Prob. 4.1 Help

(a) $x_a(t) = e^{-2(t-1)} u(t-1)$

Recall Example 4.1 in Text: also in Table 4.2

$$e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} \quad a>0$$

Also, time-shift property of Fourier Transform

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) e^{-j\omega t_0}$$

Thus:

$$X_a(\omega) = \frac{1}{2+j\omega} \underbrace{e^{-j\omega(1)}}$$

- does not affect magnitude
- only impacts phase

$$|X_a(\omega)| = \frac{1}{\sqrt{(-2)^2+\omega^2}} = \frac{1}{\sqrt{4+\omega^2}}$$

Prob. 4.1 (b)

$$x_b(t) = e^{-2|t-1|}$$

Recall Example 4.2 in Text: also in my version Table 4.2

2-sided exponential $a>0$

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2} \qquad a>0$$

Plus: time-shift property of Fourier Transform

$$X_b(\omega) = \underbrace{\frac{2(2)}{4+\omega^2}}_{\text{real-valued + strictly positive}} \underbrace{e^{-j\omega}}_{\text{only affects phase}}$$

Thus: $|X_b(\omega)| = \dfrac{4}{4+\omega^2}$ see plot in Example 4.2

Hw #6 Solutions

Prob 4.1 a/ $x(t) = e^{-2(t-1)} u(t-1)$

Using the Fourier pair $e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}$

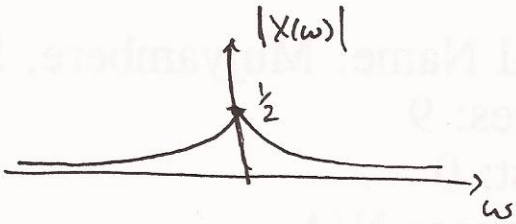
and the time shifting property $x(t-t_0) \xleftrightarrow{F.T} X(\omega) e^{-j\omega t_0}$

with $x(t) \xleftrightarrow{F.T} X(\omega)$,

we get $e^{-2(t-1)} u(t-1) \xleftrightarrow{F.T} \frac{1}{2+j\omega} e^{-j\omega \cdot 1}$

$$\Rightarrow X(\omega) = \frac{e^{-j\omega}}{2+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$



b/ $x(t) = e^{-2|t-1|}$

From Text example 4.2, the following Fourier Transform pair has been established:

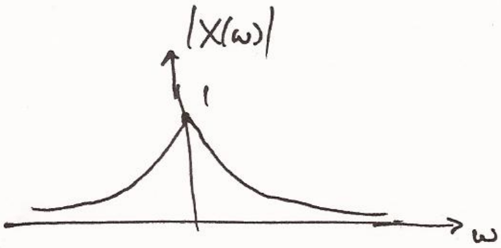
$$e^{-a|t|} \xleftrightarrow{F.T} \frac{2a}{a^2+\omega^2}$$

So using this pair and time shifting property, we obtain

$$x(t) = e^{-2|t-1|} \xleftrightarrow{F.T} e^{-j\omega \cdot 1} \frac{2 \cdot 2}{2^2+\omega^2} = X(\omega)$$

$$\Rightarrow X(\omega) = e^{-j\omega} \frac{4}{4+\omega^2}$$

and $|X(\omega)| = \frac{4}{4+\omega^2}$ $\rightarrow$



Prob 4.2 : a/ $x(t) = \delta(t+1) + \delta(t-1)$

From $\delta(t) \xleftrightarrow{F.T} 1$ and time shifting property,

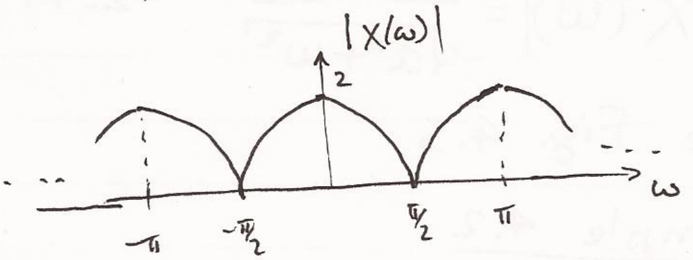
We obtain $\delta(t+1) \xleftrightarrow{F.T} 1 \cdot e^{-jw(-1)} = e^{jw}$

$\delta(t-1) \xleftrightarrow{F.T} 1 \cdot e^{-jw \cdot 1} = e^{-jw}$

So By linearity, ~~X(w)~~ = $x(t) \xleftrightarrow{F.T} X(w) = e^{jw} + e^{-jw}$

Thus, $X(\omega) = 2\cos(w)$ (by Using Euler's Identities)

So



b/ $x(t) = \frac{d}{dt}\{u(-2-t) + u(t-2)\}$

By taking derivative, we get : $x(t) = -\delta(t+2) + \delta(t-2)$

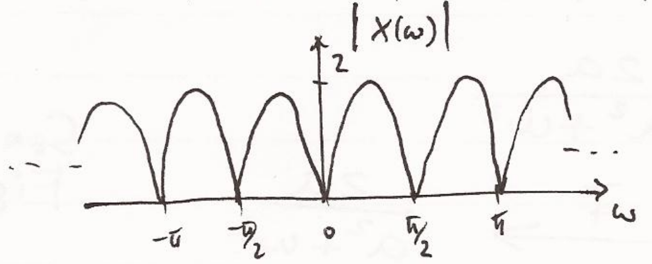
So Using the same Fourier Transform pair as in (a) and time shifting

we get $x(t) \xrightarrow{F.T} X(\omega) = -e^{jw \cdot 2} + e^{-jw \cdot 2}$

$X(\omega) = -e^{j2w} + e^{-j2w} = e^{-j2w} - e^{j2w}$

$= -2j\sin(2\omega)$

So $|X(\omega)| = |-2j\sin(2\omega)| = |2\sin(2\omega)|$



Prob 4.4

a) $X(\omega) = 2\pi\,\delta(\omega) + \pi\,\delta(\omega - 4\pi) + \pi\,\delta(\omega + 4\pi)$

From the pair $\quad 1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)$

we get that $\quad \mathcal{F}^{-1}\{2\pi\,\delta(\omega)\} = 1$

Also Using the frequency shifting property and the same pair as above, we get:

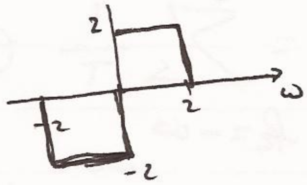
$$\pi\,\delta(\omega) \xleftrightarrow{F.T} \frac{1}{2} \quad \text{and} \quad \pi\,\delta(\omega - 4) \xleftrightarrow{F.T} \frac{1}{2} e^{j4t}$$

So $\mathcal{F}^{-1}\{\pi\,\delta(\omega - 4\pi)\} = \frac{1}{2} e^{+j4\pi t}$

and $\mathcal{F}^{-1}\{\pi\,\delta(\omega + 4\pi)\} = \frac{1}{2} e^{-j4\pi t}$

So $\mathcal{F}^{-1}\{X(\omega)\} = x(t) = 1 + \frac{1}{2} e^{j4\pi t} + \frac{1}{2} e^{-j4\pi t} = \underline{1 + \cos(4\pi t)}$

b) $X(\omega) = \begin{cases} 2 & 0 \le \omega \le 2 \\ -2 & -2 \le \omega \le 0 \\ 0 & |\omega| > 2 \end{cases} \quad \Rightarrow$



From pair $\quad \frac{\sin(Wt)}{\pi t} \xleftrightarrow{F.T} \mathrm{Rect}\left(\frac{\omega}{2W}\right)$

and Using frequency shifting property, we get the followings.

First we realize $\quad X(\omega) = -2\,\mathrm{Rect}\left(\frac{(\omega+1)}{2.1}\right) + 2\,\mathrm{Rect}\left(\frac{\omega - 1}{2.1}\right)$

So $\mathcal{F}^{-1}\{X(\omega)\} = x(t) = -2\,\frac{\sin(1.t)}{\pi t} e^{-j1.t} + 2\,\frac{\sin(1.t)}{\pi t} e^{j(+1)t}$

$$x(t) = \frac{2}{\pi t}\sin(t)\left[-e^{-jt} + e^{+jt}\right] = \frac{2}{\pi t}\sin(t)\,(+2j\sin t) = \underline{\frac{+4j\sin^2(t)}{\pi t}}$$

Prob 4.21

a) $x(t) = \left[e^{-\alpha t} \cos \omega_0 t\right] u(t)$

$$\Rightarrow x(t) = \left[e^{-\alpha t} u(t)\right] \cos(\omega_0 t) = \left[e^{-\alpha t} u(t)\right]\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$x(t) = \frac{1}{2}\left[e^{-\alpha t} u(t)\right]\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2}\left[e^{-\alpha t} u(t)\right] e^{j\omega_0 t} + \frac{1}{2}\left[e^{-\alpha t} u(t)\right] e^{-j\omega_0 t}$$

So Using the Modulation (or frequency shifting) property and the pair: $x(t) = e^{-\alpha t} u(t) \xleftrightarrow{F.T} \dfrac{1}{\alpha + j\omega}$

Recall Modulation property: if $x(t) \xleftrightarrow{F.T} X(\omega)$

then $e^{j\omega_0 t} x(t) \xleftrightarrow{F.T} X(\omega - \omega_0)$

We get:

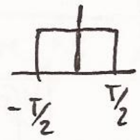
$$x(t) = \frac{1}{2}\left[e^{-\alpha t} u(t)\right] e^{j\omega_0 t} + \frac{1}{2}\left[e^{-\alpha t} u(t)\right] e^{-j\omega_0 t} \xleftrightarrow{F.T} X(\omega) = \frac{1}{2}\frac{1}{\alpha + j(\omega - \omega_0)} + \frac{1}{2}\frac{1}{\alpha + j(\omega + \omega_0)}$$

$$\Rightarrow X(\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} + \frac{1}{2(\alpha + j\omega_0 + j\omega)}$$

4.21 (c) $x(t) = \begin{cases} 1 + \cos(\pi t), & |t| < 1 \\ 0 & |t| > 1 \end{cases}$

$$x(t) = \left(1 + \cos(\pi t)\right) \text{rect}\left(\frac{t}{2}\right)$$

From basic fourier pair :



$$\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

and modulation property : $e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$

we get that $x(t)\cos(\omega_0 t) = \frac{1}{2} x(t) e^{j\omega_0 t} + \frac{1}{2} x(t) e^{-j\omega_0 t} \xleftrightarrow{\mathcal{F}} \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$

So for $T = 2$ and $\omega_0 = \pi$ and $\omega_0 = 0$, we :

$$x(t) = \left[1 + \cos(\pi t)\right] \text{rect}\left(\frac{t}{2}\right) = \text{rect}\left(\frac{t}{2}\right) + \cos(\pi t)\,\text{rect}\left(\frac{t}{2}\right)$$

we get :

$$X(\omega) = \frac{\sin\left(2\frac{\omega}{2}\right)}{\frac{\omega}{2}} + \frac{1}{2}\frac{\sin\left(2\frac{(\omega+\pi)}{2}\right)}{\frac{(\omega+\pi)}{2}} + \frac{1}{2}\frac{\sin\left(2\frac{(\omega-\pi)}{2}\right)}{\frac{\omega-\pi}{2}}$$

$$X(\omega) = \frac{\sin(\omega)}{\omega/2} + \frac{1}{2}\frac{\sin(\omega+\pi)}{\frac{(\omega+\pi)}{2}} + \frac{1}{2}\frac{\sin(\omega-\pi)}{\frac{(\omega-\pi)}{2}}$$

$$= \sin(\omega)\left[\frac{2}{\omega} - \frac{1}{\omega+\pi} - \frac{1}{\omega-\pi}\right]$$

$$= \sin(\omega)\left[\frac{2}{\omega} - \frac{2\omega}{\omega^2 - \pi^2}\right] = 2\sin(\omega)\left[\frac{1}{\omega} - \frac{\omega}{\omega^2 - \pi^2}\right]$$

Prob. 4.21 (e)

$$x(t) = t\,e^{-2t}\sin(4t)\,u(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$$

Rewrite as: $x(t) = t\left[e^{-2t}u(t)\cdot\sin(4t)\right]$

$$= t\,z(t)$$

- First, find $Z(\omega)$,

Since from table, $e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{2+j\omega}$

THUS:

$$Z(\omega) = \mathcal{F}\left\{e^{-2t}u(t)\,\sin(4t)\right\}$$

$$= \frac{1}{2j}\,\frac{1}{2+j(\omega-4)} - \frac{1}{2j}\,\frac{1}{2+j(\omega+4)}$$

by Use of modulation property

- Now, from Table :

$$t\,z(t) \overset{\mathcal{F}}{\longleftrightarrow} j\,\frac{d}{d\omega}\{Z(\omega)\}$$

THUS:

$$X(\omega) = j\,\frac{d}{d\omega}\{Z(\omega)\}$$

$$= \frac{1}{2}\,\frac{d}{d\omega}\left\{\frac{1}{2+j(\omega-4)}\right\} + \frac{1}{2}\,\frac{d}{d\omega}\left\{\frac{1}{2+j(\omega+4)}\right\}$$

$$= \frac{1}{2}\,\frac{j}{(2+j(\omega-4))^2} + \frac{1}{2}\,\frac{-j}{(2+j(\omega+4))^2}$$

Alternative Problem 4.21 f: Notice there is no time shift in Second Sinc function

Problem 4.21

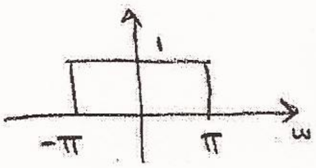
(f) $x(t) = \left(\frac{\sin \pi t}{\pi t}\right)\left(\frac{\sin 2\pi t}{\pi t}\right)$

By the multiplication property of the FT, we have that

$$X(j\omega) = \frac{1}{2\pi}\, \mathcal{F}\left\{\frac{\sin \pi t}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin 2\pi t}{\pi t}\right\}.$$

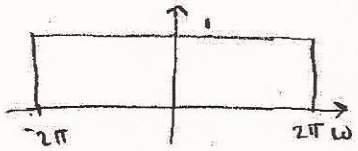
But

$$\mathcal{F}\left\{\frac{\sin \pi t}{\pi t}\right\} = \begin{cases} 1, & \text{if } |\omega| \le \pi \\ 0, & \text{otherwise} \end{cases} =$$



and

$$\mathcal{F}\left\{\frac{\sin 2\pi t}{\pi t}\right\} = \begin{cases} 1, & \text{if } |\omega| \le 2\pi \\ 0, & \text{otherwise} \end{cases} =$$



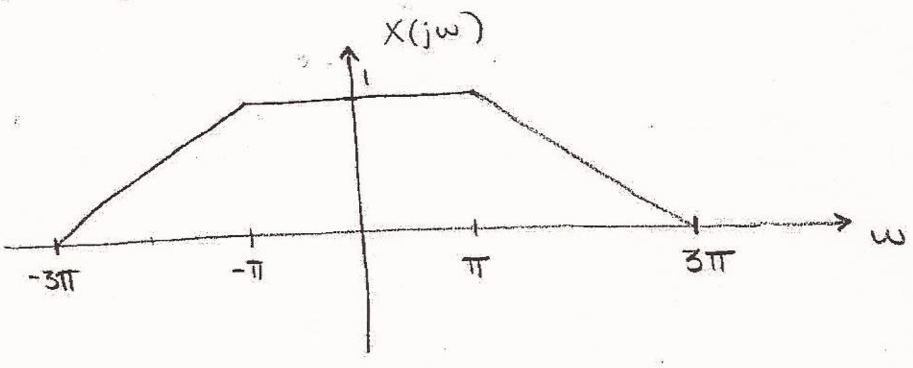
Now, the convolution of the above two spectrums is

$$\mathcal{F}\left\{\frac{\sin \pi t}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin 2\pi t}{\pi t}\right\} = \begin{cases} 2\pi, & \text{if } 0 \le |\omega| \le \pi \\ 3\pi - |\omega|, & \text{if } \pi \le |\omega| \le 3\pi \\ 0, & \text{if } |\omega| \ge 3\pi \end{cases}$$

Hence,

$$X(j\omega) = \begin{cases} 1, & \text{if } 0 \le |\omega| \le \pi \\ \frac{3}{2} - \frac{|\omega|}{2\pi}, & \text{if } \pi \le |\omega| \le 3\pi \\ 0, & \text{if } |\omega| \ge 3\pi \end{cases}$$

$=$



• Other variation on modulation property:

$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}X(\omega-\omega_0) - \frac{1}{2j}X(\omega+\omega_0)$$

Use for Probs. 4.21 (b) and (e)

<u>4.21 (b)</u> $\quad e^{-3|t|}\sin(2t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$

FT pair from Ex. 4.2 in text on pg. 292:

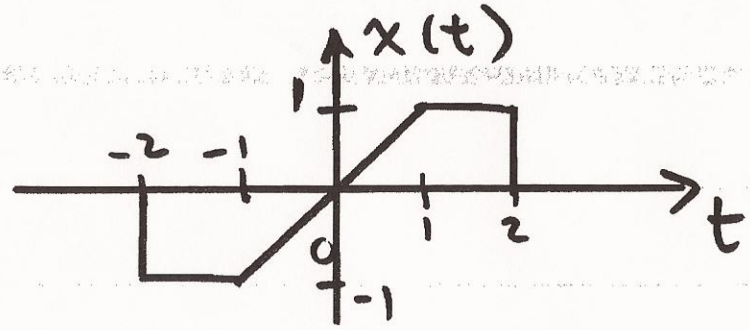
$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$$

THUS:

$$X(\omega) = \frac{1}{2j}\frac{2(3)}{(\omega-2)^2+3^2} - \frac{1}{2j}\frac{2(3)}{(\omega+2)^2+3^2}$$

$$= +3j\left\{\frac{1}{(\omega+2)^2+9} - \frac{1}{(\omega-2)^2+9}\right\}$$

Prob. 4.21 (g)



$$x(t) = -\,\mathrm{rect}\left(\frac{t+\frac{3}{2}}{1}\right) + t\,\mathrm{rect}\left(\frac{t}{2}\right) + \mathrm{rect}\left(\frac{t-\frac{3}{2}}{1}\right)$$

FT pair: $\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \dfrac{\sin\left(T\frac{\omega}{2}\right)}{\omega/2}$

FT property: $x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$

$$t\,x(t) \xleftrightarrow{\mathcal{F}} j\frac{d}{d\omega}X(\omega)$$

$$X(\omega) = -\frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}e^{j\frac{3}{2}\omega} + j\frac{d}{d\omega}\left\{\frac{\sin(\omega)}{\omega/2}\right\} + \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}e^{-j\frac{3}{2}\omega}$$

$$= -4j\,\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}\,\sin\left(\frac{3}{2}\omega\right) + j\,2\left\{\frac{\cos(\omega)\,\omega - \sin(\omega)}{\omega^2}\right\}$$

Example. Prob. 4.21 (i)

$$x(t) = \begin{cases} 1-t^2, & 0<t<1 \\ 0, & \text{otherwise} \end{cases} = (1-t^2)\,\text{rect}\left(\frac{t-\frac{1}{2}}{1}\right)$$

$$= \text{rect}\left(\frac{t-\frac{1}{2}}{1}\right) - t^2\,\text{rect}\left(\frac{t-\frac{1}{2}}{1}\right)$$

Basic FT pair: $\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \dfrac{\sin\left(\frac{\omega}{2}T\right)}{\omega/2}$ $\left(\begin{matrix} T=1 \\ \text{here}\end{matrix}\right)$

Two FT properties:

$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(\omega)e^{-j\omega t_0}$ and $t\,x(t) \xleftrightarrow{\mathcal{F}} j\dfrac{dX(\omega)}{d\omega}$

Generalization: $t^n x(t) \xleftrightarrow{\mathcal{F}} (j)^n \dfrac{d^n X(\omega)}{d\omega^n}$

Thus:

$$X(\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} e^{-j\omega\frac{1}{2}} - (-1)\frac{d^2}{d\omega^2}\left\{\frac{\sin\left(\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}}}{\omega/2}\right\}$$

★ Alternative Derivation of $X_0(\omega)$

Prob. 4.23 (cont.) $x_0(t) = \begin{cases} e^{-t}, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$

$$x_0(t) = e^{-t}\{u(t) - u(t-1)\}$$

$$= e^{-t}u(t) - e^{-1}e^{-(t-1)}u(t-1)$$

Since: $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{a+j\omega}$ $\boxed{x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(\omega)}$

We have: $X_0(\omega) = \dfrac{1}{1+j\omega}\left(1 - e^{-1}e^{-j\omega}\right)$

$$= \frac{1 - e^{-1}(\cos\omega - j\sin\omega)}{1+j\omega} \cdot \frac{(1-j\omega)}{(1-j\omega)}$$

$$0.5 \quad \underbrace{\frac{2 - 2e^{-1}\cos(\omega) - 2e^{-1}\omega\sin(\omega)}{1+\omega^2}}_{Re\{X_0(\omega)\}} + j\underbrace{\left\{\frac{-2\omega + 2e^{-1}\sin(\omega) + 2e^{-1}\omega\cos(\omega)}{1+\omega^2}\right\}}_{Im\{X_0(\omega)\}} \quad 0.5$$

4.23 Cnt'd

a) $x_1(t) = x_0(t) + x_0(-t) = 2\,\cancel{X}_{even}(t)$

thus

$$2\,x_{even}(t) = x_1(t) \xleftrightarrow{\mathcal{F}} 2\,Re\{X_0(\omega)\} = \left[\frac{2 - 2e^{-1}\cos(\omega) - 2e^{-1}\omega\sin(\omega)}{1+\omega^2}\right] = X_1(\omega)$$

b) $x_2(t) = x_0(t) - x_0(-t) = 2\,x_{odd}(t)$

Using Property $x_{odd}(t) \xleftrightarrow{F} j\,Im\{X_0(\omega)\}$

we get $x_2(t) \xleftrightarrow{F} 2\,j\,Im\{X_0(\omega)\}$

$$X_2(\omega) = j\left[\frac{-2\omega + 2e^{-1}\sin\omega + 2e^{-1}\omega\cos\omega}{1+\omega^2}\right]$$

c) $x_3(t) = x_0(t+1) + x_0(t)$

by time shifting property, we have $x_0(t+1) \xleftrightarrow{F} X_0(\omega)\,e^{j\omega}$

so $x_3(t) = x_0(t+1) + x_0(t) \xleftrightarrow{F} X_3(\omega) = X_0(\omega)\,e^{j\omega} + X_0(\omega) = \left(e^{j\omega} + 1\right)\left[\frac{1 - e^{-(1+j\omega)}}{1+j\omega}\right]$

$$\Rightarrow X_3(\omega) = \frac{\left(1+e^{j\omega}\right)\left(1-e^{-(1+j\omega)}\right)}{1+j\omega} = \frac{1+e^{j\omega} - e^{-1}\left(1+e^{-j\omega}\right)}{1+j\omega}$$

d) $x_4(t) = t\,x_0(t)$

Using the differentiation in frequency domain property: $t\,x(t) \xleftrightarrow{\mathcal{F}} j\frac{d}{d\omega}X(\omega)$

we get that $x_4(t) \xleftrightarrow{\mathcal{F}} X_4(\omega) = j\frac{d}{d\omega}X_0(\omega)$

$$X_4(\omega) = j\frac{d}{d\omega}\left[\frac{1-e^{-(1+j\omega)}}{1+j\omega}\right] = \frac{1 - 2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1+j\omega)^2}$$

ECE301

15 [redacted].

Text Prob. 4.24

(a) (1) Recall) $x(t)$: real & odd $\overset{\mathcal{F}}{\longleftrightarrow}$ $X(j\omega)$: purely imaginary & odd.

$\Rightarrow$ $x(t)$: real & odd $\longrightarrow$ $Re\{X(j\omega)\} = 0$.

Figure (a) and (d) have this property

(2) Recall) $x(t)$: real & even $\overset{\mathcal{F}}{\longleftrightarrow}$ $X(j\omega)$: purely real & even $-(*)$

$\Rightarrow$ $x(t)$: real & even $\longrightarrow$ $Im\{X(j\omega)\} = 0$.

Figure (e) and (f) have this property.

(3) Recall) $x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$ $-(**)$.

$\Rightarrow$ $x(t+\alpha)$: real & even $\longrightarrow$ There exist a real α s.t $e^{j\alpha\omega}X(j\omega)$ is real

(By $(*)$ and $(**)$).

Figure $\underbrace{(a)}_{\alpha=1}$, $\underbrace{(b), (e), \text{and } (f)}_{\alpha=0}$ have this property

(4) $\int_{-\infty}^{\infty} X(\omega)d\omega = \int_{-\infty}^{\infty} X(\omega)e^{j\omega 0}d\omega = 0$, $\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega 0}d\omega}_{x(t=0)} = 0$.

$\Rightarrow$ $x(0) = 0 \longrightarrow \int_{-\infty}^{\infty} X(\omega)d\omega = 0$.

Figure (a), (b), (c), (d), and (f) have this property

(5) Recall) $\frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega)$

$\Rightarrow$ $\left.\frac{d}{dt}x(t)\right|_{t=0} = 0 \longrightarrow \int_{-\infty}^{\infty} j\omega X(\omega) = 0$, i.e. $\int_{-\infty}^{\infty} \omega X(\omega) = 0$

Figure (b), $\underbrace{(c), (e)}$, and (f) have this property.

@ $t=0$, it has extreme points $\rightarrow$ $\left.\frac{d}{dt}x(t)\right|_{t=0} = 0$.

(6) Recall) $e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$

Let $X(\omega)$ be periodic for $-\infty<\omega<\infty$ w/ period T. Then $x(t)$ has the complex exponential Fourier series.

$$X(\omega) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 \omega}, \quad \text{where} \quad \omega_0 = 2\pi/T.$$

$\mathcal{F} \updownarrow$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \delta(t+k\omega_0) \rightarrow \text{train of impulse functions}$$

located @ $t = -k\omega_0$, $k = 0, \pm1, \pm2 \ldots$

$\Rightarrow$ $x(t)$: discrete (or sampled) signal $\longrightarrow$ $X(\omega)$: periodic

Figure (b) has this property

Prob. 4.38 We previously proved modulation property as requested in part (a):

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$X(\omega - \omega_0) = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(x(t) e^{j\omega_0 t} \right) e^{-j\omega t}\, dt$$

Since FT is unique, this proves:

$$e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$$

Part (b): prove same property using multiplication property: $x(t)\, y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega)$

So, let: $y(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0)$

Thus:

$$x(t)e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * 2\pi\delta(\omega - \omega_0)$$

$$= X(\omega) * \delta(\omega - \omega_0)$$

$$= X(\omega - \omega_0)$$

Alternative proof of modulation property.

- On an exam, simply use modulation property

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$$

  - no need to use multiplication property if multiplying by a sinewave

Prob 4.39

a/ **Duality Property**: This is subtly tricky.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

t is a dummy variable of integration
change to $\lambda = t$

$$X(\omega) = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda$$

Now, replace/substitute $\omega = t$ in eqn. above

$$g(t) = X(t) = \int_{-\infty}^{\infty} x(\lambda) e^{-jt\lambda} d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi x(\lambda) e^{-j\lambda t} d\lambda$$

change of variables: $\omega = -\lambda$ $\quad (\lambda = -\omega)$
$d\omega = -d\lambda$ $\quad (d\lambda = -d\omega)$

$$g(t) = X(t) = \frac{1}{2\pi} \int_{\infty}^{-\infty} 2\pi x(-\omega) e^{j\omega t} (-d\omega)$$

now in form of inverse Fourier Transform

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{2\pi x(-\omega)\} e^{j\omega t} d\omega$$

Thus, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

then $\underset{g(t)}{\underset{\parallel}{X(t)}} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$

i.e. $g(t) = X(t) \overset{\mathcal{F}}{\longleftrightarrow} \overset{G(\omega)}{\overset{\parallel}{2\pi x(-\omega)}}$

QED

4.39a

(b) $\mathcal{F}\{\delta(t+B)\} = e^{jB\omega}$

let $\delta(t+B) = x(t)$, then $X(\omega) = e^{jB\omega}$

by duality property, $\mathcal{F}\{g(t)\}$ where $g(t) = X(t) = e^{jBt}$

will be : $\mathcal{F}\{g(t)\} = \mathcal{F}\{X(t)\} = 2\pi x(-\omega)$

with $x(t) = \delta(t+B)$

$$\Rightarrow \mathcal{F}\{g(t)\} = \mathcal{F}\{X(t)\} = 2\pi\,\delta(-\omega+B)$$

$$\mathcal{F}\{g(t)\} = \mathcal{F}\{X(t)\} = 2\pi\,\delta(\omega-B)$$

since $\delta(\lambda) = \delta(-\lambda)$ where $\lambda = \omega - B$

QED