\pi}{4}) - \frac{\pi}{j} X_1(\omega + \tfrac{\pi}{4})\right] = \frac{1}{2j}\left(\frac{1}{1-\frac{1}{2}e^{j(\omega-\frac{\pi}{4})}} - \frac{1}{1-\frac{1}{2}e^{j(\omega+\frac{\pi}{4})}}\right)$$

$$= \frac{1}{2j}\left[\frac{1-\frac{1}{2}e^{j\frac{\pi}{4}}e^{j\omega} - 1 + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{j\omega}}{1-\frac{1}{2}e^{j\frac{\pi}{4}}e^{j\omega} - \frac{1}{2}e^{-j\frac{\pi}{4}}e^{j\omega} + \frac{1}{4}e^{j2\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{-\frac{1}{2}e^{j\omega}(e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}})}{1-\frac{1}{2}e^{j\omega}(e^{j\frac{\pi}{4}} + e^{-j\frac{\pi}{4}}) + \frac{1}{4}e^{j2\omega}}\right] = \frac{1}{2j}\left[\frac{-j\sin(\frac{\pi}{4})e^{j\omega}}{1-\cos(\frac{\pi}{4})e^{j\omega} + \frac{1}{4}e^{j2\omega}}\right]$$

$$= \frac{-\frac{1}{2\sqrt{2}}e^{j\omega}}{1 - \frac{1}{\sqrt{2}}e^{j\omega} + \frac{1}{4}e^{j2\omega}}$$

e: $x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$

Let $x_1[n] = \left(\frac{1}{2}\right)^{|n|}$ & $x_2[n] = \cos\left(\frac{\pi}{8}(n-1)\right)$

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} \left(\tfrac{1}{2}\right)^{-n} e^{-j\omega n}$$

Let $\ell = -n$

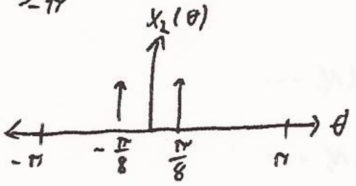
$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} e^{-j\omega}\right)^{n} + \sum_{\ell=0}^{\infty} \left(\tfrac{1}{2} e^{j\omega}\right)^{\ell} - 1$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{2}e^{j\omega}} - 1 = \frac{3}{5-4\cos(\omega)}$$

$$X_2(\omega) = e^{-j\omega}\, \pi \sum_{\ell} \left[\delta\left(\omega - \tfrac{\pi}{8} - 2\pi\ell\right) + \delta\left(\omega + \tfrac{\pi}{8} - 2\pi\ell\right)\right]$$

$$X(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(\theta)\, X_1(\omega-\theta)\, d\theta = \frac{e^{-j\omega}}{2\pi}\left[\pi X_1\left(\omega-\tfrac{\pi}{8}\right) + \pi X_1\left(\omega+\tfrac{\pi}{8}\right)\right]$$

$$= \frac{e^{-j\omega}}{2}\left[\frac{3}{5-4\cos\left(\omega-\frac{\pi}{8}\right)} + \frac{3}{5-4\cos\left(\omega+\frac{\pi}{8}\right)}\right]$$



f: $x[n] = \begin{cases} n & -3 \le n \le 3 \\ 0 & \text{o w} \end{cases}$ $\quad x[n] = n\, x_1[n] \quad x_1[n] = \begin{cases} 1 & -3 \le n \le 3 \\ 0 & \text{o w} \end{cases}$

$$X_1(\omega) = \frac{\sin\left(\omega\left(3+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$ from Table 5-2

$$X(\omega) = j\frac{d}{d\omega} X_1(\omega) = j\frac{d}{d\omega}\, \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} = j\left[\frac{\sin\left(\frac{1}{2}\omega\right)\frac{7}{2}\cos\left(\frac{7}{2}\omega\right) - \sin\left(\frac{7}{2}\omega\right)\frac{1}{2}\cos\left(\frac{1}{2}\omega\right)}{\sin^2\left(\frac{1}{2}\omega\right.}\right]$$

g: $x[n] = \sin\left(\frac{\pi}{2}n\right) + \cos(n)$

$$X(\omega) = \frac{\pi}{j}\sum_k\left[\delta\left(\omega-\tfrac{\pi}{2}-2\pi k\right) - \delta\left(\omega+\tfrac{\pi}{2}-2\pi k\right)\right] + \pi\sum_k\left[\delta(\omega-1-2\pi k) + \delta(\omega+1-2\pi k)\right]$$

$$= \pi\sum_k\left[\tfrac{1}{j}\delta\left(\omega-\tfrac{\pi}{2}-2\pi k\right) - \tfrac{1}{j}\delta\left(\omega+\tfrac{\pi}{2}-2\pi k\right) + \delta(\omega-1-2\pi k) + \delta(\omega+1-2\pi k)\right]$$

h: $x[n] = \sin\left(\frac{5\pi}{3}n\right) + \cos\left(\frac{7\pi}{3}n\right)$

$$= -\sin\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}n\right)$$

$$= \pi \sum_k \left[\frac{1}{j}\delta\left(w - \frac{\pi}{3} - 2\pi k\right) + \frac{1}{j}\delta\left(w + \frac{\pi}{3} - 2\pi k\right) + \delta\left(w - \frac{\pi}{3} - 2\pi k\right) + \delta\left(w + \frac{\pi}{3} - 2\pi k\right)\right]$$

$$= \pi \sum_k \left[\left(\frac{j-1}{j}\right)\delta\left(w - \frac{\pi}{3} - 2\pi k\right) + \left(\frac{j+1}{j}\right)\delta\left(w + \frac{\pi}{3} - 2\pi k\right)\right]$$

i: $x[n]$ periodic with $N = 6$

$x_0[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{ow} \end{cases}$ Let $x_1[n] = \begin{cases} 1 & |n| \le 2 \\ 0 & \text{ow} \end{cases}$

$x_1[n]$ periodic with period 6 $\quad N_1 = 2$

$X_1(w) = 2\pi \sum_k a_k \delta\left(w - \frac{2\pi k}{2}\right)$ from Table 5.2

$$a_k = \begin{cases} \dfrac{\sin\left[\left(\frac{2\pi k}{2}\right)\left(2 + \frac{1}{2}\right)\right]}{6 \sin\left[\frac{\pi k}{6}\right]} & k \neq 0, \pm N, \pm 2N, \dots \\ \dfrac{5}{6} & k = 0, \pm N, \pm 2N, \dots \end{cases}$$

$x[n] = x_1[n-2]$

$X(w) = e^{-j2w} X_1(w)$

j: $x[n] = (n-1)\left(\frac{1}{3}\right)^{|n|}$

$= n\left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{3}\right)^{|n|}$

Let $x_0[n] = \left(\frac{1}{3}\right)^{|n|}$

$$X_0(\omega) = \sum_n \left(\tfrac{1}{3}\right)^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{3}\right)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} \left(\tfrac{1}{3}\right)^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{3}e^{-j\omega}\right)^n + \sum_{l=0}^{\infty} \left(\tfrac{1}{3}e^{j\omega}\right)^l - 1$$

$$= \frac{1}{1-\frac{1}{3}e^{-j\omega}} + \frac{1}{1-\frac{1}{3}e^{j\omega}} - 1$$

$$= \frac{1-\frac{1}{3}e^{j\omega} + 1 - \frac{1}{3}e^{-j\omega} - \left(1 - \frac{1}{3}e^{j\omega} - \frac{1}{3}e^{-j\omega} + \frac{1}{9}\right)}{1 - \frac{1}{3}e^{j\omega} - \frac{1}{3}e^{-j\omega} + \frac{1}{9}}$$

$$= \frac{\frac{8}{9} - \frac{2}{3}\cos(\omega) + \frac{2}{3}\cos(\omega)}{\frac{10}{9} - \frac{2}{3}\cos(\omega)} = \frac{\frac{8}{9}}{\frac{10}{9} - \frac{2}{3}\cos(\omega)} = \frac{8}{10-6\cos(\omega)} = 8\left(10-6\cos(\omega)\right)^{-1}$$

$$X(\omega) = j\frac{d}{d\omega}X_0(\omega) - X_0(\omega)$$

$$= j\left[-8\left(10-6\cos(\omega)\right)^{-2} 6\sin(\omega)\right] - \frac{8}{10-6\cos(\omega)}$$

$$= \frac{-j48\sin(\omega)}{\left(10-6\cos(\omega)\right)^2} - \frac{8}{10-6\cos(\omega)} = \frac{-80+48\cos(\omega) - j48\sin(\omega)}{\left(10-6\cos(\omega)\right)^2}$$

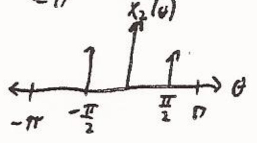
k: $x[n] = \frac{\sin\left(\frac{\pi n}{5}\right)}{\pi n}\cos\left(\frac{7\pi}{2}n\right) = \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n}\cos\left(\frac{3\pi}{2}n\right) = \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n}\cos\left(\frac{\pi}{2}n\right)$

Let $x_1[n] = \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n}$ & $x_2[n] = \cos\left(\frac{\pi}{2}n\right)$

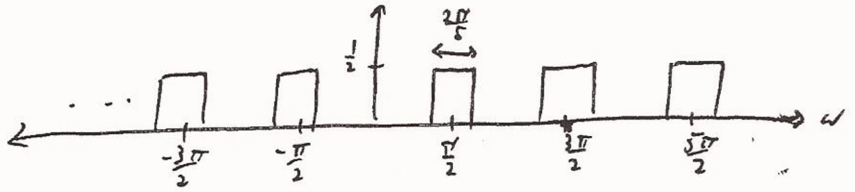
$$X_1(\omega) = \begin{cases} 1 & 0 \le |\omega| \le \frac{\pi}{5} \\ 0 & \text{ow} \end{cases}$$ by Table 5.2

$$X_2(\omega) = \pi\sum_k\left[\delta\left(\omega - \tfrac{\pi}{2} - 2\pi k\right) + \delta\left(\omega + \tfrac{\pi}{2} - 2\pi k\right)\right]$$

$$X(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(\theta)X_1(\omega-\theta)\,d\theta = \frac{1}{2\pi}\pi\left(X_1\left(\omega-\tfrac{\pi}{2}\right) + X_1\left(\omega+\tfrac{\pi}{2}\right)\right)$$



$$= \begin{cases} \frac{1}{2} & |\omega - \frac{\pi}{2}| \le \frac{\pi}{5},\ |\omega + \frac{\pi}{2}| \le \frac{\pi}{5} \\ 0 & \text{ow} \end{cases}$$ periodic with period 2π



Chap 5 DTFT Problems Help/Examples

Prob. 5.21 (a) Recall basic DTFT pair and basic DTFT property (Tables 5.1 and 5.2)

$$u[n]-u[n-N] \overset{DTFT}{\longleftrightarrow} e^{-j\frac{(N-1)}{2}\omega}\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$x[n-n_0] \overset{DTFT}{\longleftrightarrow} X(\omega)e^{-j\omega n_0}$$

$$x[n]=u[n-2]-u[n-6]=\tilde{x}[n-2]$$

where: $\tilde{x}[n]=u[n]-u[n-4]$

Thus:

$$u[n-2]-u[n-6] \overset{DTFT}{\longleftrightarrow} e^{-j2\omega}\, e^{-j\frac{(4-1)}{2}\omega}\frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$=e^{-j\frac{7}{2}\omega}\frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}$$

5.21 (b) $x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1] = \left(\frac{1}{2}\right)^{-n} u[-(n+1)]$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-(n+1)} u[-(n+1)]$$

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

$$x[-n] \xleftrightarrow{DTFT} X(-\omega) \qquad x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(\omega)$$

$n_0 = -1$ here

End result:

$$X(\omega) = \frac{1}{2} e^{-j\omega(-1)} \frac{1}{1-\frac{1}{2}e^{-j(-\omega)}}$$

$$= \frac{1}{2} \frac{e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}$$

5.21 (c) $x[n] = \left(\frac{1}{3}\right)^{|n|} u[-n-2] = \left(\frac{1}{3}\right)^{|n|} \underbrace{u[-(n+2)]}$

turns on at $n=-\infty$

shuts off at $n=-1$

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-(n+2)] \Longrightarrow$$

$$= \left(\frac{1}{3}\right)^{2}\left(\frac{1}{3}\right)^{-(n+2)} u[-(n+2)]$$

since $|n| = -n$ for $n<0$

From previous problem:

$$X(w) = \frac{1}{9} e^{-jw(-2)} \frac{1}{1-\frac{1}{3}e^{-j(-w)}}$$

$$= \frac{1}{9} \frac{e^{j2w}}{1-\frac{1}{3}e^{jw}}$$

Prob. 5.21 (d) $x[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[-n]$

$$= 2^n u[-n] \left\{ \frac{1}{2j} e^{j\frac{\pi}{4}n} - \frac{1}{2j} e^{-j\frac{\pi}{4}n} \right\}$$

$$= \left(\frac{1}{2}\right)^{-n} u[-n] \left\{ \text{''} \qquad \text{''} \right\}$$

From 5.21 (b) and (c) plus freq. shift prop. and linearity

$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(\omega - \omega_0)$$

$$X(\omega) = \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{-j(-(\omega - \frac{\pi}{4}))}} - \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{-j(-(\omega + \frac{\pi}{4}))}}$$

$$= \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{j(\omega - \pi/4)}} - \frac{1}{2j} \frac{1}{1 - \frac{1}{2} e^{j(\omega + \pi/4)}}$$

<u>5.21 (e)</u> $x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$

Example 5.2
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$$a^{|n|} \xleftrightarrow{\text{DTFT}} \frac{1-a^2}{1-2a\cos\omega + a^2}$$

$$x[n] = \left(\frac{1}{2}\right)^{|n|} \frac{1}{2} e^{j\frac{\pi}{8}n} e^{-j\frac{\pi}{8}} + \frac{1}{2}\left(\frac{1}{2}\right)^{|n|} e^{j\frac{\pi}{8}} e^{-j\frac{\pi}{8}n}$$

Thus:

$$X(\omega) = \frac{1}{2} e^{-j\frac{\pi}{8}} \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\left(\omega - \frac{\pi}{8}\right) + \left(\frac{1}{2}\right)^2}$$

$$+ \frac{1}{2} e^{j\frac{\pi}{8}} \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\left(\omega + \frac{\pi}{8}\right) + \left(\frac{1}{2}\right)^2}$$

$$= \frac{3}{8}\left\{ \frac{e^{-j\pi/8}}{\frac{5}{4} - \cos\left(\omega - \frac{\pi}{8}\right)} + \frac{e^{j\pi/8}}{\frac{5}{4} - \cos\left(\omega + \frac{\pi}{8}\right)} \right\}$$

Prob. 5.21 (f) $x[n] = \begin{cases} n, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$

$x[n] = n\,\tilde{x}[n]$ where $\tilde{x}[n] = u[n+3] - u[n-4]$

The DTFT of $\tilde{x}[n]$ is $\tilde{X}(\omega) = \dfrac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$

Table 5.2

and using this prop. from Table 5.1

$$n\,x[n] \overset{DTFT}{\longleftrightarrow} j\,\frac{dX(\omega)}{d\omega}$$

End result:

$$X(\omega) = j\,\frac{d}{d\omega}\left\{\frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right\}$$

$$= j\,\frac{1}{\sin^2\left(\frac{\omega}{2}\right)}\left\{\frac{7}{2}\cos\left(\frac{7}{2}\omega\right)\sin\left(\frac{1}{2}\omega\right) - \sin\left(\frac{7}{2}\omega\right)\frac{1}{2}\cos\left(\frac{\omega}{2}\right)\right\}$$

5.21 (g) $x[n] = \sin\left(\frac{\pi}{2}n\right) + \cos(n)$

$$X(\omega) = \frac{\boxed{2\pi}}{2j}\,\delta\left(\omega - \frac{\pi}{2}\right) - \frac{\boxed{2\pi}}{2j}\,\delta\left(\omega + \frac{\pi}{2}\right)$$

for

$-\pi < \omega < \pi$

$$+ \frac{\boxed{2\pi}}{2}\,\delta(\omega - 1) + \frac{\boxed{2\pi}}{2}\,\delta(\omega + 1)$$

repeats every 2π

5.21 (h) $x[n] = \sin\left(\frac{5\pi}{3}n\right) + \cos\left(\frac{7\pi}{3}n\right)$

$$= \sin\left(\left(\frac{5\pi}{3} - \frac{6\pi}{3}\right)n\right) + \cos\left(\left(\frac{7\pi}{3} - \frac{6\pi}{3}\right)n\right)$$

$$= -\sin\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{3}n\right)$$

$$X(\omega) = \boxed{2\pi}\left(\frac{1}{2} - \frac{1}{2j}\right)\delta\left(\omega - \frac{\pi}{3}\right) + \left(\frac{1}{2} + \frac{1}{2j}\right)\delta\left(\omega + \frac{\pi}{3}\right)\boxed{2\pi}$$

for $-\pi < \omega < \pi$ repeats every 2π

5.21 (i) $x[n] = \tilde{x}[n-6]$ $\tilde{x}[n] = u[n] - u[n-5]$

From 5.21 (a)

$$X(\omega) = \underbrace{e^{-j6\omega} \; e^{-j\frac{(5-1)}{2}\omega}} \; \frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$= e^{-j8\omega}$$

5.21 (j) $x[n] = (n-1)\left(\frac{1}{3}\right)^{|n|}$

$$= n\left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{3}\right)^{|n|}$$

$$X(\omega) = j\frac{d}{d\omega}\left\{\frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2 - \frac{2}{3}\cos(\omega)}\right\} - \frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2 - \frac{2}{3}\cos\omega}$$

simplification left to the reader :)

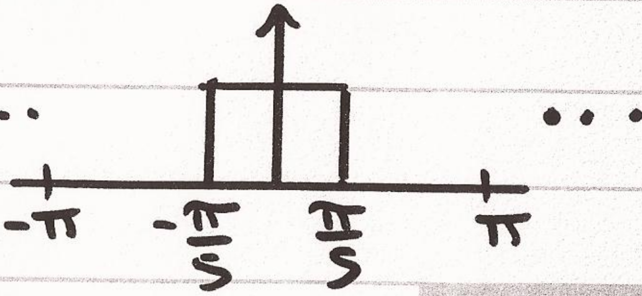
5.21 (k) $$x[n] = \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n}\cos\left(\frac{7\pi}{2}n\right)$$

$$\cos\left(\frac{7\pi}{2}n\right) = \cos\left(\left(\frac{7\pi}{2} - \frac{8\pi}{2}\right)n\right) = \cos\left(-\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right)$$

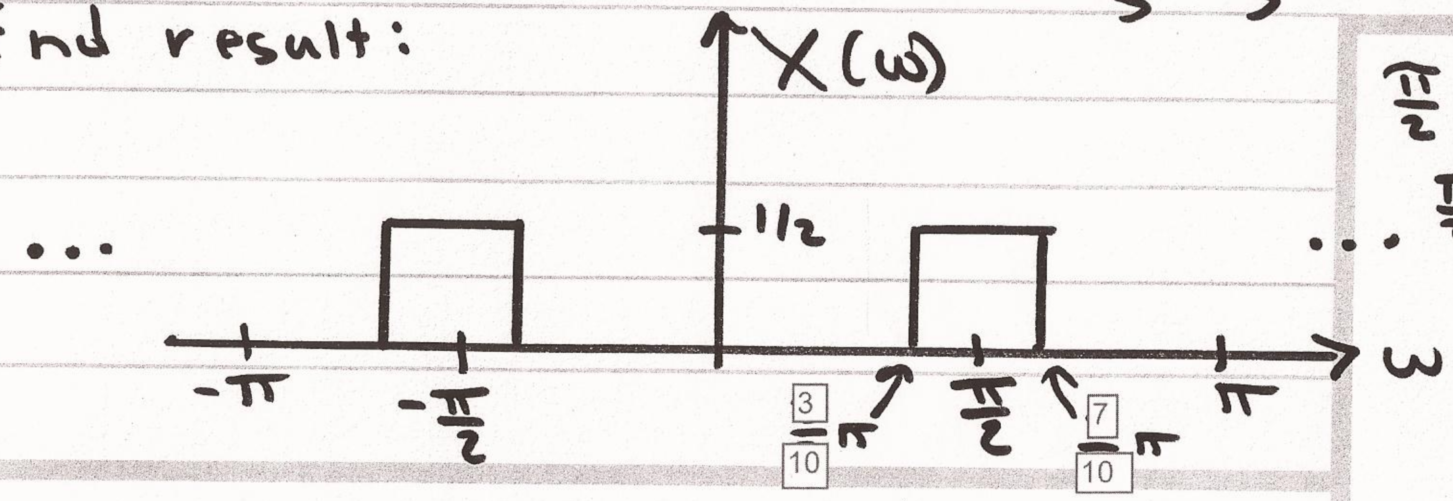
Thus:

$$X(\omega) = \frac{1}{2}\tilde{X}\left(\omega - \frac{\pi}{2}\right) + \frac{1}{2}\tilde{X}\left(\omega + \frac{\pi}{2}\right)$$

where: $$\frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow}$$



End result:



$$\frac{\pi}{2} - \frac{\pi}{5} = \frac{3}{10}\pi$$

$$\frac{\pi}{2} + \frac{\pi}{5} = \frac{7}{10}\pi$$

5.35) $y[n] - ay[n-1] = bx[n] + x[n-1]$ $\quad a \in \mathbb{R}$ $\quad a$ real $\quad |a|<1$

a) $|H(\omega)| = 1 \quad \forall \omega$

$$Y(\omega) - a e^{-j\omega} Y(\omega) = b X(\omega) + e^{-j\omega} X(\omega)$$

$$(1 - a e^{-j\omega}) Y(\omega) = (b + e^{-j\omega}) X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}}$$

$$|H(\omega)| = 1 \Rightarrow |H(\omega)|^2 = 1 \Rightarrow |b + e^{-j\omega}|^2 = |1 - a e^{-j\omega}|^2$$

$$(b + \cos(\omega))^2 + \sin^2(\omega) = (1 - a\cos(\omega))^2 + a^2 \sin^2(\omega)$$

$$b^2 + 2b\cos(\omega) + \cos^2(\omega) + \sin^2(\omega) = 1 - 2a\cos(\omega) + a^2\cos^2(\omega) + a^2 \sin^2(\omega)$$

$$b^2 + 2b\cos(\omega) + 1 = 1 - 2a\cos(\omega) + a^2$$

For this equation to hold for all ω, $b = -a$

5.35 / Alternative Solution

Prob. 5.35 All-Pass Filters ⇒ "Pet Problem"

①

$$y[n] - a\,y[n-1] = b\,x[n] + x[n-1]$$

Take DTFT of both sides:

$$Y(\omega)\left(1 - a e^{-j\omega}\right) = X(\omega)\left(b + e^{-j\omega}\right)$$

Convolution prop. dictates $Y(\omega) = H(\omega)\,X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}}$$

So, we have DTFT of $h[n] = H(\omega)$ frequency response of system without even determining impulse response $h[n]$

Consider $b = -a$: (where a is real-valued)

$$H(w) = \frac{-a + e^{-jw}}{1 - a e^{-jw}} = e^{-jw} \frac{(1 - a e^{jw})}{1 - a e^{-jw}}$$

since a is real-valued

$$\frac{1 - a e^{jw}}{1 - a e^{-jw}} = \frac{c}{c^*}$$

for any complex no. c, $\frac{c}{c^*}$ has magnitude 1

$\Rightarrow$ easy to see in polar form

$$\frac{c}{c^*} = \frac{|c| e^{j\angle c}}{|c| e^{-j\angle c}} = e^{j2\angle c}$$

and since $|ab| = |a||b|$, we have

$$|H(w)| = |e^{-jw}| \left| \frac{1 - a e^{jw}}{1 - a e^{-jw}} \right| = 1 \text{ for all } w$$

• <u>Thus:</u> $y[n] - a\,y[n-1] = -a\,x[n] + x[n-1]$

is an all-pass (magnitude) filter for any value of a (real-valued) $|H(\omega)| = 1 \;\forall \omega$

• Thus, for any sinewave into this system, amplitude is unchanged ⇒ just phase changes

$$Ae^{j(\omega_0 n + \theta)} \longrightarrow \boxed{\text{All-Pass Filter}} \longrightarrow Ae^{j(\omega_0 n + \theta + \phi(\omega_0))}$$

$$\phi(\omega_0) = \angle H(\omega_0)$$

• For arbitrary input $x[n]$, consider:

$$x[n] \longrightarrow \boxed{\text{All-Pass Filter}} \longrightarrow y[n]$$

$\boxed{\text{Consider Parseval's Theorem}}$

<u>note</u>: all-pass filter does <u>NOT</u> mean $y[n] = x[n]$

$$x[n] \longrightarrow \boxed{\text{All-Pass Filter}} \longrightarrow y[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad\qquad E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega \qquad\qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Y(\omega)|^2 \, d\omega$$

- since $Y(\omega) = H(\omega)\,X(\omega)$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

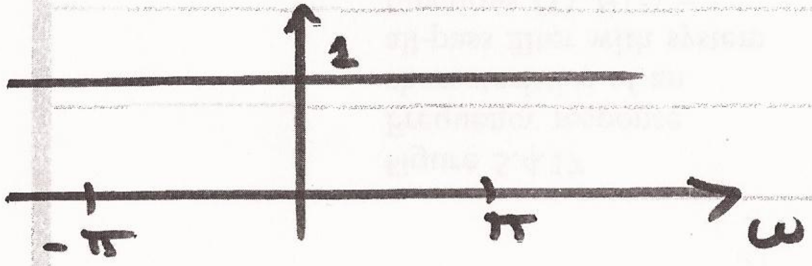
$|Y(\omega)| = |X(\omega)|$ For all-pass filter

Thus, $E_y = E_x$ for all-pass filter

(pet problem :-))

$$H(\omega) = \frac{-a + e^{-j\omega}}{1 - a e^{-j\omega}} = e^{-j\omega} \frac{(1 - a e^{j\omega})}{1 - a e^{-j\omega}}$$

$|H(\omega)| = 1 \quad \forall \omega$



$\angle H(\omega) = ?$

$= -\omega + 2 \angle (1 - a e^{j\omega})$

$1 - a e^{j\omega}$

$= (1 - a\cos\omega) - j a \sin\omega$

$$\angle(1 - a e^{j\omega}) = -\tan^{-1}\left\{\frac{a \sin\omega}{1 - a\cos\omega}\right\}$$

Thus:

$$\angle H(\omega) = -\omega - 2\tan^{-1}\left\{\frac{a \sin\omega}{1 - a\cos\omega}\right\}$$

$-1 < a < 1$ for stability (later = Chap. 10)

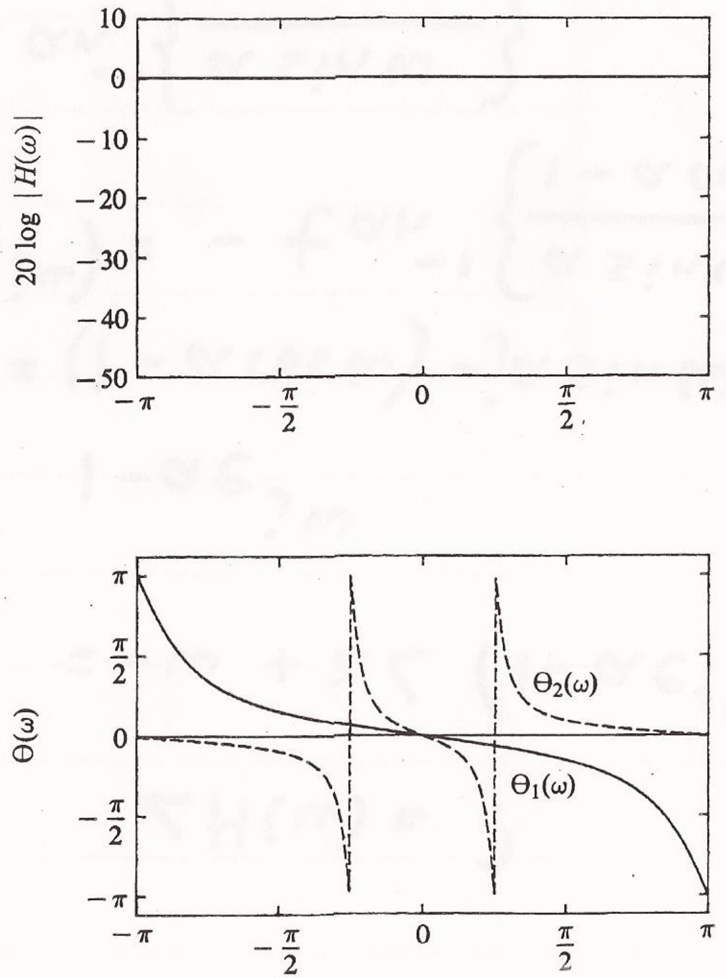


Figure 5.4.17 Frequency response characteristics of an all-pass filter with system functions (1) $H(z) = (0.6 + z^{-1})/(1 + 0.6z^{-1})$, (2) $H(z) = (r^2 - 2r\cos\omega_0 z^{-1} + z^{-2})/(1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})$, $r = 0.9$, $\omega_0 = \pi/4$.