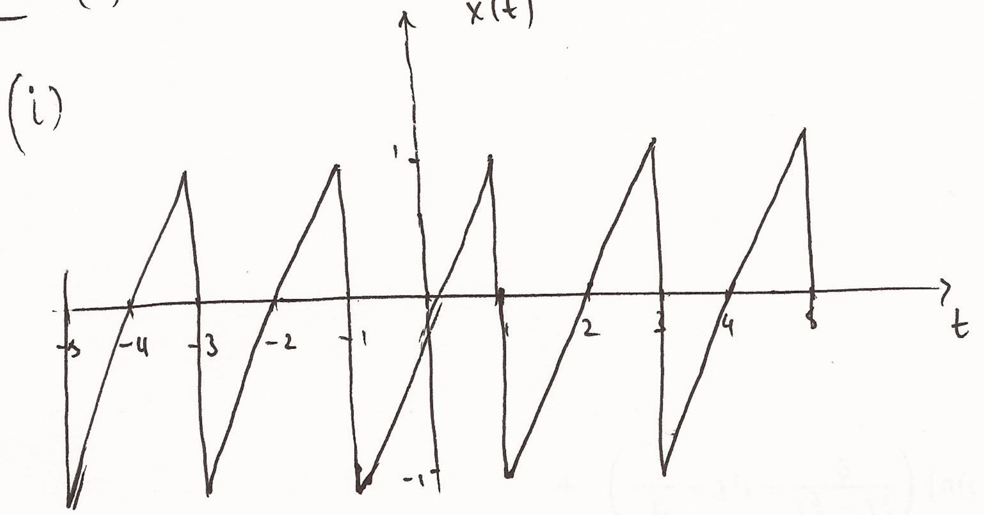
# ECE 301 Spring 2019

## Hw 5 Solutions

Prob 3.22 (a)

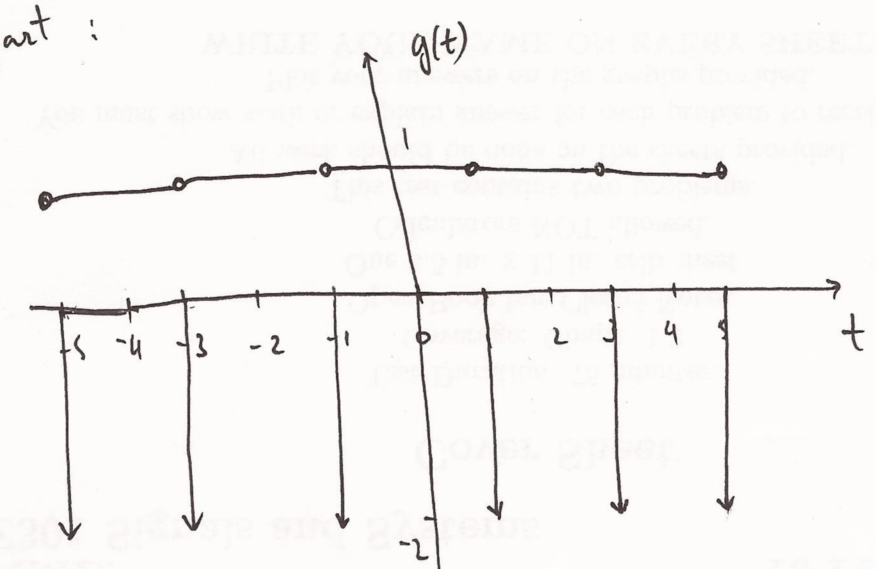
(i)



$$x(t) \xleftrightarrow{FS} a_k$$

Let $g(t) = \dfrac{d\,x(t)}{dt} \xleftrightarrow{F.S} b_k$

By taking $\dfrac{d\,x(t)}{dt}$, we get the following : A train of deltas + Constant part :



$$\text{So } b_k = \underbrace{\frac{\sin\left(\frac{2\pi k\, 1}{2}\right)}{k\pi}}_{\text{F.S of ●—● part}} + \underbrace{(-2)\left(\frac{1}{2}\right) e^{-jk\frac{2\pi}{2}(1)}}_{\text{F.s of train of deltas that are shifted}}$$

⇒ We make Use of linearity and time shifting properties

⇒ (↓ ↓)

$$b_k = 0 - 1\, e^{-jk\pi}$$

$$= -e^{-jk\pi}$$

$$= -(-1)^k \qquad k \neq 0$$

By differentiation property.

$$jk\frac{2\pi}{T} a_k = b_k$$

$$a_k = \frac{\cancel{2}}{\cancel{2}k\pi j} b_k = \frac{b_k}{k\pi j}$$

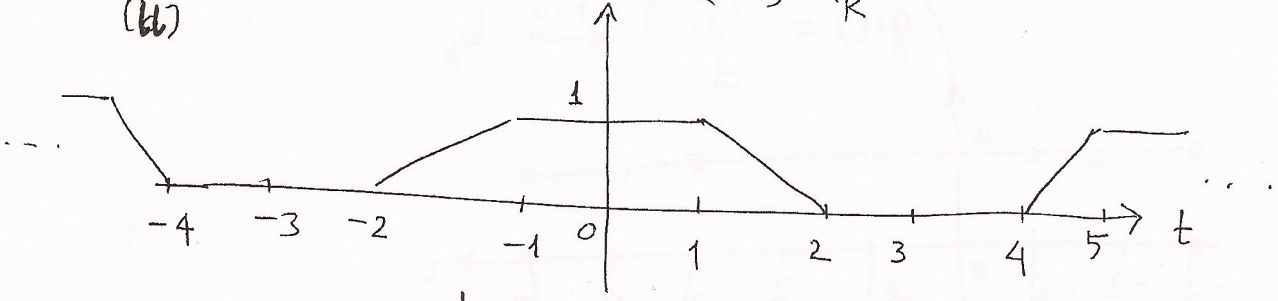
$$= \frac{-(-1)^k}{k\pi j}$$

$$= \frac{j(-1)^k}{k\pi} \qquad k \neq 0$$

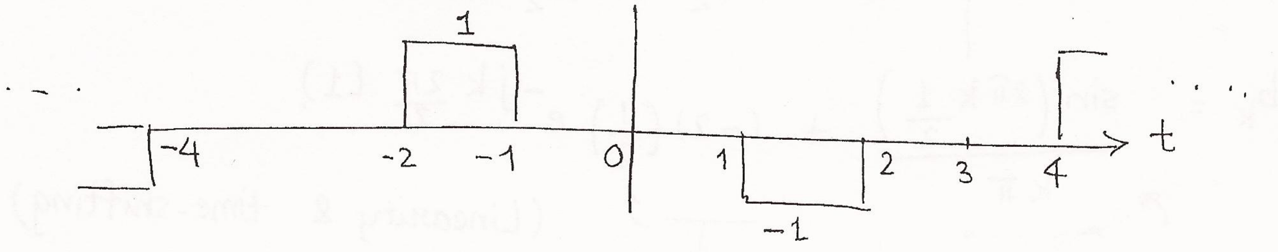
For $k = 0$,

$$a_k = a_0 = 0$$ (average value of $x(t)$ over one period)

(ii) $x(t) \xleftrightarrow{F.S} a_k$



$$\frac{dx(t)}{dt} = g(t) \xleftrightarrow{F.S} b_k$$



$$b_k = \underbrace{\frac{\sin\left(\frac{k\pi\, 2\, \frac{1}{2}}{6}\right)}{k\pi}\, e^{-j\frac{2\pi k}{6}\left(\frac{-3}{2}\right)}}_{+\text{ pulse}} + \underbrace{-\frac{\sin\left(\frac{k\pi\, 2\cdot \frac{1}{2}}{6}\right)}{k\pi}\, e^{-j\frac{2\pi k}{6}\left(\frac{3}{2}\right)}}_{-\text{ pulse}}$$

(By Linearity & time shifting)

$$= \frac{\sin \frac{k\pi}{6}}{k\pi}\left(e^{\frac{j2\pi k}{4}} + (-e^{-j\frac{2\pi k}{4}})\right)$$

$$= \frac{\sin \frac{k\pi}{6}}{k\pi}\,(j2)(\sin \frac{k\pi}{2})$$

$$= \frac{j2}{k\pi}\,\sin\left(\frac{k\pi}{6}\right)\sin\left(\frac{k\pi}{2}\right) \qquad k \neq 0$$

By differentiation property.

$$a_k = \frac{1}{jk\frac{2\pi}{6}}\, b_k$$

$$= \frac{1}{\cancel{j}k\frac{\pi}{3}}\,\frac{\cancel{j}2}{k\pi}\,\sin\left(\frac{k\pi}{6}\right)\sin\left(\frac{k\pi}{2}\right)$$

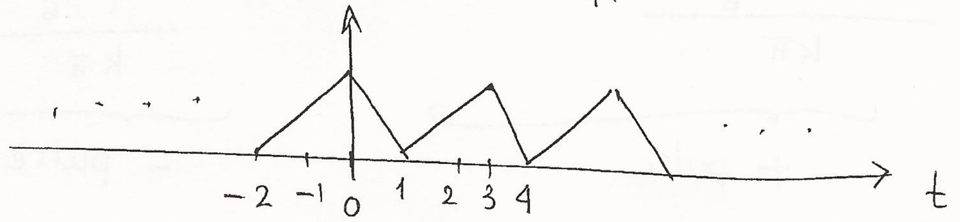
$$= \frac{6}{k^2\pi^2}\,\sin\left(\frac{k\pi}{6}\right)\sin\left(\frac{k\pi}{2}\right) \qquad k \neq 0$$

For $k=0$,

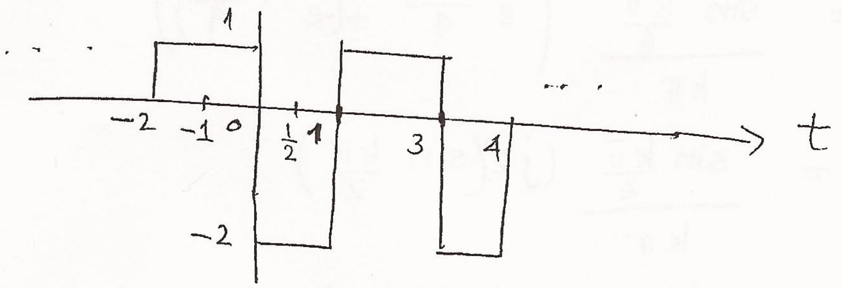
$$a_k = a_0 = \frac{3}{6} = \frac{1}{2} \qquad k=0$$

(iii)

$$x(t) \xleftrightarrow{F.S} a_k$$



$$g(t) = \frac{d\,x(t)}{dt} \xleftrightarrow{F.S} b_k$$



For $k \neq 0$

$$b_k = \underbrace{\frac{1}{k\pi}\sin\left(\frac{k\pi\, 2(1)}{3}\right)e^{-jk\frac{2\pi}{3}(-1)}}_{+\text{ pulse}} \underbrace{-\frac{2}{k\pi}\sin\left(\frac{k\pi\, 2\left(\frac{1}{2}\right)}{3}\right)e^{-jk\frac{2\pi}{3}\left(\frac{1}{2}\right)}}_{-\text{ pulse.}}$$

(By linearity & time shift)

By differentiation property,

$$a_k = \frac{1}{jk\frac{2\pi}{3}}\, b_k$$

$$= \frac{1}{k\pi}\frac{3}{jk2\pi}\left[\sin\left(\frac{2k\pi}{3}\right)e^{jk\frac{2\pi}{3}} - 2\sin\left(\frac{k\pi}{3}\right)e^{-jk\frac{\pi}{3}}\right]$$

~~$= \frac{3}{jk^2 2\pi^2}\sin\left(\frac{k\pi}{3}\right)\left[e^{jk\frac{2\pi}{3}} - 2e^{-jk\frac{\pi}{3}}\right]$~~

$$= \frac{3}{2j\,k^2\pi^2}\left[\frac{\left(e^{j\frac{2\pi k}{3}} - e^{-j\frac{2\pi k}{3}}\right)}{2j}e^{jk\frac{2\pi}{3}} - 2\frac{\left(e^{jk\frac{\pi}{3}} - e^{-jk\frac{\pi}{3}}\right)}{2j}e^{-jk\frac{\pi}{3}}\right]$$

$$a_k = \frac{-3}{4\pi^2 k^2}\left[\left(e^{j\frac{4\pi k}{3}} - 1\right) - 2\left(1 - e^{-j\frac{2\pi k}{3}}\right)\right]$$

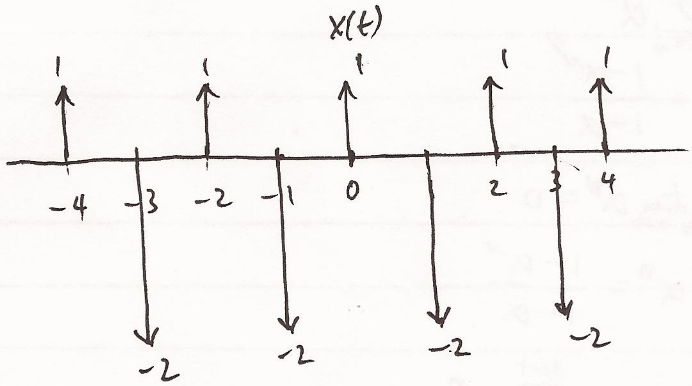
$$a_k = \frac{-3}{4\pi^2 k^2}\left[e^{j\frac{4\pi k}{3}} - 1 - 2 + 2e^{-j\frac{2\pi k}{3}}\right]$$

$$\underline{\underline{a_k = \frac{9}{4\pi^2 k^2} - \frac{9}{4\pi^2 k^2} e^{-j\frac{2k\pi}{3}}}} \qquad k \neq 0$$

For $k=0$ , $a_0$ is just the average value $= \frac{3}{3} = 1$

so $\underline{\underline{a_0 = 1}}$

(iv)



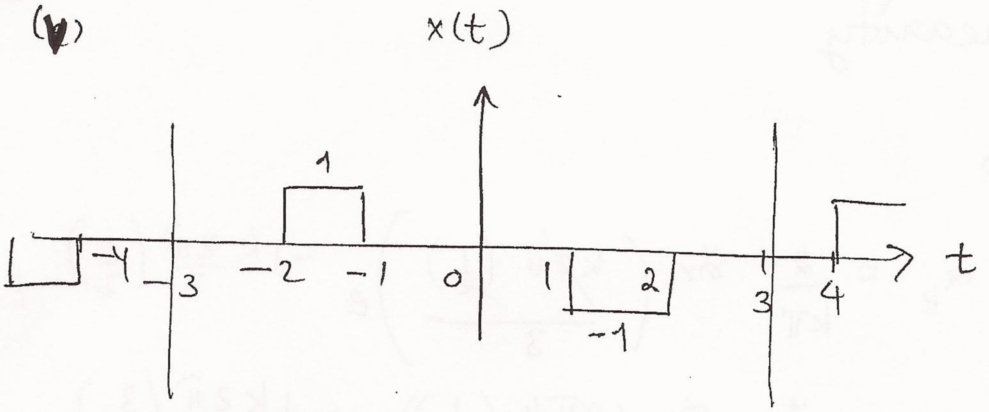
By linearity + time shift and Using F.S of train of deltas,

$$a_k = \frac{1}{2} + (-2)\,\frac{1}{2}\,e^{-j\frac{2\pi}{2}k(1)}$$

$$a_k = \frac{1}{2} - e^{-jk\pi}$$

$$a_k = \frac{1}{2} - (-1)^k \qquad \text{for all } k$$

(v)



from (ii)

$$a_k = \frac{j2}{k\pi} \sin\left(\frac{k\pi}{6}\right) \sin\left(\frac{k\pi}{2}\right) \qquad k \neq 0$$

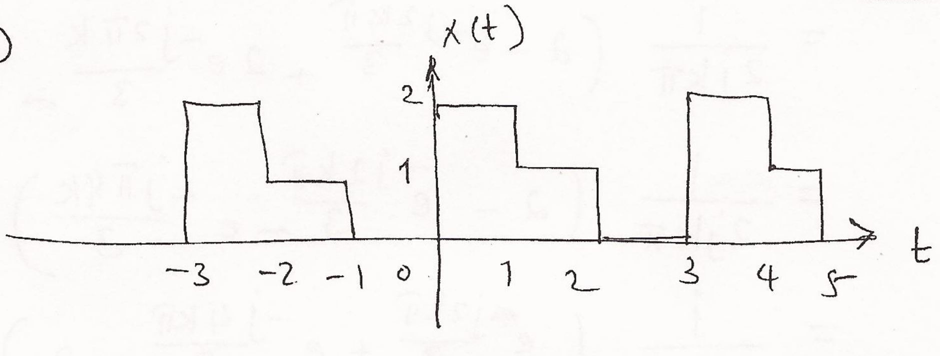
$$= \frac{j}{k\pi}\left[\cos\frac{\pi}{3}k - \cos\frac{2\pi}{3}k\right]$$

$\hookrightarrow$ $\boxed{2 \sin A \sin B = \cos(A-B) + \cos(A+B)}$

k=0,

$a_k = 0 \leftarrow$

(vi)



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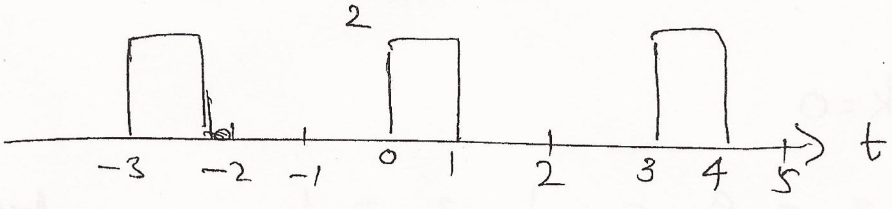

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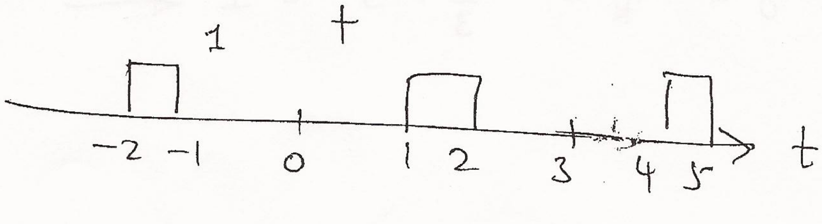


By Linearity

$k \neq 0$

$$a_k = \frac{2}{k\pi}\sin\left(\frac{k\cancel{2}\pi\left(\frac{1}{\cancel{2}}\right)}{3}\right)e^{-jk\frac{2\pi}{3}\left(\frac{1}{2}\right)} +$$

$$\frac{1}{k\pi}\sin\left(\frac{\cancel{2}\pi k\left(\frac{1}{\cancel{2}}\right)}{3}\right)e^{-jk\frac{2\pi}{3}\left(\frac{3}{2}\right)}$$

$$= \frac{\left(\sin\frac{\pi k}{3}\right)}{k\pi}\left[2e^{-j\frac{k\pi}{3}} + e^{-jk\pi}\right] \leftarrow$$

$$= \frac{1}{k\pi}\,\frac{\left(e^{j\frac{k\pi}{3}} - e^{-j\frac{k\pi}{3}}\right)}{2j}\left[2e^{-j\frac{k\pi}{3}} + e^{-jk\pi}\right]$$

$$= \frac{1}{2jk\pi}\left(2 - e^{-j\frac{2k\pi}{3}} + 2e^{-j\frac{2\pi k}{3}} - e^{-j\frac{\pi 4k}{3}}\right)$$

$$= \frac{1}{2jk\pi}\left(2 - e^{-j\frac{2k\pi}{3}} - e^{-j\frac{\pi 4k}{3}}\right)$$

$$= \frac{j}{2k\pi}\left(e^{-j\frac{2k\pi}{3}} + e^{-j\frac{4k\pi}{3}} - 2\right)$$

$k = 0$

$a_0 = a_k = \frac{1}{3}\cdot 3 = 1$ ← Avg over one period.

## Prob 3.22 (b)

$$x(t) = e^{-t} \qquad -1 < t < 1 \qquad , T = 2$$

So
$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t} \cdot e^{-j\frac{2\pi}{2}kt}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+jk\pi)t}\, dt$$

$$= \frac{1}{2}\cdot\frac{-1}{1+jk\pi}\, e^{-(1+jk\pi)t}\Big|_{-1}^{1}$$

$$= \frac{-1}{2(1+jk\pi)}\left[e^{-(1+jk\pi)} - e^{(1+jk\pi)}\right]$$

$$= \frac{-1\cdot e^{jk\pi}}{2(1+jk\pi)}\left[e^{-1} - e^{1}\right] \qquad \text{Since } e^{-jk\pi} = e^{jk\pi} = (-1)^k$$

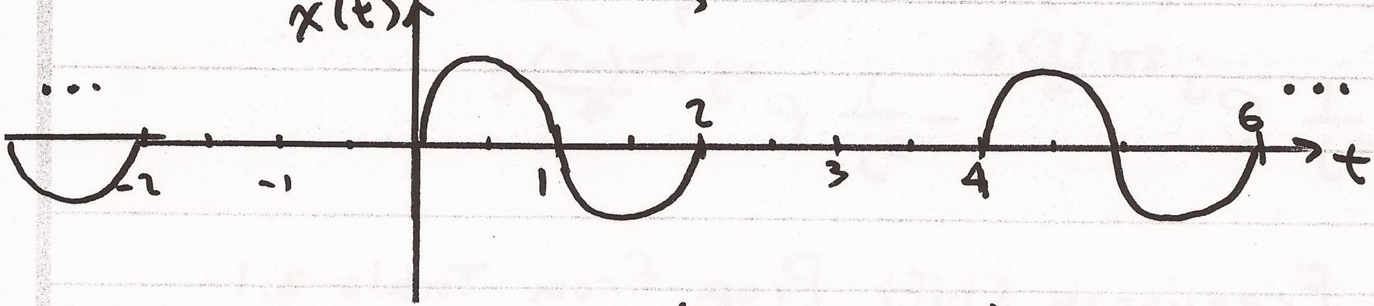
$$= \frac{(-1)(-1)^k}{2(1+jk\pi)}\left[e^{-1} - e\right]$$

$$= \frac{(-1)^k}{2(1+jk\pi)}\left[e - e^{-1}\right] \qquad \text{for all } k$$

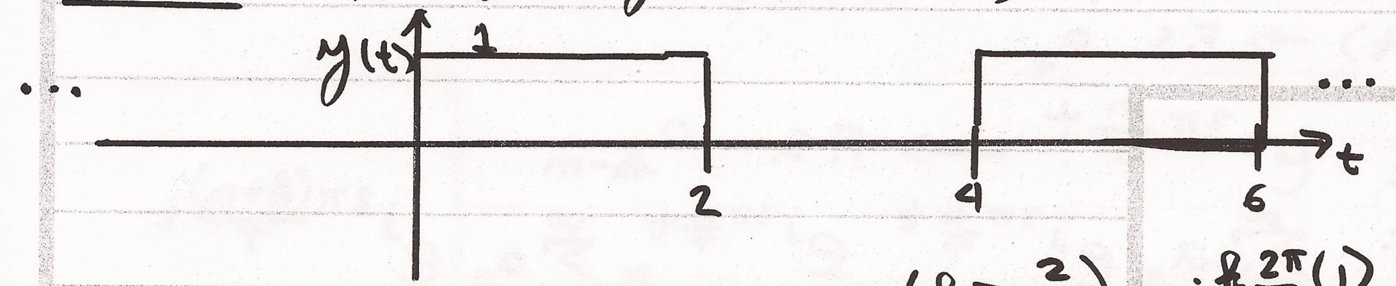
## Prob. 3.22 (c)

$$x(t) = \begin{cases} \sin(\pi t), & 0 \le t \le 2 \\ 0, & 2 \le t \le 4 \end{cases}$$

period $= T$ $= 4$



**TRICK** : $x(t) = y(t) \sin(\pi t) =$ where:



F.S. for y(t) above: $b_k = \frac{\sin(k\pi \frac{2}{4})}{k\pi} e^{-jk\frac{2\pi}{4}(1)}$

$$b_k = \frac{\sin(k\frac{\pi}{2})}{k\pi}(-j)^k \qquad \boxed{j = e^{j\frac{\pi}{2}}}$$ ★

**Note:** $\sin(\pi t) = \sin\left(2\pi \frac{2}{4} t\right)$

$$= \frac{1}{2j} e^{j 2\pi \frac{(2)}{4} t} - \frac{1}{2j} e^{+j 2\pi \frac{(-2)}{4} t}$$

**Also:** Frequency Shift Prop. from Table 3.1

$x(t) \rightarrow$ F.S. $a_k$

**then** $e^{j 2\pi \frac{m}{T} t} \longrightarrow$ F.S. $a_{k-m}$

Proof: $\sum_{k=-\infty}^{\infty} a_k e^{j 2\pi \frac{k}{T} t} \, e^{j 2\pi \frac{m}{T} t} = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi \frac{(k+m)}{T} t}$

change of variables $k' = k + m$
$\Rightarrow k = k' - m$

$$= \sum_{k'=-\infty}^{\infty} a_{k'-m} e^{j 2\pi \frac{k'}{T} t}$$

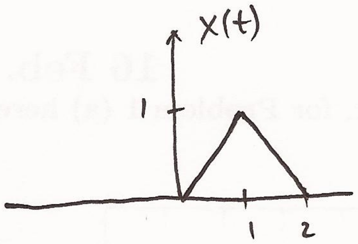
• Thus: $$x(t) = y(t)\left\{\overset{m=2}{\frac{1}{2j}e^{j\frac{2\pi(2)}{4}t}} - \overset{m=-2}{\frac{1}{2j}e^{j\frac{2\pi(-2)}{4}t}}\right\}$$

and

$$a_k = \frac{1}{2j}\, b_{k-2} - \frac{1}{2j}\, b_{k+2}$$

$$= \frac{1}{2j}\,\frac{\sin\left[(k-2)\frac{\pi}{2}\right]}{(k-2)\pi}(-j)^{k-2} - \frac{1}{2j}\,\frac{\sin\left[(k+2)\frac{\pi}{2}\right]}{(k+2)\pi}(-j)^{k+2}$$

Prob 3.24



$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$$

$T = 2$

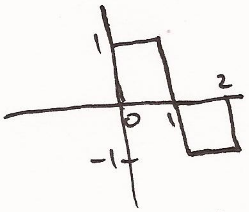
a| $a_0 = \text{Avg over the period} = \frac{\text{Area}}{2} = \frac{\frac{2 \cdot 1}{2}}{2} = \frac{1}{2}$

alternatively $a_0 = \frac{1}{2}\int_T x(t)\,dt$

$$= \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 (2-t)\,dt\right] = \frac{1}{2}$$

b| let $g(t) = \frac{d\,x(t)}{dt}$, then $g(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases}$

$T = 2$



so $g(t) \xleftrightarrow{FS} b_k = \underbrace{\frac{\sin\left(\frac{2k\pi}{2}\,\frac{1}{2}\right)}{k\pi}}_{+\text{ pulse}} \overset{\text{shift by } t_0 = \frac{1}{2}}{e^{-j\frac{2\pi}{2}k\left(\frac{1}{2}\right)}} - \underbrace{\frac{\sin\left(\frac{2k\pi}{2}\,\frac{1}{2}\right)}{k\pi}}_{-\text{ pulse}} \overset{\text{shift by } t_0 = \frac{3}{2}}{e^{-j\frac{2\pi}{2}k\left(\frac{3}{2}\right)}}$

$$\Rightarrow b_k = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi} e^{-j\frac{\pi}{2}k} - \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} e^{-j\frac{3\pi}{2}k}$$

Since) $e^{-j\frac{3\pi}{2}k} = e^{j\frac{\pi}{2}k}$

$$b_k = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}\left[e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}\right]$$

$$b_k = \frac{\frac{1}{2j}\left[e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right]}{k\pi}\left[e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}\right]$$

$$= \frac{1}{2jk\pi}\left[1 - e^{j\pi k} - e^{-j\pi k} + 1\right] = \frac{1}{2jk\pi}\left[2 - 2e^{-j\pi k}\right]$$ Since $e^{j\pi k} = e^{-j\pi k}$

$$b_k = \frac{1}{jk\pi}\left[1 - e^{-j\pi k}\right]$$ for all $k \neq 0$

c/ By differentiation property, if $x(t) \xleftrightarrow{F.S} a_k$

then $\frac{dx(t)}{dt} \xleftrightarrow{F.S} jk\frac{2\pi}{T}a_k$

So since $\frac{dx(t)}{dt} \xleftrightarrow{F.S} b_k$

it follows that $b_k = jk\frac{2\pi}{T}a_k = \frac{1}{jk\pi}\left[1 - e^{-j\pi k}\right]$

$$\Rightarrow a_k = \frac{b_k}{jk\frac{2\pi}{T}} = \frac{b_k}{jk\frac{2\pi}{2}} = \frac{b_k}{jk\pi} = \frac{\frac{1}{jk\pi}\left[1 - e^{-j\pi k}\right]}{jk\pi}$$

$$a_k = \frac{1}{(jk\pi)^2}\left[1 - e^{-j\pi k}\right] = \underline{\underline{\frac{-1}{k^2\pi^2}\left[1 - e^{-j\pi k}\right]}}$$

## Prob 3.25

a/ $x(t) = \cos(4\pi t) = \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right)$

$$= \frac{1}{2} e^{j\frac{2\pi}{\frac{1}{2}}t} + \frac{1}{2} e^{-j\frac{2\pi}{\frac{1}{2}}t} = \frac{1}{2} e^{j(1)\frac{2\pi}{\frac{1}{2}}t} + \frac{1}{2} e^{j(-1)\frac{2\pi}{\frac{1}{2}}t}$$

So for $k=1$, $a_1 = \frac{1}{2}$ and $k=-1$, $a_{-1} = \frac{1}{2}$

for all other $k \neq -1, 1$, $a_k = 0$

b/ $y(t) = \sin(4\pi t) = \frac{1}{2j}\left[e^{j4\pi t} - e^{-j4\pi t}\right] = \frac{1}{2j} e^{j(1)\frac{2\pi}{\frac{1}{2}}t} - \frac{1}{2j} e^{j(-1)\frac{2\pi}{\frac{1}{2}}t}$

thus $a_1 = \frac{1}{2j}$, $a_{-1} = \frac{-1}{2j}$ and $a_k = 0$ for all $k \neq -1, 1$

c/ $z(t) = x(t)\,y(t) \xleftrightarrow{F.S} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$ where $x(t) \xleftrightarrow{FS} a_k$, $y(t) \xleftrightarrow{FS} b_k$

So since $a_k$ and $b_k$ have values only at $k = -1, 1$, we get

$a_l = \left\{\frac{1}{2}, \frac{1}{2}\right\}$ for $l = 1, -1$ and $b_{k-l} = \left\{-\frac{1}{2j}, \frac{1}{2j}\right\}$ for $k-l = -1, 1$

thus if $k - l = -1 \Rightarrow k = -1 + l$ where $l = -1, 1$

$\Rightarrow k = -1-1 = \underline{\underline{-2}}$ or $k = -1+1 = \underline{\underline{0}}$

if $k - l = 1 \Rightarrow k = 1 + l \Rightarrow k = 1-1 = 0$ or $k = 1+1 = 2$

$\Rightarrow k = -2$, $c_{-2} = a_{-1} b_{-2-(-1)} + a_1 b_{-2-1} = a_{-1} b_{-1} + a_1 b_{-3} = \frac{1}{2}\cdot\frac{-1}{2j} + \frac{1}{2}\cdot 0 = \frac{-1}{4j}$

$k = 0$, $c_0 = a_{-1} b_{-1} + a_1 b_1 = \frac{1}{2}\left(-\frac{1}{2j}\right) + \frac{1}{2}\cdot\frac{1}{2j} = -\frac{1}{4j} + \frac{1}{4j} = 0$

$k = 2$, $c_2 = a_{-1} b_{2-(-1)} + a_1 b_{2-1} = a_{-1} b_3 + a_1 b_1 = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2j} = \frac{1}{4j}$

Hence Coefficients of $z(t)$ are: $c_{-2} = -\frac{1}{4j}$, $c_2 = \frac{1}{4j}$, $c_k = 0$ for all other $k \neq -2, 2$

d/ By direct expansion of $z(t)$, we get:

remember $T = \frac{1}{2}$

$$z(t) = \cos(4\pi t)\sin(4\pi t)$$

$$= \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right]\left[\frac{e^{j4\pi t} - e^{-j4\pi t}}{2j}\right]$$

$$= \frac{1}{4j}\left[e^{j8\pi t} - e^{\cancel{0}} + e^{\cancel{0}} - e^{-j8\pi t}\right]$$

$$= \frac{1}{4j}\left[e^{j8\pi t} - e^{-j8\pi t}\right]$$

$$= \frac{1}{4j}\left[e^{j\frac{2\pi}{\frac{1}{2}}(2)t} - e^{j\frac{2\pi}{\frac{1}{2}}(-2)t}\right]$$

$$= \frac{1}{4j}e^{j(2)\frac{2\pi}{\frac{1}{2}}t} - \frac{1}{4j}e^{j(-2)\frac{2\pi}{\frac{1}{2}}t}$$

Hence $k = 2$, $a_2 = \frac{1}{4j}$, $k = -2$, $a_{-2} = \frac{-1}{4j}$

$\forall k \neq -2, 2$, $a_k = 0$

**Prob. 3.40** $x(t) \longrightarrow$ F.S. $a_k$ ⑧

(a) $y(t) = x(t-t_0) + x(t+t_0)$ $b_k = ?$

$$b_k = a_k e^{-j 2\pi \frac{k}{T} t_0} + a_k e^{-j 2\pi \frac{k}{T}(-t_0)}$$

$$= 2 a_k \cos\left(2\pi \frac{k}{T} t_0\right)$$

(b) $y(t) = \mathcal{E}v\{x(t)\} = \frac{1}{2} x(t) + \frac{1}{2} \underbrace{x(-t)}_{\text{see Table 3.1 time-reverse property}}$

$$b_k = \frac{1}{2} a_k + \frac{1}{2} a_{-k}$$

(c) $y(t) = Re\{x(t)\} = \frac{1}{2} x(t) + \frac{1}{2} \underbrace{x^*(t)}_{\text{see Table 3.1 conjugation property}}$

$$b_k = \frac{1}{2} a_k + \frac{1}{2} a^*_{-k}$$

(d) $y(t) = \frac{d^2}{dt^2} x(t)$

$b_k = \left(jk\frac{2\pi}{T}\right)^2 a_k = -\left(k\frac{2\pi}{T}\right)^2 a_k$

(e) $y(t) = x(3t-1) = x\left(3\left(t-\frac{1}{3}\right)\right)$

First: $z(t) = x(3t)$

Time-Scaling does not change F.S. coeffts.
just the period

Here, new period for $z(t)$ is $\frac{T}{3}$

Then: $y(t) = z\left(t-\frac{1}{3}\right)$

Thus: $b_k = a_k e^{-j\frac{2\pi k}{T/3}\left(\frac{1}{3}\right)} = a_k e^{-j2\pi\frac{k}{T}}$