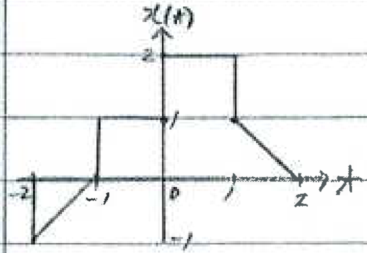
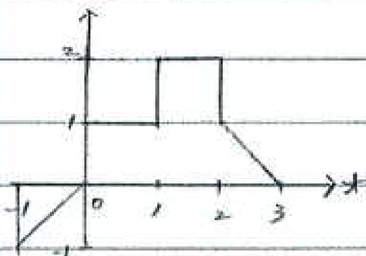
ECE301 Spring 2024

HW #1 Solutions.

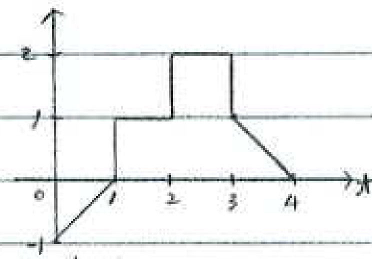
Text Prob 1.21.



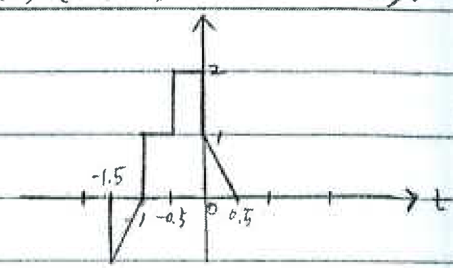
(a) $x(t-1)$



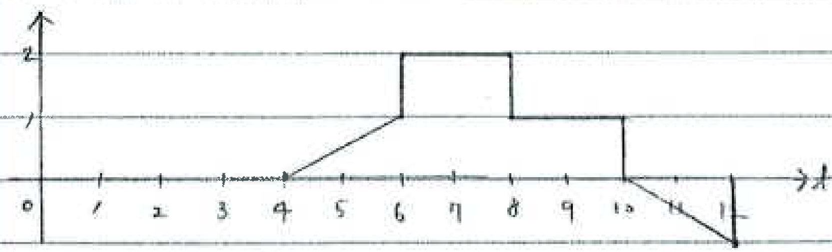
(b) $x(t-2)$



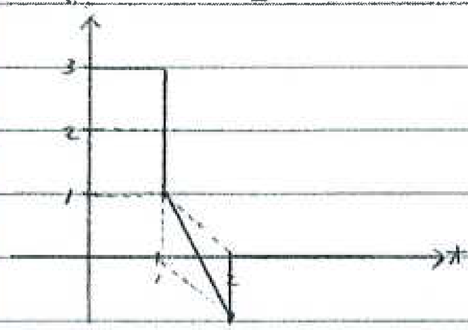
(c) $x(2t+1) = x(2(t+0.5))$.



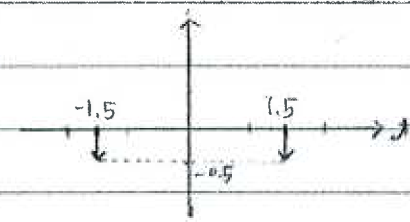
(d) $x(4-\frac{t}{2}) = x(-\frac{1}{2}(t-8))$



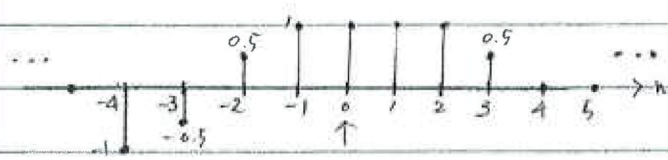
(e) $[x(t) + x(-t)]u(t)$



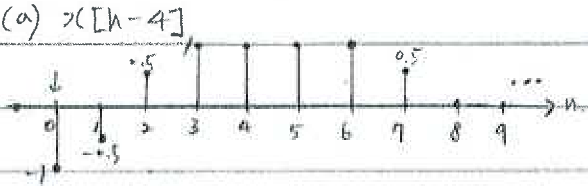
(f) $x(t)\,[\delta(t+\frac{3}{2}) - \delta(t-\frac{3}{2})]$



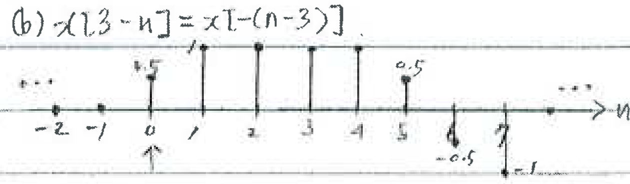
Text Prob 1.22.



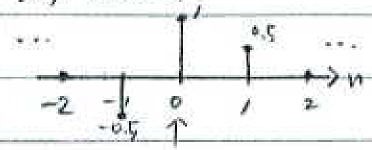
(a) $x[n-4]$



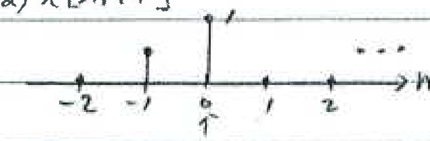
(b) $x[3-n] = x[-(n-3)]$.



(c) $x[3n]$



(d) $x[3n+1]$

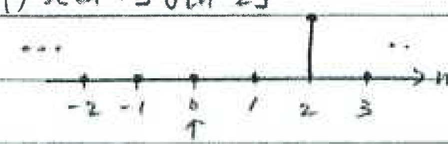


(e) $x[n]\,u[3-n] = x[n]\,u[-(n-3)]$

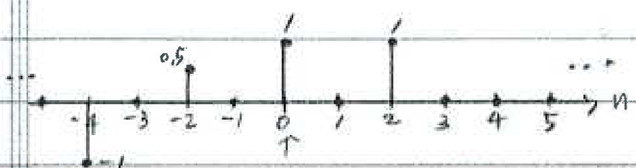
$u[3-n] = u[-(n-3)]$

$\therefore x[n]\,u[3-n] = x[n]$

(f) $x[n-2]\,\delta[n-2]$



(g) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n] = 0.5\left(1+(-1)^n\right)x[n] = \begin{cases} x[n], & n = \text{even } \# \\ 0, & n = \text{odd } \# \end{cases}$



Text Prob 1.34. (a) $\sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{-1} x[n] + x[0] + \sum_{n=1}^{\infty} x[n] = x[0] + \sum_{n=1}^{\infty}\{x[n] + x[-n]\}$

Recall) If $x[n]$ is odd signal,

then $x[n] + x[-n] = 0$ and $x[0] = 0$

$\therefore \sum_{n=-\infty}^{\infty} x[n] = 0$

(b) Let $y[n] = x_1[n]\,x_2[n]$.

Then $y[-n] = x_1[-n]\,x_2[-n]$

$= -x_1[n]\,x_2[n]$ since $x_1[n]$: odd & $x_2[n]$: even

$= -y[n]$

Recall) If $x[n]$ is even signal, then $x[n] = x[-n]$.

$\therefore$ If $x_1[n]$ is an odd signal and $x_2[n]$ is an even signal,

$x_1[n]\,x_2[n]$ is an odd signal.

Prob. 1.34 (c)

$$x[n] = x_e[n] + x_o[n]$$

$$x^2[n] = x_e^2[n] + 2x_e[n]x_o[n] + x_o^2[n]$$

$$\sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=-\infty}^{\infty} x_e^2[n] + 2\sum_{n=-\infty}^{\infty} x_e[n]\,x_o[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$

product of even and odd signal is odd

Thus: $$\sum_{n=-\infty}^{\infty} x_e[n]\,x_o[n] = 0$$

Thus: $$\sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$

Prob. 1.34 (d) Similarly for CT:

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))^2\,dt$$

$$= \int_{-\infty}^{\infty} x_e^2(t)\,dt + \int_{-\infty}^{\infty} 2x_e(t)\,x_o(t)\,dt + \int_{-\infty}^{\infty} x_o^2(t)\,dt$$

$x_e(t)\,x_o(t)$ is an odd function such that

$$\int_{-\infty}^{\infty} x_e(t)\,x_o(t)\,dt = 0$$

Thus: $$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} x_e^2(t)\,dt + \int_{-\infty}^{\infty} x_o^2(t)\,dt$$

Text Prob 1.26

(a) $x[n] = \sin\left(\frac{6\pi}{7}n + 1\right)$

do not have any factors in common.

Note) $x[n]$ is periodic if $\omega_0 = 2\pi m/N$ for m and N = positive integer

$$N = \frac{2\pi m}{\omega_0} = \frac{7}{3} m$$

$\Rightarrow$ When $m=3$ (makes N the smallest integer),

$N = \boxed{7}$

$\therefore$ The signal is periodic with fundamental period $N = \boxed{7}$

(b) $x[n] = \cos\left(\frac{n}{8} - \pi\right)$

$$N = \frac{2\pi m}{\omega_0} = 16\pi m$$

$\Rightarrow$ There is NO integer m that can make N an integer.

$\therefore$ The signal is NOT periodic.

(c) $x[n] = \cos\left(\frac{\pi}{8}n^2\right)$

Grader: don't grade part (c) of this problem: don't take any pts off

$$x[n+N] = \cos\left(\frac{\pi}{8}(n+N)^2\right) = \cos\left(\frac{\pi}{8}n^2 + \frac{\pi}{8}N^2 + \frac{\pi}{4}nN\right).$$

To show $x[n] = x[n+N]$,

$$\frac{\pi}{8}N^2 + \frac{\pi}{4}nN = 2\pi k, \quad \forall n \ \& \ k \in \mathbb{Z}$$

$$= 2\pi(l+m) = 2\pi l + 2\pi m, \quad l, m \in \mathbb{Z}.$$

i) $\frac{\pi}{8}N^2 = 2\pi l$

$\Rightarrow N^2 = 16 l$, $N = 4\sqrt{l}$

When $\sqrt{l} \in \mathbb{Z}$, N = multiple of 4. — (*)

ii) $\frac{\pi}{4}nN = 2\pi m$

$\Rightarrow N = 8\frac{m}{n}$

When $\frac{m}{n} \in \mathbb{Z}$, N = multiple of 8. — (**)

$\Rightarrow N = 8$ to satisfy both condition (*) and (**)

$\therefore$ The signal is periodic with fundamental period $N = 8$.

(d) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$

Recall) $\cos A \cos B = 0.5(\cos(A-B) + \cos(A+B))$

$$x[n] = \frac{1}{2}\{\underset{①}{\cos(\frac{\pi}{4}n)} + \underset{②}{\cos(\frac{3\pi}{4}n)}\}$$

i) Regarding term-① ii) Regarding term-②

$N_1 = \frac{8}{3}m_1$, m_1, N_1 = positive integers $N_2 = 8m_2$, m_2, N_2 = posive integers

⇒ The positive integers, $m_1 = 3$ and $m_2 = 1$, makes the positive integer $N = N_1 = N_2 = 8$ the smallest.

∴ The signal is periodic with fundamental period $N=8$.

(e) $x[n] = \underset{①}{2\cos(\frac{\pi}{4}n)} + \underset{②}{\sin(\frac{\pi}{8}n)} - \underset{③}{2\cos(\frac{\pi}{2}n + \frac{\pi}{6})}$

i) Regarding term-①

$N_1 = 8m_1$, for m_1, N_1 = positive integers.

ii) Regarding term-②

$N_2 = 16m_2$, for m_2, N_2 = positive integers.

iii) Regarding term-③

$N_3 = 4m_3$, for m_3, N_3 = positive integers.

⇒ The positive integers, $m_1 = 2$, $m_2 = 1$, $m_3 = 4$, makes the positive integer $N = N_1 = N_2 = N_3 = 16$ the smallest.

∴ The signal is periodic w/ fundamental period $N = 16$.

Text Prob 1.36.

(a) Note) If $x[n]$ is periodic with period N,

then $x[n+N] = x[n]$, $\forall n \in \mathbb{Z}$ & $N \in \mathbb{Z}$

If $x[n]$ is periodic, $e^{j\omega_0(n+N)T} = e^{j\omega_0 nT}$, where $\omega_0 = 2\pi/T_0$, $\forall n \in \mathbb{Z}$.
$- (*)$

From $(*)$, we obtain

$$e^{j\omega_0 NT} = 1 = e^{j2\pi k}, \quad k \in \mathbb{Z}.$$

This implies that

$$\frac{2\pi}{T_0} NT = 2\pi k \Rightarrow \frac{T}{T_0} = \frac{k}{N} = \text{a rational number.}$$

(b) If $T/T_0 = p/q$ where $p, q \in \mathbb{Z}$,

then $x[n] = e^{j2\pi n(p/q)}$.

Since $x[n]$ is periodic with period N,

$$\exp\left(j2\pi(p/q)(n+N)\right) = \exp\left(j2\pi(p/q)n\right), \quad \forall n \in \mathbb{Z}.$$
$- (**)$

From $(**)$, we can get

$$\exp\left(j2\pi(p/q)N\right) = 1 = \exp(j2\pi k), \quad k \in \mathbb{Z}.$$

Note) N_0 : the smallest N satisfying $x[n+N] = x[n]$ $\forall n \in \mathbb{Z}$.

$\therefore N_0 = \dfrac{q}{\gcd(p,q)}$. $\Rightarrow$ [When $\frac{T}{T_0} = \frac{p}{q}$ = rational, q is one period of $x[n]$, but not necessarily the fundamental period, and since p and q may have a common divisor, we need to divide q by the greatest common divisor $\gcd(p,q)$ to find fundamental period]

Now, the fundamental frequency of $x[n]$ is ...

$$\frac{2\pi}{N_0} = \frac{2\pi}{q}\gcd(p,q)$$

$$= \frac{2\pi}{p}\frac{p}{q}\gcd(p,q) = \frac{2\pi}{p}\frac{T}{T_0}\gcd(p,q)$$

$$= \frac{\omega_0 T}{p}\gcd(p,q)$$

(c) $p/\gcd(p,q)$ periods of $x(t)$ are needed.

1.36.

(c) From part (b), we have that $N = \dfrac{q}{\gcd(p,q)}$

In other words, there are N samples in a period of $x[n]$.

These samples are equally spaced in time (T units of time apart).

thus the time it takes to obtain the N samples forming a single period of $x[n]$ is : $N.T = \dfrac{q}{\gcd(p,q)} \cdot T$,

Corresponding to $\dfrac{N.T}{T_0}$ periods of $x(t)$.

$$\Rightarrow \frac{N.T}{T_0} = \frac{q}{\gcd(p,q)} \cdot \frac{T}{T_0} \qquad \text{where } \frac{T}{T_0} = \frac{p}{q}$$

$$\text{So } \frac{N.T}{T_0} = \frac{\cancel{q}}{\gcd(p,q)} \cdot \frac{p}{\cancel{q}} = \underline{\underline{\frac{p}{\gcd(p,q)}}}$$

Hence $\dfrac{p}{\gcd(p,q)}$ periods of $x(t)$ are needed!

Text Prob. 1.54

(a) i) For $\alpha=1$,

$$\sum_{n=0}^{N-1}\alpha^n=\sum_{n=0}^{N-1}1=\underbrace{1+\ldots+1}_{N}=N.$$

ii) For $\alpha\neq1$ & $\alpha\in\mathbb{C}$,

we can write

$$1-\alpha^N=(1+\alpha^1+\ldots+\alpha^{N-1})-(\alpha^1+\ldots+\alpha^{N-1}+\alpha^N)$$

$$=\sum_{n=0}^{N-1}\alpha^N-\sum_{n=0}^{N-1}\alpha^{N+1}$$

$$=\sum_{n=0}^{N-1}\alpha^n-\alpha\sum_{n=0}^{N-1}\alpha^n$$

$$=(1-\alpha)\sum_{n=0}^{N-1}\alpha^n$$

Therefore, $\displaystyle\sum_{n=0}^{N-1}\alpha^n=\frac{1-\alpha^N}{1-\alpha}$.

(b) For $|\alpha|<1$, $\displaystyle\lim_{N\to\infty}\alpha^N=0$

From (a), $\displaystyle\sum_{n=0}^{N-1}\alpha^n=\frac{1-\alpha^N}{1-\alpha}$.

Therefore, $\displaystyle\sum_{n=0}^{\infty}\alpha^n=\lim_{N\to\infty}\sum_{n=0}^{N-1}\alpha^n$

$$=\lim_{n\to\infty}\left(\frac{1-\alpha^n}{1-\alpha}\right)=\frac{1}{1-\alpha}.$$

Prob. 1.54 (c) Show: $\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}$

for $|\alpha| < 1$

$$\frac{d}{d\alpha}\left\{ \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} = (1-\alpha)^{-1} \right\}$$

take derivative wrt α on both sides, using chain rule on right hand side

$$\sum_{n=0}^{\infty} n\alpha^{n-1} = (-1)(1-\alpha)^{-2}(-1) = \frac{1}{(1-\alpha)^2}$$

multiply by α on both sides yields result

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}$$

Prob. 1.54 (d)

$$\sum_{n=k}^{\infty} \alpha^n = \sum_{n=0}^{\infty} \alpha^n - \sum_{n=0}^{k-1} \alpha^n$$

$$= \frac{1}{1-\alpha} - \frac{1-\alpha^k}{1-\alpha}$$

$$= \frac{\alpha^k}{1-\alpha}$$