Fourier Transform Examples. Hmwk. 7 Help

Prob. 4.10 Find FT of: 1

a/ $$x(t) = t\left(\frac{\sin(Wt)}{\pi t}\right)^2$$

- For Prob 4.10, $W=1$ => special case
- At least 2 ways to solve problem
- Method 1:

$$x(t) = t \, \frac{\sin(Wt)}{\pi t} \, \frac{\sin(Wt)}{\pi t}$$

$$= \frac{1}{\pi} \, \frac{\sin(Wt)}{\pi t} \sin(Wt)$$

recall:

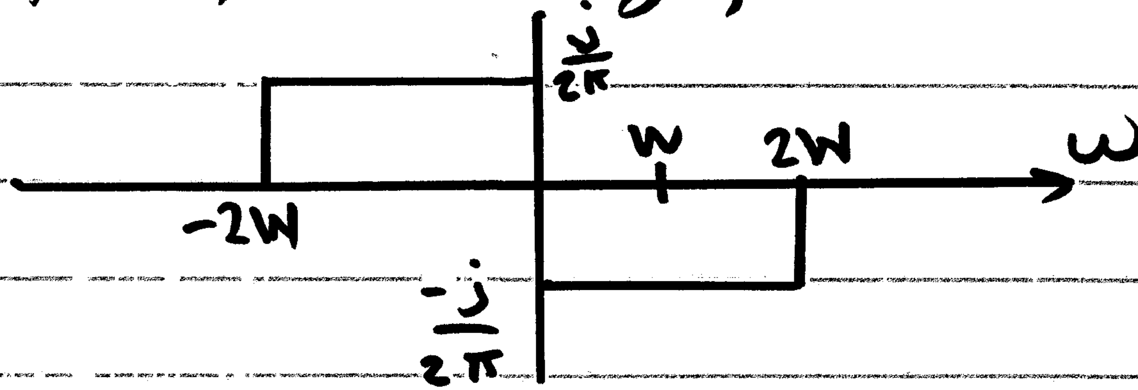
$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \text{rect}\left(\frac{\omega}{2W}\right)$$

$$x(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j} X(\omega - \omega_0) - \frac{1}{2j} X(\omega + \omega_0)$$

• Thus: (2)

$$X(\omega) = \frac{1}{\pi}\left\{\frac{1}{2j}\,\mathrm{rect}\left(\frac{\omega - W}{2W}\right) - \frac{1}{2j}\,\mathrm{rect}\left(\frac{\omega + W}{2W}\right)\right\}$$

$\Rightarrow$ purely imaginary, since $x(t)$ is odd fn.

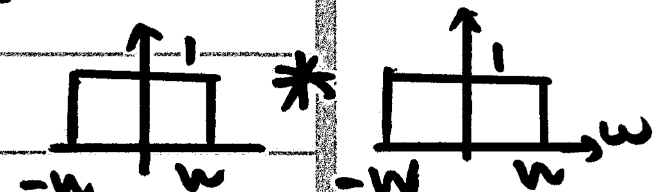


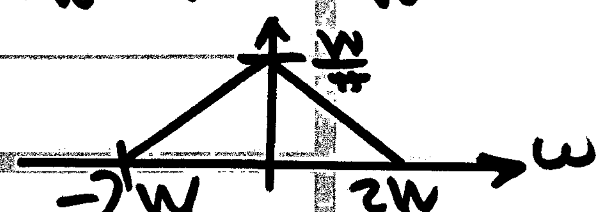
Method 2:

• recall: $x(t)\,y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega)$

• thus: $x^2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * X(\omega)$

$$\left\{\frac{\sin(Wt)}{\pi t}\right\}^2 \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}$$





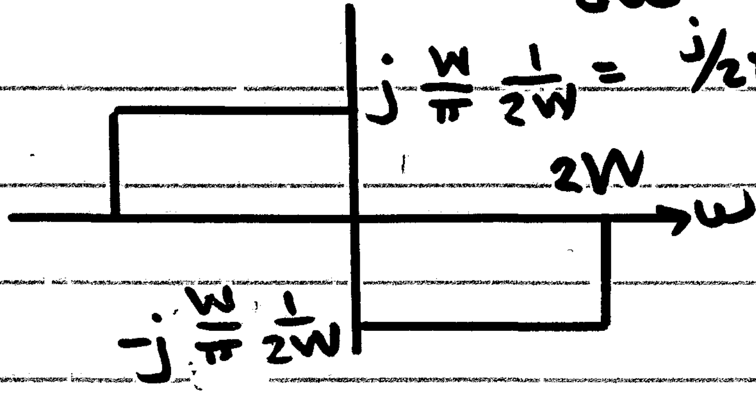
On crib sheet, have FT of $\frac{\sin(W_1 t)}{\pi t}\,\frac{\sin(W_2 t)}{\pi t}$

• recall: $t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\dfrac{dX(\omega)}{d\omega}$

(3)

• Thus:

slope $= \dfrac{\frac{W}{\pi}}{2W}$

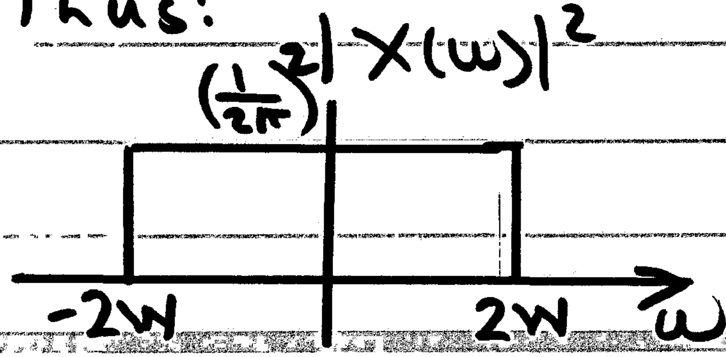


• same answer!

4.10(b) Find: $A = \displaystyle\int_{-\infty}^{\infty}\left\{t\left(\frac{\sin(Wt)}{\pi t}\right)^2\right\}^2 dt$

Parseval's Theorem: $\displaystyle\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$

Thus:



Area under $= \left(\dfrac{1}{2\pi}\right)^2 (4W) \cdot \dfrac{1}{2\pi}$

$= \dfrac{W}{\pi^2} \cdot \dfrac{1}{2\pi} = \dfrac{W}{2\pi^3} = \dfrac{1}{2\pi^3}$

Since $W = 1$

Prob. 4.12 (a) Find FT of $x(t) = t e^{-|t|}$

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· again: $t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j \dfrac{d X(\omega)}{d\omega}$

Example 4.2 in text $e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{2a}{\omega^2 + a^2}$

Thus:
$$t e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} j \frac{d}{d\omega}\left\{\frac{2}{\omega^2+1}\right\} = j \frac{-2(2\omega)}{(\omega^2+1)^2}$$

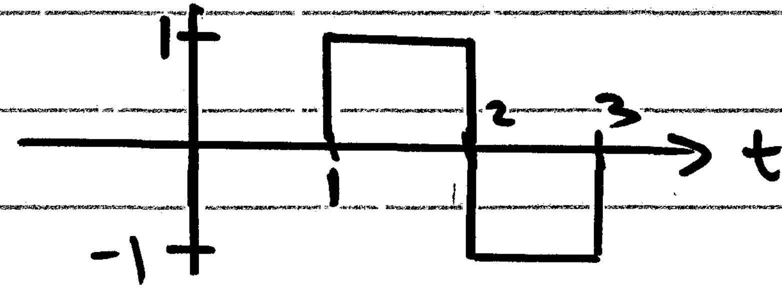
(b) $t e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{-4\omega j}{(1+\omega^2)^2}$

Recall Duality Property: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$

Thus:
$$\frac{-4jt}{(1+t^2)^2} \overset{\mathcal{F}}{\longleftrightarrow} \omega e^{-|\omega|}\,2\pi \Rightarrow \frac{4t}{(1+t^2)^2} \overset{\mathcal{F}}{\longleftrightarrow} j\omega e^{-|\omega|} \cdot 2\pi$$

Prob. 4.27 $x(t) = u(t-1) - 2u(t-2) + u(t-3)$



$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$ } replicates $x(t)$ above every T secs $\Rightarrow$ periodic signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{k2\pi}{T}t}\,dt = \frac{1}{T}\int_{-\infty}^{\infty}\left\{x(t)\,\mathrm{rect}\left(\frac{t}{T}\right)e^{-j\omega t}\right\}dt$$

evaluated at $\omega = k\frac{2\pi}{T}$

Thus: $a_k = \frac{1}{T}\mathcal{F}\{\text{one period}\}\Big|_{\omega = k\frac{2\pi}{T}}$

(6)

- In this case: $x(t)$ is one period of $\tilde{x}(t)$

- $$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow a_k = \frac{1}{T} X\left(k\frac{2\pi}{T}\right)$$

- recall: $$\text{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin\left(\frac{T}{2}\omega\right)}{\frac{\omega}{2}} = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega)$$

- Thus: $$X(\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} \left\{ e^{-j\frac{3}{2}\omega} - e^{-j\frac{5}{2}\omega} \right\}$$

$$= \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} e^{-j2\omega} \left\{ e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right\} \frac{2j}{2j}$$

$$= 4j \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega} e^{-j2\omega}$$

$$a_k = \frac{1}{T} 4j \frac{\sin^2\left(\frac{1}{2}k\frac{2\pi}{T}\right)}{k\,2\pi/T} e^{-j2k\frac{2\pi}{T}}$$

Prob 4.28

a) $p(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 n t}$

Using the pair $e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$ and linearity, we get

$$P(\omega) = \sum_{n=-\infty}^{\infty} 2\pi a_n \delta(\omega - n\omega_0)$$

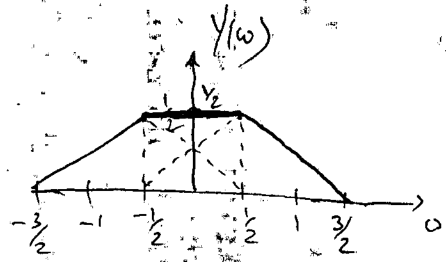
, and so

$$y(t) = x(t)\,p(t) \xleftrightarrow{F} Y(\omega) = \frac{1}{2\pi}\{X(\omega) * P(\omega)\} = \frac{1}{2\pi}\left\{X(\omega) * \sum_{n=-\infty}^{\infty} 2\pi a_n \delta(\omega - n\omega_0)\right\}$$

$$\Rightarrow Y(\omega) = \underline{\underline{\sum_{n=-\infty}^{\infty} a_n X(\omega - n\omega_0)}}$$

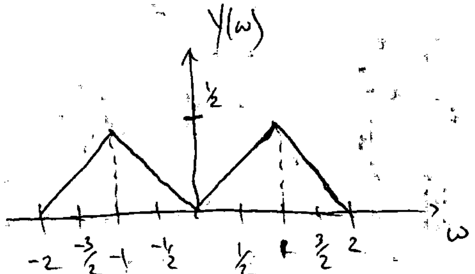
b) i) $p(t) = \cos(t/2) \xleftrightarrow{F} P(\omega) = \pi\{\delta(\omega-\tfrac{1}{2}) + \delta(\omega+\tfrac{1}{2})\}$

So $Y(\omega) = \frac{1}{2\pi}\{X(\omega) * P(\omega)\} = \frac{1}{2}\{X(\omega-\tfrac{1}{2}) + X(\omega+\tfrac{1}{2})\}$



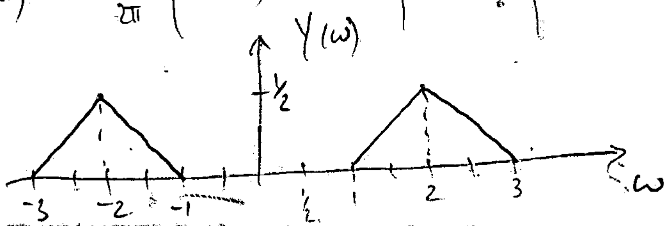
ii) $p(t) = \cos(t) \xleftrightarrow{F} P(\omega) = \pi\{\delta(\omega-1) + \delta(\omega+1)\}$

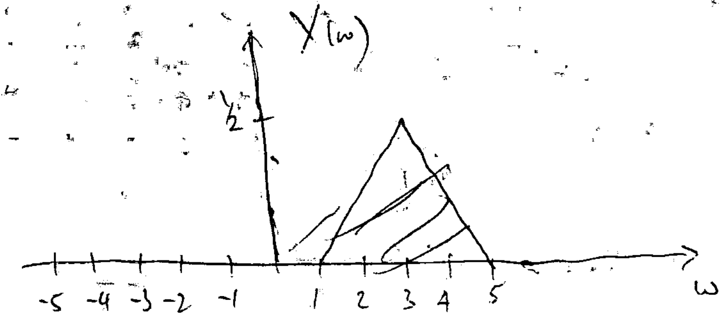
$Y(\omega) = \frac{1}{2\pi}\{X(\omega) * P(\omega)\} = \frac{1}{2}\{X(\omega-1) + X(\omega+1)\}$



iii) $p(t) = \cos(2t) \xleftrightarrow{F} P(\omega) = \pi\{\delta(\omega-2) + \delta(\omega+2)\}$

$\Rightarrow Y(\omega) = \frac{1}{2\pi}\{X(\omega) * P(\omega)\} = \pi\{X(\omega-2) + X(\omega+2)\}$





Prob. 4.28. (Cnt'd)

For parts (i)-(iii), just use FT property

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$

- For parts (vi)-(viii), review Chap. 7 "Basic Sampling Theory" Handout

$$x(t)\,p(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}X(\omega) * P(\omega)$$

$$p(t)=\sum_{k=-\infty}^{\infty}\delta(t-nT) \overset{\mathcal{F}}{\longleftrightarrow} P(\omega)=?$$

It's in Table 4.2 on page 329

$$=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\frac{2\pi}{T}t} \overset{\mathcal{F}}{\longleftrightarrow} =\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\frac{2\pi}{T}\right)$$

periodic signal =
FS expansion
FS coeffs $a_k=\frac{1}{T}$

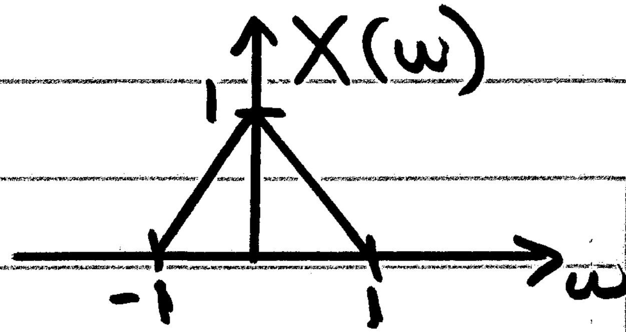
Thus:

$$x(t)\,p(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\cancel{\pi}} X(\omega) * \frac{\cancel{2}\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega) * \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$

For this problem,



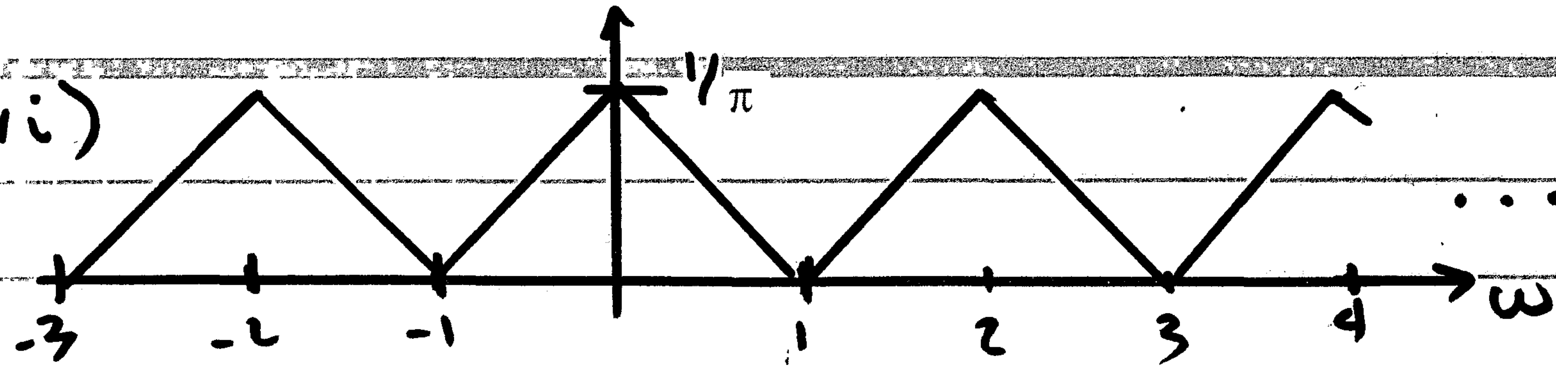
and

(vi) $T=\pi \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

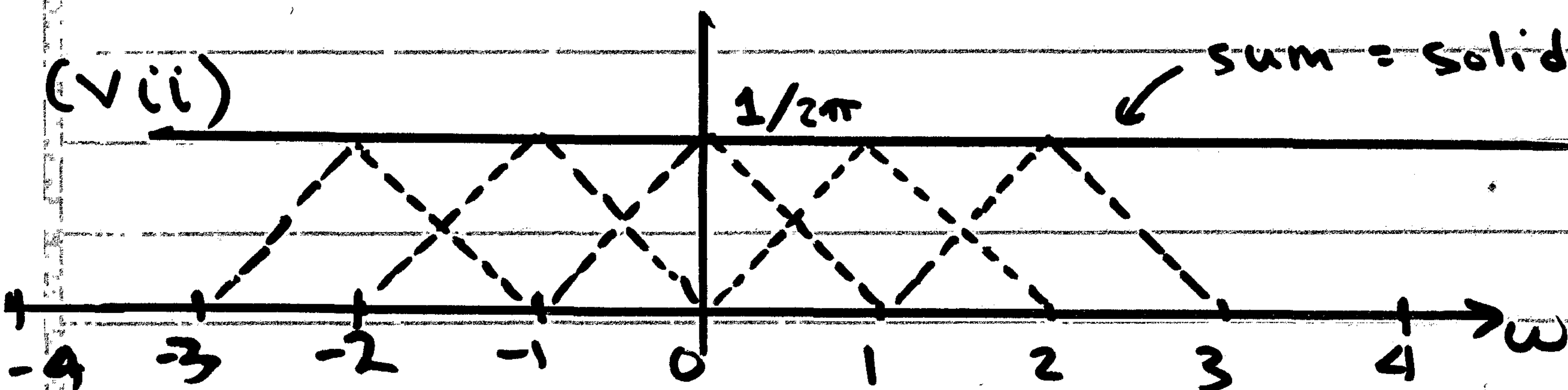
(vii) $T=2\pi \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

(viii) $T=4\pi \Rightarrow \frac{2\pi}{T} = \frac{2\pi}{4\pi} = \frac{1}{2}$

(vi)



(vii)



(viii) same answer as (vii)

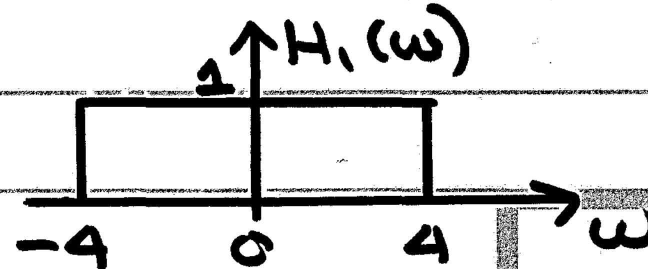
You get the double the number of triangles as occurring in in the answer to (vii), which can be viewed as the answer to (vii) plus the answer to (vi) shifted to the right by 1/2. That doubles the amplitude BUT the value of 1/T for (viii) is half of the value of 1/T for (vii) -- so you get the same amplitude.

Prob. 4.32 $\quad h(t) = \dfrac{\sin(4(t-1))}{\pi(t-1)}$

View as: $h(t) = \dfrac{\sin(4t)}{\pi t} * \delta(t-1)$

$$x(t) \longrightarrow \boxed{h_1(t) = \frac{\sin(4t)}{\pi t}} \xrightarrow{y_1(t)} \boxed{h_2(t) = \delta(t-1)} \longrightarrow y(t) = y_1(t-1)$$

$h_1(t) = \dfrac{\sin(4t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$



Recall for complex sinewave:

$$e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega_0)e^{j\omega_0 t}$$

$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$

- See Handout on Response of LTI Systems to Sinewaves. Generalizations

$$A\cos(\omega_0 t+\theta) \longrightarrow \boxed{h(t)} \longrightarrow A\,|H(\omega_0)|\cos(\omega_0 t+\theta+\angle H(\omega_0))$$

$$H(\omega)$$

$$H(\omega_0) = |H(\omega_0)|\,e^{j\angle H(\omega_0)}$$

polar form

$$A\sin(\omega_0 t+\theta) \longrightarrow \boxed{h(t)} \longrightarrow A\,|H(\omega_0)|\sin(\omega_0 t+\theta+\angle H(\omega_0))$$

For sum of sinewaves, invoke linearity

$\Rightarrow$ superposition

$\Rightarrow$ do one at a time and then sum

(a) $x_1(t) = \cos\left(6t + \frac{\pi}{2}\right)$

since $H(6) = 0$, $y_1(t) = 0$

(b) $x_2(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \sin(k3t)$

$$= \underbrace{(1)\sin(0\cdot t)}_{=0} + \underbrace{\frac{1}{2}\sin(3t)}_{\text{Passes}} + \underbrace{\frac{1}{4}\sin(6t) + \frac{1}{8}\sin(9t) + \ldots}_{\text{all rejected}}$$

since $H(\omega) = 0$
for $\omega > 4$

Final answer:

$$y_2(t) = \frac{1}{2}\sin(3(t-1))$$

c) $x_3(t) = \dfrac{\sin[4(t+1)]}{\pi(t+1)} \xleftrightarrow{\mathcal{F}} X_3(\omega) = \begin{cases} e^{j\omega} & |\omega| < 4 \\ 0 & \text{elsewhere} \end{cases}$

So $X_3(\omega)$ passes thru system (i.e $H_1(\omega)$) and is ultimately delayed by one.

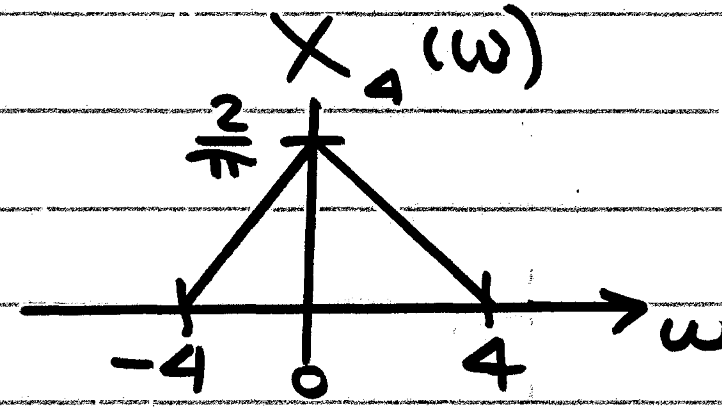
thus $y_3(t) = x_3(t-1) = \dfrac{\sin[4(t-1+1)]}{\pi(t-1+1)}$

$$y_3(t) = \frac{\sin(4t)}{\pi t}$$

(d) $x_4(t) = \left(\frac{\sin(2t)}{\pi t}\right)^2$

See handout on Fourier Transforms involving product of sinc functions

$$\left(\frac{\sin(2t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow}$$



Since $X_4(\omega) = 0$ for $\omega > 4$, passes right thru system $\Rightarrow$ ultimately delayed by 1

$$y_4(t) = \left(\frac{\sin(2(t-1))}{\pi(t-1)}\right)^2$$