Hmwk 4 Soluion SP 2024

Example. Prob. $\rightarrow$ $\qquad y[n] = x[n] * h[n]$

$$x[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-10]$$

$$h[n] = \begin{cases} 1, & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases} = u[n] - u[n-(N+1)]$$

Find N given $y[4] = 5$ $\quad y[14] = 0$

From convolution of two DT rectangles result

on bottom of pg. 2:

length of $y[n] = 10 + (N+1) - 1 = N + 10$

is nonzero from $n=0$ to $n = N+10-1 = N+9$

So: $N + 9 < 14$ $\quad$ so $\quad N < 5$

So $N_1 = N$ and $N_2 = 9$ at bottom of pg. 2

Check $N = 4 = N_1$, $\Rightarrow$ $y[N_1] = N_1 + 1$ $\Rightarrow$ $y[4] = 5$ ✓ checks

Answer: $N = 4$

Prob. 2.21 (a) done in class notes

$$\underbrace{\alpha^n u[n]}_{x[n]} * \underbrace{\beta^n u[n]}_{h[n]} = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, \beta^{n-k} u[n-k] = \left\{ \sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^k u[-(k-n)] \right\} \beta^n$$

flipped around $k=0$ axis and slid to right by n

$$= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k = \beta^n \left\{ \frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}} \right\}$$

$$= \frac{\beta}{\beta-\alpha}\beta^n u[n] + \frac{\alpha}{\alpha-\beta}\alpha^n u[n]$$ when $\alpha \neq \beta$

(b) when $\alpha = \beta$

$$y[n] = \alpha^n \sum_{k=0}^{n} (1)^k = (n+1)\alpha^n u[n]$$

no part (c) for Hmwk.

Prob. 2.21 part (d) $\quad y[n] = x[n] * h[n]$

$$x[n] = u[n] - u[n-5] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Define:

$$\tilde{h}[n] = u[n] - u[n-6] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

Then: $h[n] = \tilde{h}[n-2] + \tilde{h}[n-11]$

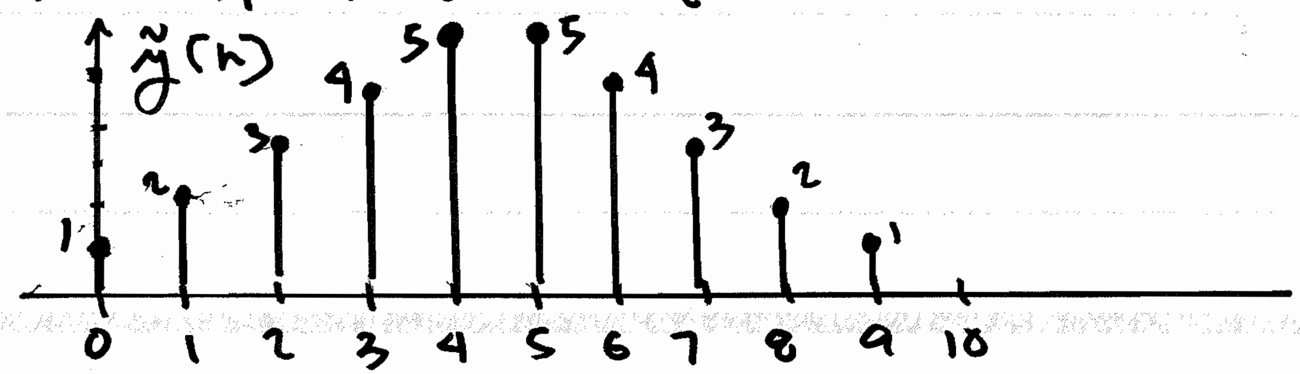
It follows: define $\tilde{y}[n] = x[n] * \tilde{h}[n]$

Then:

$$y[n] = \tilde{y}[n-2] + \tilde{y}[n-11]$$

where: $\tilde{y}[n]$ determined from bottom of pg. 2

with $N_1 = 4$ and $N_2 = 5$



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$$\tilde{y}[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$$

$$y[n] = \tilde{y}[n-2] + \tilde{y}[n-11]$$

$$\begin{array}{cccccccccccccc} n: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ = \{ & 0, & 0, & 1, & 2, & 3, & 4, & 5, & 5, & 4, & 3, & 2, & 1\} \end{array}$$

$$+ \{ \qquad 1, 2, 3, 4, 5, 5, 4, 3, 2, 1$$

Final answer:

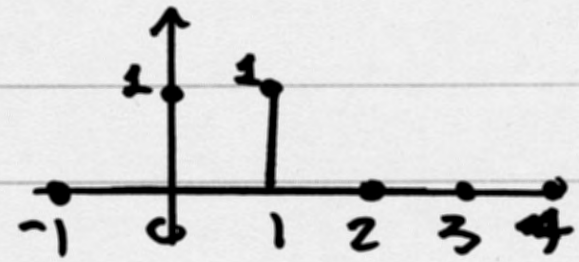
$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{0}, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, \underset{\substack{\uparrow}}{2}, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$$

single point where the two terms overlap at $n=11$

Prob. 2.24

$$h_2[n] = u[n] - u[n-2]$$

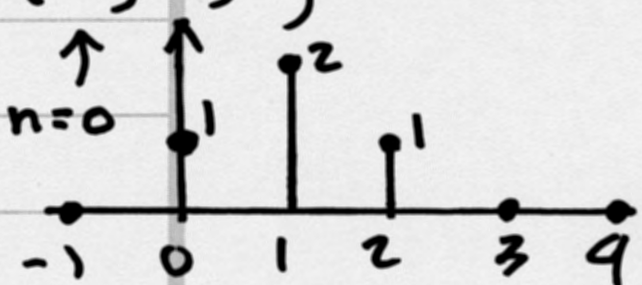
$$= \{\underset{\substack{\uparrow \\ n=0}}{1}, 1\}$$



Three LTI systems in series:

$$h_0[n] = h_1[n] * h_2[n] * h_2[n]$$

$$= h_1[n] * \underbrace{\{(u[n]-u[n-2]) * (u[n]-u[n-2])\}}_{h_3[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 1\}}$$



Recall DT convolution method discussed in class notes

$$h_0[n] = h_1[n] * \overbrace{\{1, 2, 1\}}^{h_3(n)}$$

$$= \sum_{k=0} h_1[k]\, h_3[n-k]$$

First: how "long" is $h_1[n]$? call it N

Given: $h_0[n]$ is of "length" 7

$$= \{1, 5, 10, 11, 8, 4, 1\}$$

$\uparrow$ $n=0$ $\qquad$ $\uparrow$ $n=6$

Thus:

$$N + 3 - 1 = 7 \implies N = 5$$

which means it's only nonzero over $0 \le n \le 4$

n:	0	1	2	3	4	5	6
$h_1[0]\,x$	1	2	1	0	0	0	0
$h_1[1]\,x$	0	1	2	1	0	0	0
$h_1[2]\,x$	0	0	1	2	1	0	0
$h_1[3]\,x$	0	0	0	1	2	1	0
$h_1[4]\,x$	0	0	0	0	1	2	1
$h_0[n]$	1	5	10	11	8	4	1

$n=0$:

$h_1[0]\cdot 1 = 1 \Rightarrow \boxed{h_1[0] = 1}$

$n=1$:

$2h_1[0] + 1\cdot h_1[1] = 5 \quad \Big| \quad h_1[1] = 5 - 2h_1[0] = 3$

$\boxed{h_1(1) = 3}$

$n=2$:

$1\cdot h_1[0] + 2\cdot h_1[1] + h_1[2] = 10$

$\Rightarrow h_1[2] = 10 - 1 - 2(3) = 3 \qquad \boxed{h_1[2] = 3}$

$n=3$:

$1\cdot h_1[1] + 2\cdot h_1[2] + 1\cdot h_1[3] = 11$

$\Rightarrow h_1[3] = 11 - 3 - 2(3) = 2 \qquad \boxed{h_1[3] = 2}$

$n=4$: $h_1[2] + 2h_1[3] + h_1[4] = 8$

$h_1[4] = 8 - 3 - 2(2) = 1 \qquad \boxed{h_1[4] = 1}$

$h_1[n] = \{\underset{\uparrow \atop n=0}{1}, 3, 3, 2, \underset{\uparrow \atop n=4}{1}\}$

Example. Prob. 2.26 $\quad y[n] = x_1[n] * x_2[n] * x_3[n]$

$$x_1[n] = \left(\tfrac{1}{2}\right)^n u[n] \qquad x_2[n] = u[n+3] \qquad x_3[n] = \delta[n] - \delta[n-1]$$

(a) $x_1[n] * x_2[n] = \,?\, = z[n]$

$\alpha = \frac{1}{2} \quad \beta = 1$ Example 2.3 in text

$$r_1 = \frac{\beta}{\beta-\alpha} = \frac{1}{1-\frac{1}{2}} = 2 \qquad r_2 = \frac{\alpha}{\alpha-\beta} = \frac{1/2}{1/2-1} = -1$$

$$\underbrace{x_1[n] * x_2[n]} = 2\,u[n+3] - \left(\tfrac{1}{2}\right)^{n+3} u[n+3]$$

(b) $z[n] * x_3[n]$ $\Rightarrow$ easiest way

$$z[n] * x_3[n] = z[n] * \{\delta[n] - \delta[n-1]\}$$

$$= z[n] - z[n-1]$$

$$= 2\,u[n+3] - \left(\tfrac{1}{2}\right)^{n+3} u[n+3] - 2\,u[n+2] + \left(\tfrac{1}{2}\right)^{n+2} u[n+2]$$

$$= \delta[n+3] + \left(-\tfrac{1}{8} + \tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^n u[n+2]$$

$$= \delta[n+3] + \tfrac{1}{8}\left(\tfrac{1}{2}\right)^n u[n+2]$$

You do parts (c) and (d)

Prob. 2.26 (cont.)

Ans to (b) further simplifies as

$$z[n] = \frac{1}{8}\left(\frac{1}{2}\right)^n u[n+3] = \left(\frac{1}{2}\right)^{n+3} u[n+3]$$

Now (c) and (d):

First convolve $x_2[n]$ and $x_3[n]$

$$u[n+3] * \left(\delta[n] - \delta[n-1]\right)$$

$$= u[n+3] * \delta[n] - u[n+3] * \delta[n-1]$$

$$= u[n+3] - u[n+2] = \delta[n+3]$$

Next, convolve this with $x_1[n]$

$$\left(\frac{1}{2}\right)^n u[n] * \delta[n+3] = \left(\frac{1}{2}\right)^{n+3} u[n+3]$$

- Demonstrates associativity of DT convolution:

$$\left(x_1[n] * x_2[n]\right) * x_3[n] = x_1[n] * \left(x_2[n] * x_3[n]\right)$$

Prob. 2.28

Causality: $h[n] = 0$ for $n < 0$

Stability: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

(a) $h[n] = \left(\frac{1}{3}\right)^n u[n]$ causal

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1 - 1/3} < \infty$$ stable

(b) $h[n] = (.8)^{-2}\delta[n+2] + (.8)^{-1}\delta[n+1] + .8^n u[n]$

$h[n] \neq 0$ for $n < 0$ not causal

$$\sum_{n=-2}^{\infty} |h[n]| = \frac{1}{.64} + \frac{1}{.8} + \frac{1}{1-.8} < \infty$$ stable

(c) $h[n] = \left(\frac{1}{2}\right)^n u[-n]$ not causal

$= (2)^{|n|} u[-n]$ not stable

Approaching infinity as $n \to -\infty$

(d) $h[n] = (5)^n u[3-n] = (5)^n u[-(n-3)]$

$$= \left(\frac{1}{5}\right)^{|n|} u[-n] + 5\delta[n] + 5\delta[n-1] + 5^2\delta[n-2] + 5^3\delta[n-3]$$

<u>not</u> causal $\qquad$ <u>stable</u>

(e) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[n-1]$

<u>causal</u> $\qquad$ <u>not</u> stable $\qquad$ diverges as $n \to \infty$

(f) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + \underbrace{(1.01)^n u[-(n-1)]}_{\text{not causal}}$

$$= \left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1}{1.01}\right)^{|n|} u[-n] + 1.01\,\delta[n-1]$$

$\sum = \dfrac{1}{1-\frac{1}{2}}$ (take absolute value) $\qquad \sum = \dfrac{1}{1-\frac{1}{1.01}}$

$$= \frac{1}{1-\frac{1}{1.01}}$$

stable!

(g) $h[n] = n\left(\frac{1}{3}\right)^n u[n]$ can include $n=0$ since $h[0]=0$

causal!!

$$\sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{1/3}{\left(1-\frac{1}{3}\right)^2} < \infty \quad \text{Stable!}$$

See part (c) of Prob. 1.54 from Hmwk. 1