Hmwk 3 Solution   SP 2021

Prob. 2.11

$$x(t) \rightarrow \overset{\text{LTI}}{\boxed{h(t)}} \rightarrow y(t)$$

$$\frac{dx(t)}{dt} \rightarrow \boxed{h(t)} \rightarrow \text{show } z(t) = \frac{dy(t)}{dt}$$

- $x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \frac{dx(t)}{dt}$ is an LTI system

- Thus: $x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \boxed{h(t)} \rightarrow z(t)$

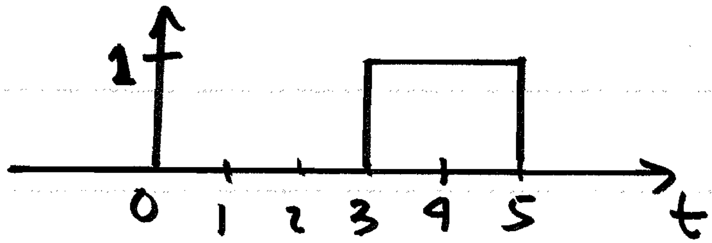
- Invoking associativity & commutativity:

$$x(t) \rightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{\frac{d}{dt}} \rightarrow z(t)$$

Thus: $z(t) = \frac{dy(t)}{dt}$ where: $y(t) = x(t) * h(t)$

dong part (b) first:

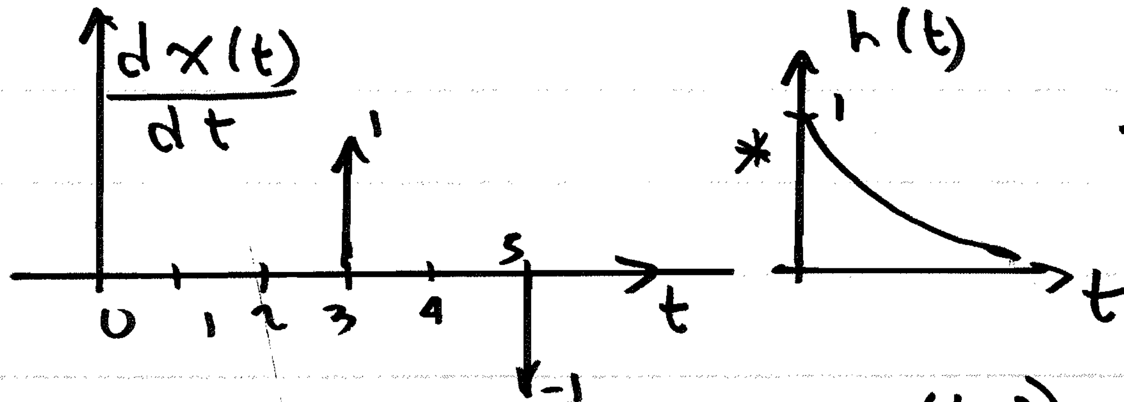
$$x(t) = u(t-3) - u(t-5) = \mathrm{rect}\left(\frac{t-4}{2}\right) \qquad h(t) = e^{-3t}u(t)$$



- Suppose you were interested in $z(t) = \dfrac{d\,y(t)}{dt}$

  where $y(t) = x(t) * h(t)$

- It's certainly easier to first differentiate $x(t)$ and then convolve with $h(t)$:



$= z(t)$

$$z(t) = h(t-3) - h(t-5) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$$

$$= \frac{d}{dt}y(t) \qquad \text{where: } y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Previous page was solution to part (b)

Now, let's do part (a)

$$y(t) = x(t) * h(t) \qquad x(t) = u(t-3) - u(t-5)$$

$$h(t) = e^{-3t}u(t)$$

First:

$$e^{-3t}u(t) * \{u(t) - u(t-2)\}$$

$$= e^{-3t}u(t) * u(t) - e^{-3t}u(t) * u(t-2)$$

$$= \left(\frac{1}{-3-0}e^{-3t} + \frac{1}{0-(-3)}e^{0t}\right)u(t) -$$

$$= \frac{1}{3}\left(1 - e^{-3t}\right)u(t) - \frac{1}{3}\left(1 - e^{-3(t-2)}\right)u(t-2)$$

Then: ans to part (a)

$$y(t) = \frac{1}{3}\left(1 - e^{-3(t-3)}\right)u(t-3) - \frac{1}{3}\left(1 - e^{-3(t-5)}\right)u(t-5)$$

(c) derivative of this ans for (a) should be ans to (b)

- have to use product rule:

Sifting Prop. of Delta Fn.

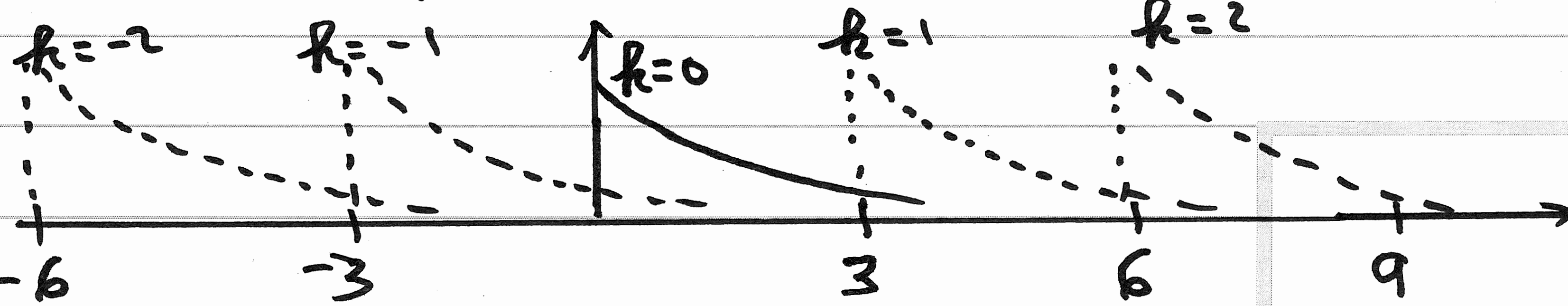
$$
\begin{aligned}
\frac{d}{dt} y(t) = {} & \frac{1}{3}\left(1 - e^{-3(t-3)}\right)\delta(t-3) \quad \Big\} = 0 \\
& - (-3)\frac{1}{3} e^{-3(t-3)} u(t-3) \\
& - \frac{1}{3}\left(1 - e^{-3(t-5)}\right)\delta(t-5) \quad \Big\} = 0 \\
& + (-3)\frac{1}{3} e^{-3(t-5)} u(t-5) \\
= {} & e^{-3(t-3)} u(t-3) - e^{-3(t-5)} u(t-5)
\end{aligned}
$$

<u>Prob. 2.12</u> $y(t) = e^{-t}u(t) * \sum_{k=-\infty}^{\infty} \delta(t-k3)$

<u>Recall:</u> $x(t) * \delta(t-T) = x(t-T)$

$$y(t) = \sum_{k=-\infty}^{\infty} e^{-(t-k3)} u(t-k3)$$

$$= e^{-t} \sum_{k=-\infty}^{\infty} (e^{3})^{k} u(t-k3)$$



only $k \le 0$ contribute in $0 \le t < 3$:

$$y(t) = e^{-t} \sum_{k=-\infty}^{0} (e^{3})^{k}$$

change of variables:

$k' = -k$

$$y(t) = e^{-t} \sum_{k'=\infty}^{0} (e^3)^{-k'} = e^{-t} \sum_{k'=0}^{\infty} \left(\frac{1}{e^3}\right)^{k'}$$

$$= e^{-t}\left(\frac{1}{\underbrace{1 - \frac{1}{e^3}}_{A}}\right)$$

answer

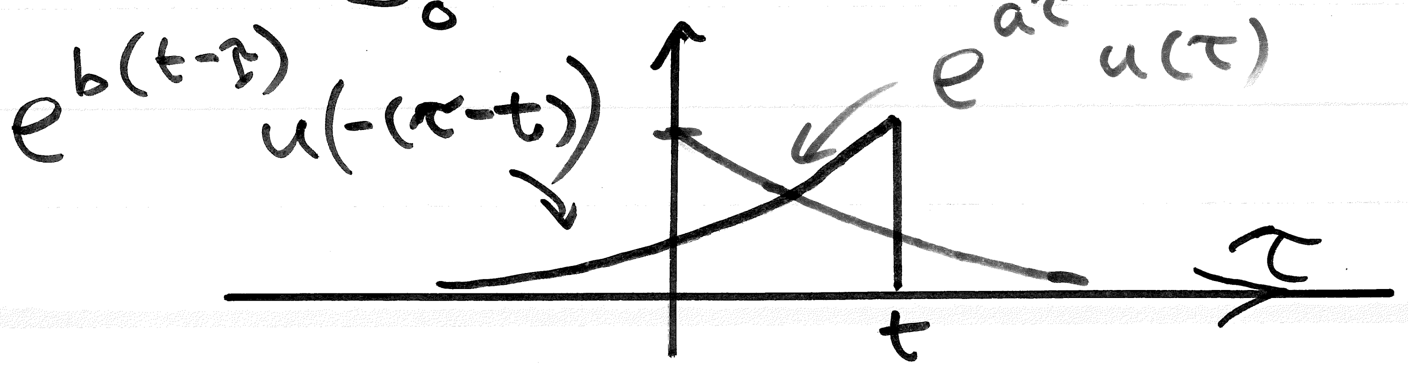
Prob. 2.22 (a) $(a=-\alpha,\ b=-\beta)$

$$e^{at}u(t) * e^{bt}u(t) = ? \quad \text{for } b \neq a$$

$$y(t) = \int_{-\infty}^{\infty} e^{a\tau} u(\tau)\, e^{b(t-\tau)} u(t-\tau)\, d\tau$$

$$= e^{bt} \int_{-\infty}^{\infty} e^{(a-b)\tau} u(\tau)\, u(-(\tau-t))\, d\tau$$

$$= e^{bt} \int_{0}^{t} e^{(a-b)\tau}\, d\tau \quad u(t)$$



$$y(t) = e^{bt}\left\{\frac{1}{a-b} e^{(a-b)\tau}\Big]_0^t\right\}$$

$$= \frac{e^{bt}}{a-b}\left\{e^{(a-b)t} - e^{0}\right\}$$

$$= \left\{\frac{e^{at} - e^{bt}}{a-b}\right\} u(t) \quad = y(t)$$

$$= e^{at}u(t) * e^{bt}u(t) \qquad a \neq b$$

$$= y(t) \quad \text{(notation)} = y(t)$$

Prob. 2.22 (a) cont.

· consider case $\alpha = \beta$:

$$e^{at}u(t) * e^{at}u(t) = ? \qquad (a = -\alpha)$$

$$y(t) = \int_0^t e^{a\tau} e^{a(t-\tau)}\, d\tau = \left\{ \int_0^t 1\, d\tau \right\} e^{at}$$

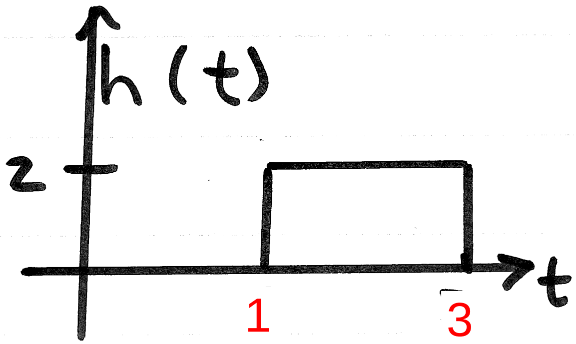
$$= e^{at} \cdot \tau \Big]_0^t = e^{at}(t-0) = t e^{at}$$

· Substituting: $a = -\alpha$

$$y(t) = t e^{-\alpha t} u(t) \quad \Leftarrow \text{ans}$$

Prob. 2.22 (c)

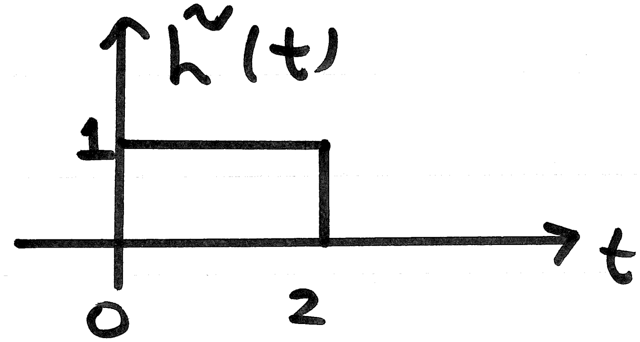
- **Convolve:** $x(t)$ with



$$y(t) = x(t) * h(t)$$

- **Instead:**

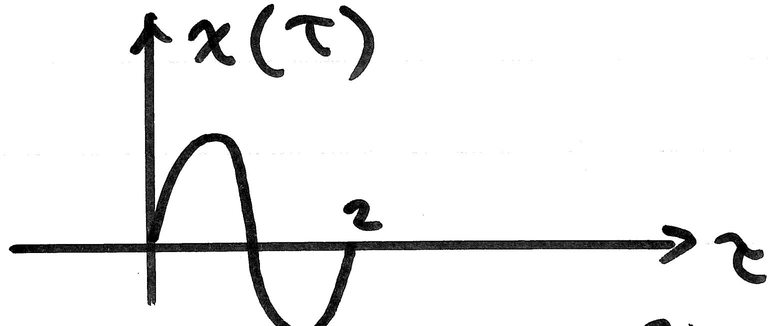
$$\tilde{y}(t) = x(t) * \tilde{h}(t)$$



- **Then:**

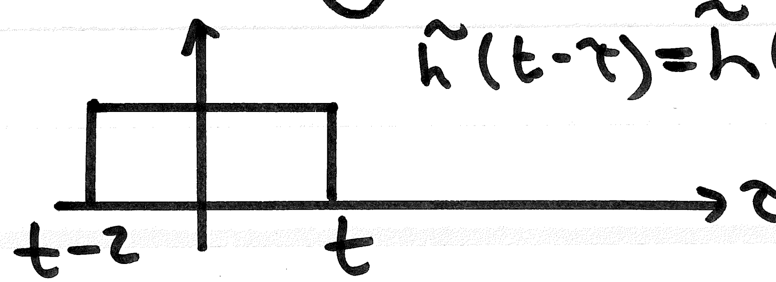
$$\boxed{y(t) = 2\,\tilde{y}(t-1)}$$

Method 1 :



For $0 < t < 2$:

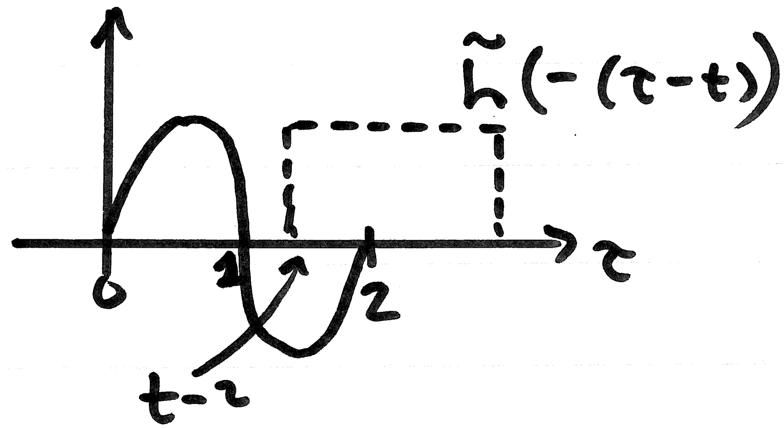
$$\tilde{h}(t-\tau) = \tilde{h}(-(\tau - t))$$



$$\tilde{y}(t) = \int_0^t \sin(\pi\tau)\,d\tau = \frac{-1}{\pi}\cos(\pi\tau)\Big]_0^t$$

$$= -\frac{1}{\pi}\{\cos(\pi t)-1\} = \frac{1}{\pi}\{1-\cos(\pi t)\}$$

<u>For $2<t<4$:</u>



$$\tilde{y}(t) = \int_{t-2}^{2} \sin(\pi\tau)\,d\tau = -\frac{1}{\pi}\cos(\pi\tau)\Big]_{t-2}^{2}$$

$$= -\frac{1}{\pi}\{\cos(2\pi)-\cos(\pi(t-2))\}$$

$$= \frac{-1}{\pi}\{1-\cos(\pi t)\}$$

Summarizing:

$$\tilde{y}(t) = \frac{1}{\pi}\left\{1-\cos(\pi t)\right\}\left(u(t)-u(t-2)\right) - \frac{1}{\pi}\left\{1-\cos(\pi t)\right\}\left(u(t-2)-u(t-4)\right)$$

- Thus:

$$y(t) = 2\,\tilde{y}(t-1)$$

$$= \frac{2}{\pi}\left\{1-\cos(\pi(t-1))\right\}\left\{u(t-1)-u(t-3)\right\} - \frac{2}{\pi}\left\{1-\cos(\pi(t-1))\right\}\left\{u(t-3)-u(t-5)\right\}$$

$$= \frac{2}{\pi}\left\{1+\cos(\pi t)\right\}\left\{u(t-1)-u(t-3)\right\} - \frac{2}{\pi}\left\{1+\cos(\pi t)\right\}\left\{u(t-3)-u(t-5)\right\}$$

Since $\cos(\theta - \pi) = -\cos(\theta)$

and $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

$$1 - \cos(\pi(t-1)) = 1 + \cos(\pi t)$$
$$= 2\cos^2\left(\frac{\pi}{2}t\right)$$

Thus:

$$y(t) = \begin{cases} \frac{4}{\pi}\cos^2\left(\frac{\pi}{2}t\right) & 1<t<3 \\ -\frac{4}{\pi}\cos^2\left(\frac{\pi}{2}t\right) & 3<t<5 \end{cases}$$

$= 0$ otherwise

$t<1 \quad t>5$

Help for Prob. 2.22 (d)

Convolve:

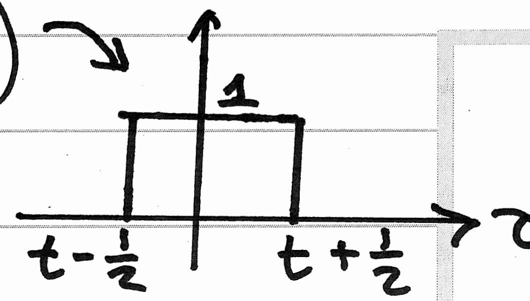
$$x(t) = at + b \qquad h(t) = \frac{4}{3}\,\text{rect}\left(t - \frac{1}{2}\right) - \frac{1}{3}\delta(t-2)$$

First, consider:

$$\tilde{y}(t) = x(t) * \text{rect}(t)$$

$$\text{rect}(t) = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > 1/2 \end{cases}$$

$$\text{rect}(t-\tau) = \text{rect}\left(-(\tau - t)\right)$$



$$\tilde{y}(t) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (a\tau + b)\, d\tau = \text{(on next page)}$$

$$\tilde{y}(t) = a\frac{\tau^2}{2} + b\tau \Big]_{t-\frac{1}{2}}^{t+\frac{1}{2}}$$

$$= \frac{a}{2}\left(t^2 + t + \frac{1}{4}\right) + b\left(t + \frac{1}{2}\right) - \frac{a}{2}\left(t^2 - t + \frac{1}{4}\right) - b\left(t - \frac{1}{2}\right)$$

$$= at + b$$

Thus: $(at+b) * \mathrm{rect}(t) = at + b$ interesting result

In this problem, we have (invoking linearity & TI)

$$(at+b) * \left(\frac{4}{3}\mathrm{rect}\left(t - \frac{1}{2}\right) - \frac{1}{3}\delta(t-2)\right)$$

$$= \frac{4}{3}\left(a\left(t - \frac{1}{2}\right) + b\right) - \frac{1}{3}\left(a(t-2) + b\right)$$

$$= at + b + \left(-\frac{2}{3}a + \frac{2}{3}a\right)$$

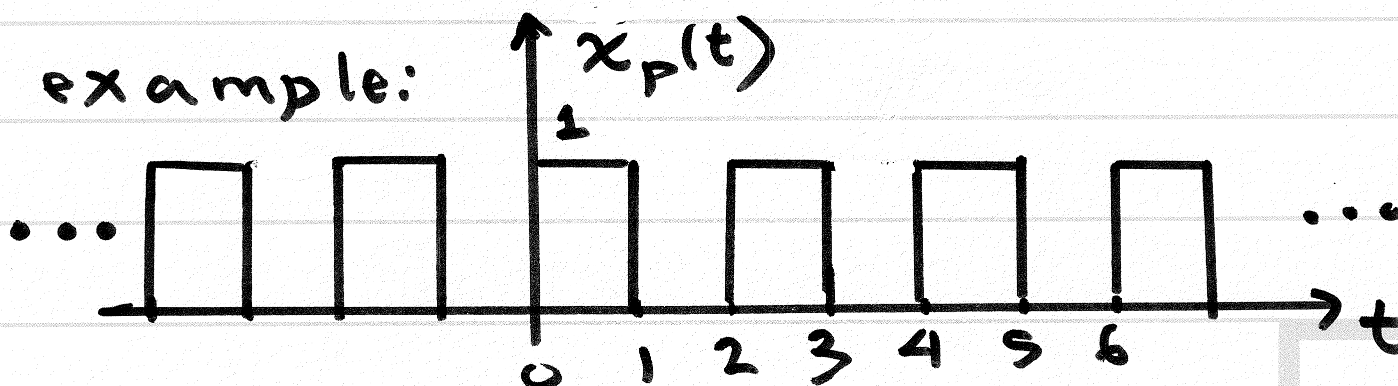
$= at + b$ (final answer)

A Solution Approach to Prob. 2.22 (e)

- First, note: any periodic signal with period T can be expressed as

$$x_p(t) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

- For example:



$$x_p(t) = \sum_{k=-\infty}^{\infty} \text{rect}\left(t - \tfrac{1}{2} - k2\right)$$

- where: $\text{rect}\left(t - \frac{1}{2}\right) = u(t) - u(t-1)$

- Consider periodic signal $x_p(t)$ as input to LTI system with impulse response $h(t)$

- $y_p(t)$ will also be periodic with period T
- simple to show using superposition/distributive property of convolution and time-invariance

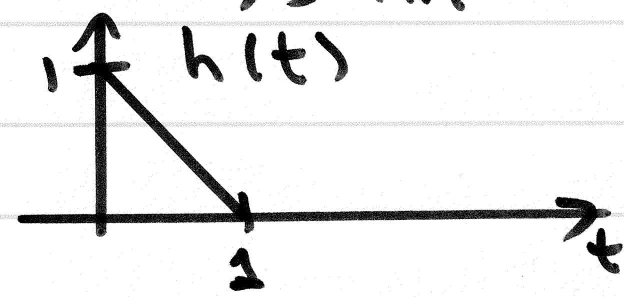
$$y_p(t) = x_p(t) * h(t) = \left\{ \sum_{k=-\infty}^{\infty} x(t-kT) \right\} * h(t)$$

$$= \sum_{k=-\infty}^{\infty} x(t-kT) * h(t)$$

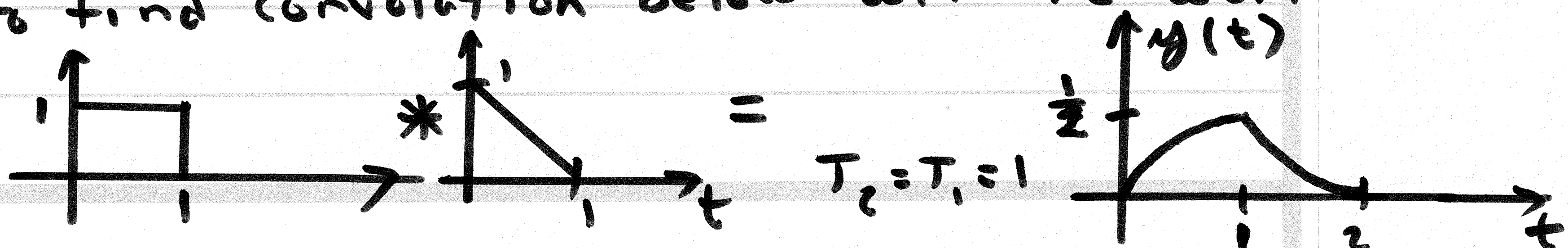
$$= \sum_{k=-\infty}^{\infty} y(t-kT)$$

where: $y(t) = x(t) * h(t)$

- So, to find the output $y_p(t)$, you only have to convolve one period of $x_p(t)$ with $h(t)$ to form $y(t) = x(t) * h(t)$ and then repeat $y(t)$ every T secs
- **Continuing example:** Suppose $x_p(t)$ = periodic train of rectangular pulses was input to LTI system with impulse response



- We can use ramp-down triangle convolution result to find convolution below with no work



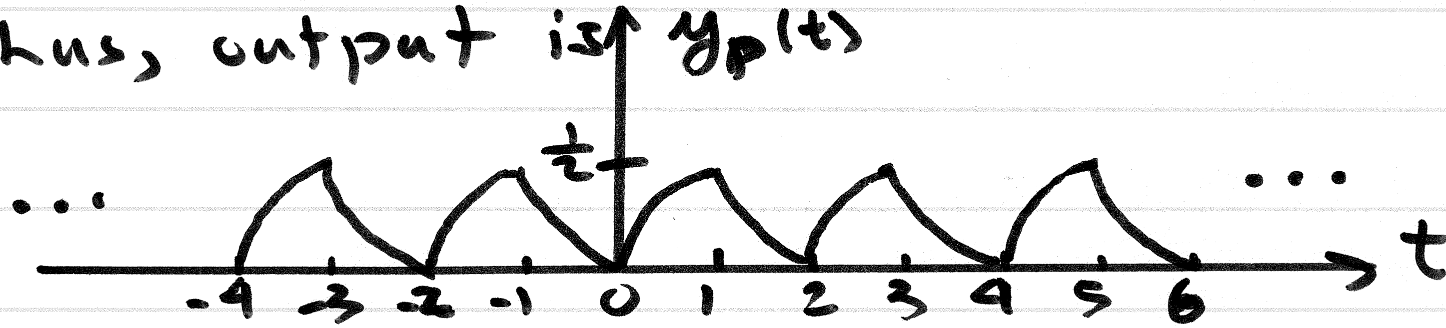
$$y(t) = \begin{cases} -\frac{t^2}{2} + t, & 0 < t < 1 \\ +\frac{t^2}{2} - 2t + 2, & 1 < t < 2 \end{cases}$$

concave down

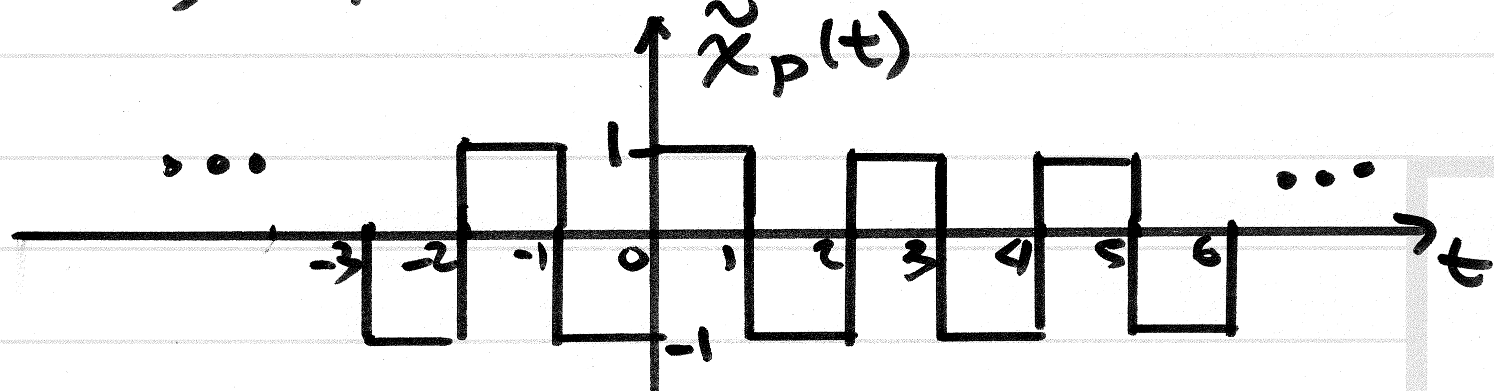
concave up

④

• Thus, output is



• Now, suppose input was instead



• <u>Observe</u>: $\tilde{x}_p(t) = x_p(t) - x_p(t-1)$

where $x_p(t)$ is plotted on pg. 2

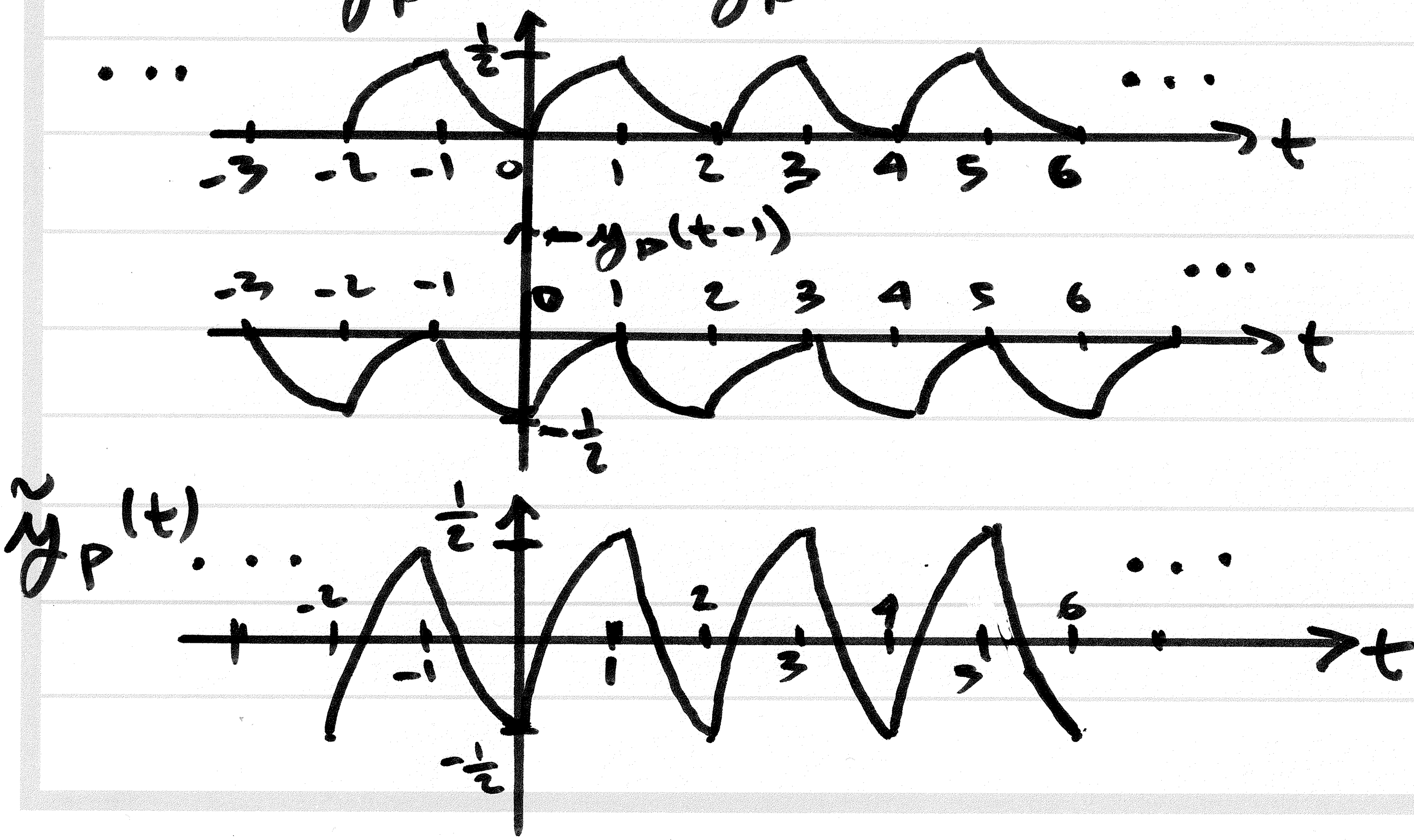
• Invoking LTI: $\tilde{y}_p(t) = \tilde{x}_p(t) * h(t)$

5

$$\tilde{y}_p(t) = \left(x_p(t) - x_p(t-1)\right) * h(t)$$

$$= x_p(t) * h(t) - x_p(t-1) * h(t)$$

$$= y_p(t) - y_p(t-1)$$



- The input for Prob. 2.22 (e) was $\tilde{x}_p(t+\frac{1}{2})$

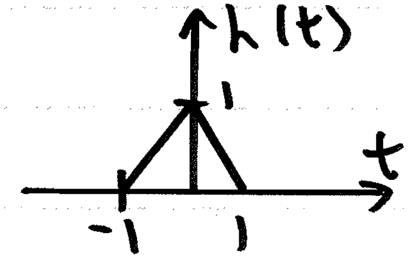
$$z(t) = \tilde{x}_p(t+\tfrac{1}{2}) * h(t)$$ (6)

$$= \tilde{y}_p(t+\tfrac{1}{2})$$

- Just take answer $\tilde{y}_p(t)$ at bottom of page 5 and shift to left by 1/2

- This whole problem was solved without having to do a convolution ⇒ but rather by using a known convolution result and invoking concepts of linearity and time-invariance

Prob. 2.23 See Fig. P2.23 on pg 143



$y(t) = x(t) * h(t)$

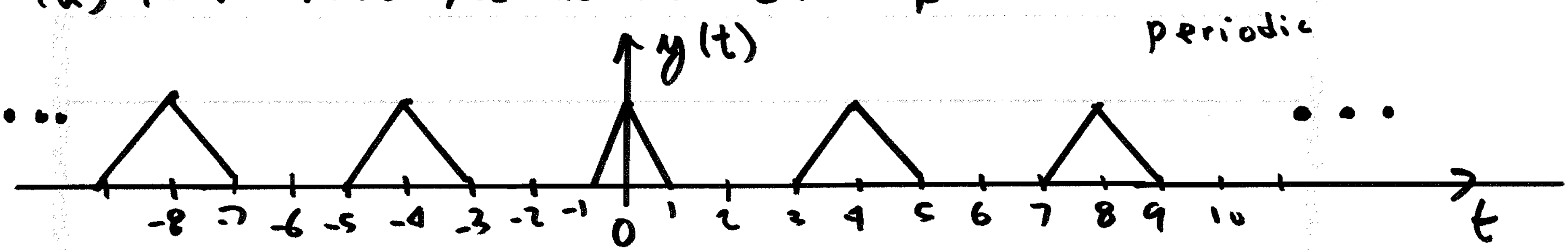
where: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

$$y(t) = \left\{ \sum_{k=-\infty}^{\infty} \delta(t-kT) \right\} * h(t)$$

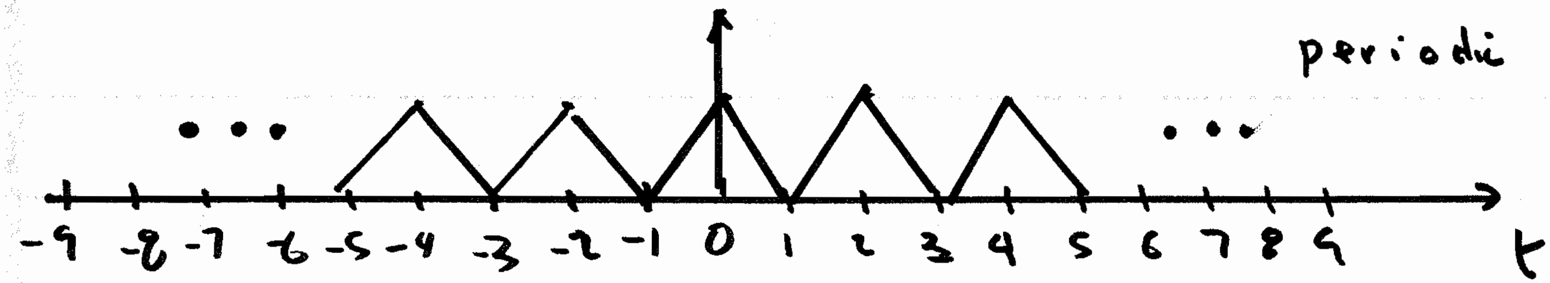
dist. prop: $= \sum_{k=-\infty}^{\infty} \left\{ \delta(t-kT) * h(t) \right\} = \sum_{k=-\infty}^{\infty} h(t-kT)$

Plot for (a) $T=4$, (b) $T=2$, (c) $T=1.5$, (d) $T=1$

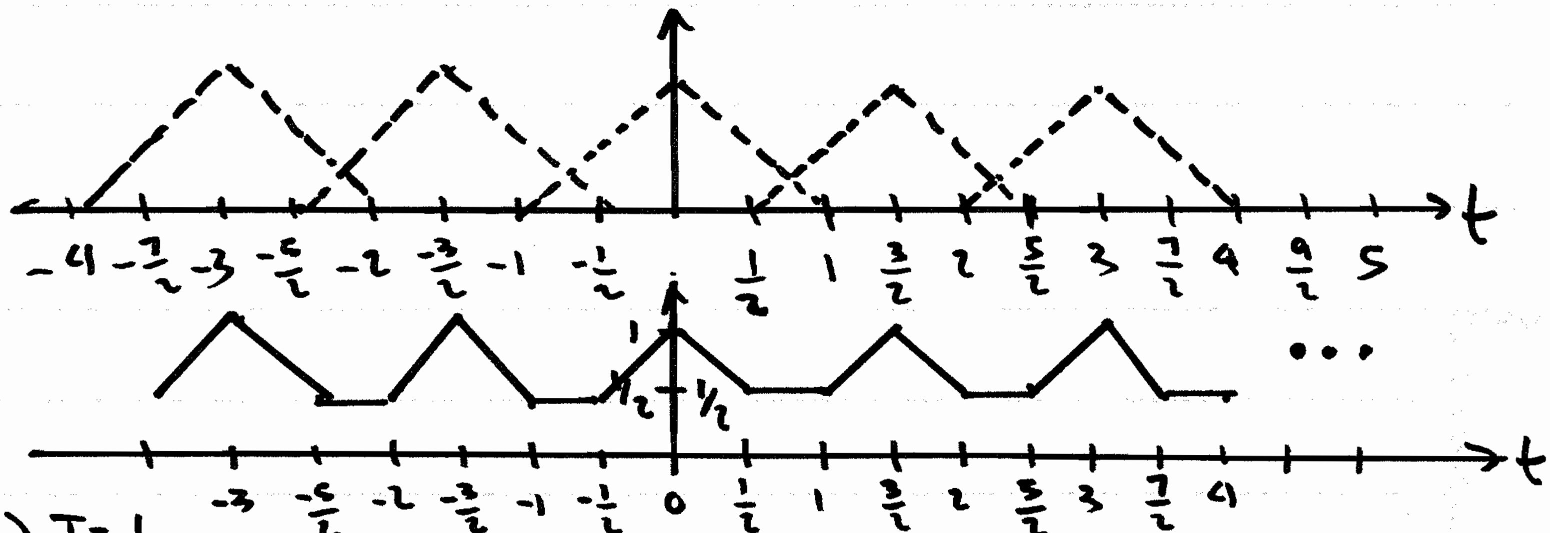
(a) $T=4$ triangles do not overlap



(b) T=2 triangles just touch each other

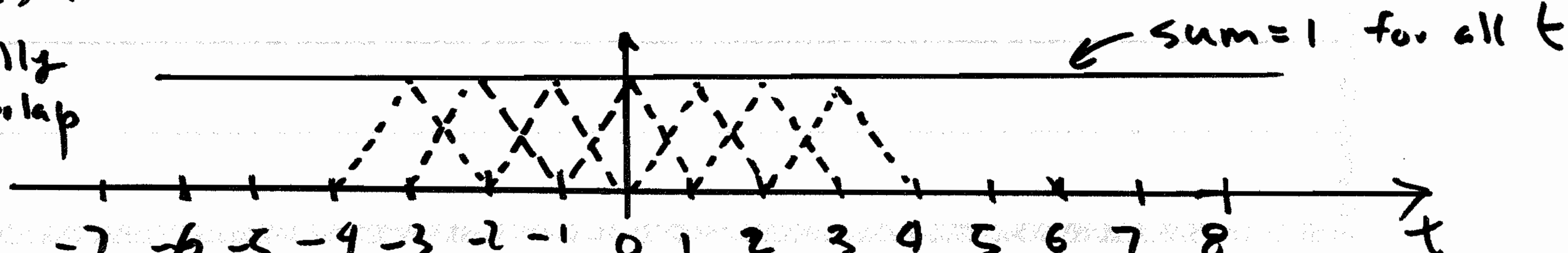


(c) T= 3/2 triangles partially overlap



(d) T=1

Fully overlap

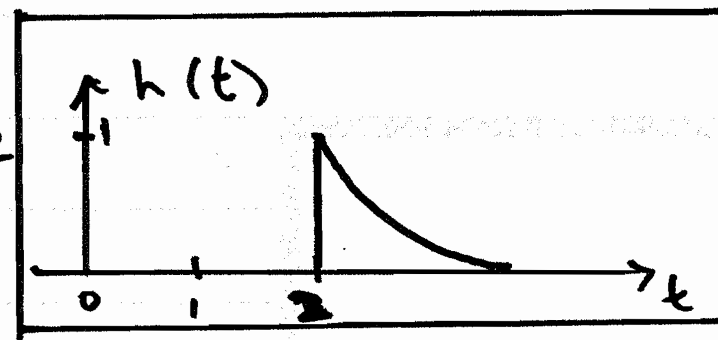


Prob. 2.40 $\quad y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau-2)\, d\tau$

$h(t) = ?$ $\quad$ let: $x(t) = \delta(t) \Rightarrow x(\tau-2) = \delta(\tau-2)$

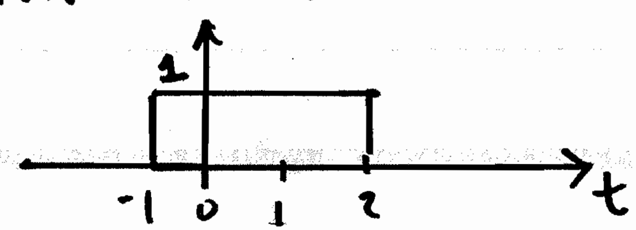
$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau-2)\, d\tau = \int_{-\infty}^{t} e^{-(t-2)} \delta(\tau-2)\, d\tau$$

$$= e^{-(t-2)} \underbrace{\int_{-\infty}^{t} \delta(\tau-2)\, d\tau}_{=1 \text{ , if } t>2}$$



$$h(t) = e^{-(t-2)} u(t-2)$$

Find: $y(t)$ when $x(t) = u(t+1) - u(t-2) = \mathrm{rect}\left(\frac{t-1/2}{3}\right)$



Recall: Example 2.6 : $e^{-at}u(t) * u(t) =$

pp. 98-99 $= \frac{1}{a}(1-e^{-at})u(t)$

Special case of: $e^{-\alpha t}u(t) * e^{-\beta t}u(t)$ with $\beta = 0$

Hmwk. Prob. 2.22 (a)

Thus: $(u(t) - u(t-T)) * e^{-at}u(t)$

$$= \frac{1}{a}(1-e^{-at})u(t) - \frac{1}{a}\left(1-e^{-a(t-T)}\right)u(t-T)$$

$$= \frac{1}{a}\left(u(t)-u(t-T)\right) - \frac{1}{a}e^{-at}\left(u(t) - e^{aT}u(t-T)\right)$$

Recall: If: $y(t) = x(t) * h(t)$

Then $x(t-t_1) * h(t-t_2) = y(t-(t_1+t_2))$

- $x(t)$ is $(u(t)-u(t-3))$ shifted to the right by $t_1=-1$
- $h(t)$ is $e^{-2t}u(t)$ shifted to the right by $t_2=2$
- With $a=2$ and $T=3$: $(u(t)-u(t-3)) * e^{-2t}u(t)$ is

$$\frac{1}{2}\left(u(t)-u(t-3)\right)-\frac{1}{2}e^{-2t}\left(u(t)-e^{2\cdot 3}u(t-3)\right)$$

- Thus, answer is above shifted to right by

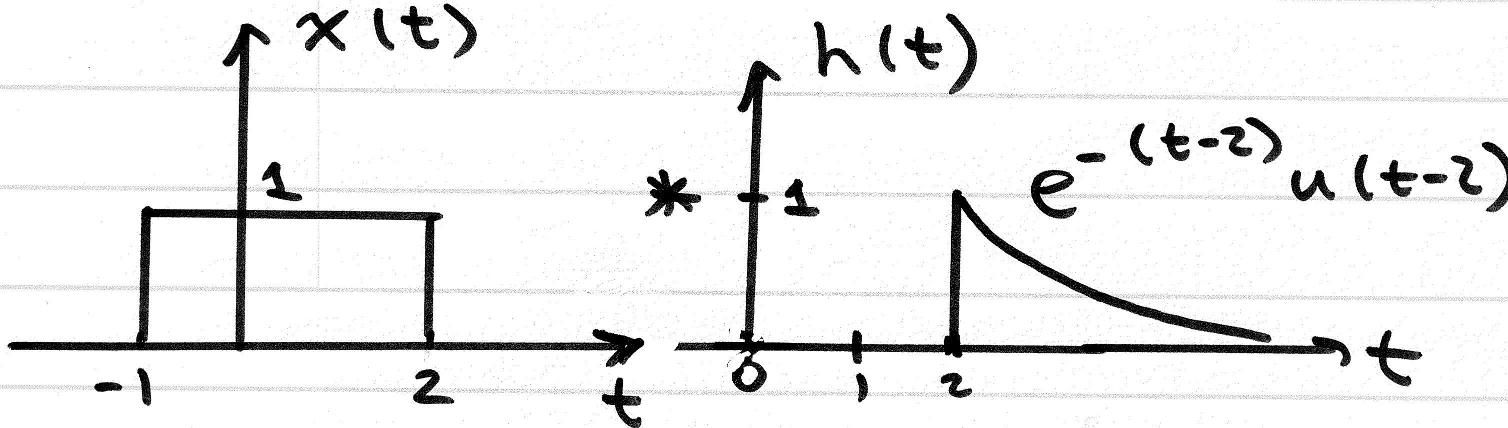
$$t_1+t_2=-1+2=1$$

- **Answer:**

$$y(t)=\frac{1}{2}\left(u(t-1)-u(t-4)\right)-e^{-2(t-1)}\left(u(t-1)-e^{6}u(t-4)\right)$$

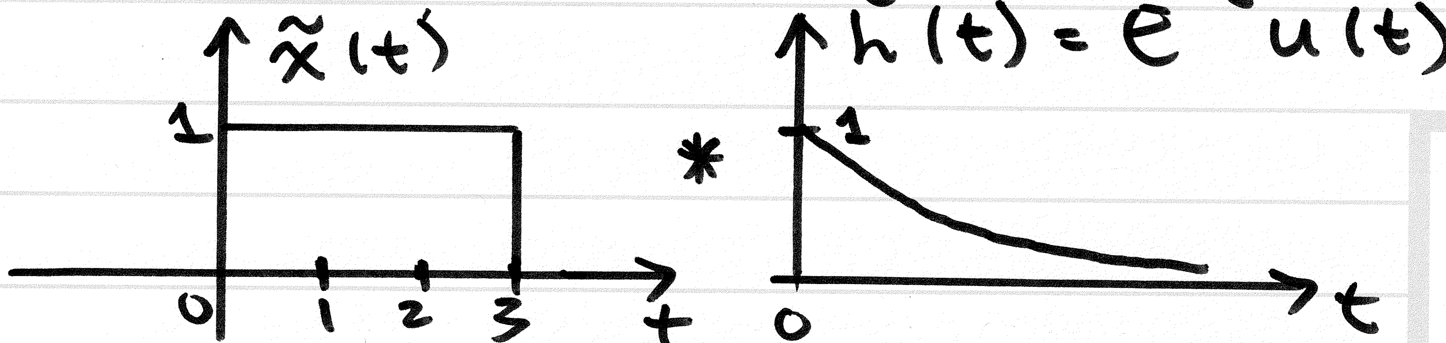
Prob. 2.40 (b)

Prob. 2.40 (b)



$$y(t) = x(t) * h(t) = ?$$

Instead, convolve:



Note:

$$\tilde{x}(t) = u(t) - u(t-3)$$

Find $\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)$ using known convolution results. Then, observe:

$$y(t) = \tilde{x}(t+1) * \tilde{h}(t-2)$$

$$= \tilde{y}\Big(t - (\underset{t_1=-1}{-1} + \underset{t_2=2}{2})\Big)$$

$$= \tilde{y}(t-1)$$

- where, again, $\tilde{y}(t)$ is the convolution result when both things being convolved start at $t=0$
- Now, basic convolution result from Prob. 2.22(a)

$$e^{-\alpha t} u(t) * e^{-\beta t} u(t) = \frac{1}{\beta - \alpha}\left\{e^{-\alpha t} - e^{-\beta t}\right\} u(t)$$

Text

- With $\beta = 0$, we have result from Example 2.6

$$\left(e^{-\alpha t} u(t)\right) * u(t) = \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) u(t)$$

- Now back to problem:

$$\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t) = \tilde{h}(t) * \tilde{x}(t)$$

$$= e^{-t} u(t) * \{u(t) - u(t-3)\}$$

$a=1$

$$= e^{-t} u(t) * u(t) - e^{-t} u(t) * u(t-3)$$

$a=1$

$$= \frac{1}{1}\left(1 - e^{-at}\right) u(t) - \frac{1}{1}\left(1 - e^{-a(t-3)}\right) u(t-3)$$

- Then, finally $y(t) = \tilde{y}(t-1)$:

$$y(t) = \left(1 - e^{-(t-1)}\right) u(t-1) - \left(1 - e^{-(t-4)}\right) u(t-4)$$