Interpretation of Phase of Fourier Transform

- $X(\omega)$ is generally complex-valued
- In polar form: $X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)}$

  magnitude is a function of frequency

  phase is a function of frequency
- Consider impulse response of LTI System

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega) = |H(\omega)|\, e^{\angle H(\omega)} = \text{frequency response of system}$$

$$A e^{j(\omega_0 t + \theta)} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega_0)\, e^{j(\omega_0 t + \theta)} A$$

$$= \left(|H(\omega_0)|\, A\right) e^{j(\omega_0 t + \theta + \angle H(\omega_0))}$$

- sinewave at output generally has a new amplitude <u>and</u> a new phase

- In the case of a sinewave:

time-shift $\Longleftrightarrow$ equivalent to $\Longleftrightarrow$ phase-shift

$$\cos(\omega_0(t-t_0)) = \cos(\omega_0 t - \omega_0 t_0)$$

- phase-shift depends on frequency ω_0
- higher frequency yields larger phase-shift
- Recall: time-shift property of Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

<u>where:</u> $\omega_0 = 2\pi \frac{1}{T}$ and T = period

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} a_k e^{-j(k\omega_0)t_0} e^{jk\omega_0 t}$$

- **observe:** higher harmonic freq. = larger phase-shift = (larger value of k)

Recall: Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} X(k\Delta\omega)\, e^{j(k\Delta\omega)t}$$

• $$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t_0}\, e^{j\omega t}\, d\omega$$

$$= \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} X(k\Delta\omega)\, e^{-jk\Delta\omega t_0}\, e^{jk\Delta\omega t}$$

$\Rightarrow$ sinewave at frequency $k\Delta\omega$ gets phase-shifted by $-k\Delta\omega t_0$ due to the time-shift

- Consider case of $x(t)$ real-valued & even-symmetric

$$x(-t) = x(t)$$

- In this case, $X(\omega)$ is real-valued for all ω (but can be negative-valued)
- Consider case of Gaussian signal: $e^{-\frac{1}{2}\frac{t}{\sigma^2}}$

$$x(t) = e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} X(\omega) = \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2}{2\, 1/\sigma^2}}$$

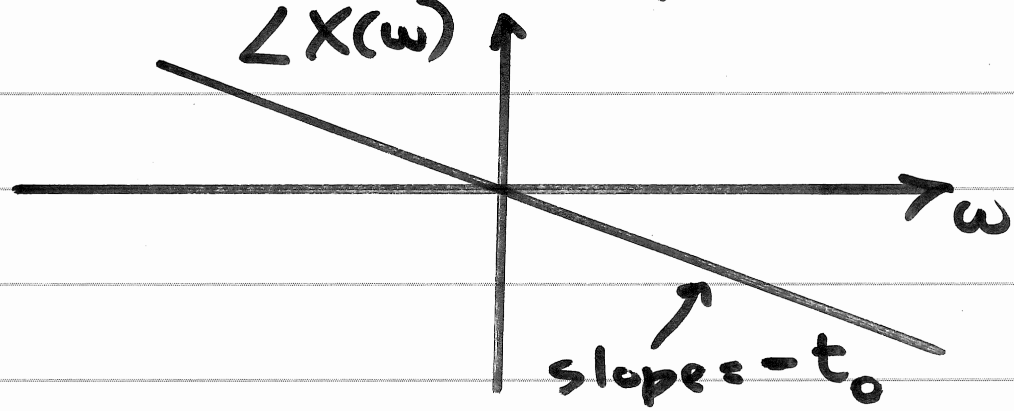
- <u>In this case:</u> $X(\omega) > 0$ for all ω as well as real-valued
  strictly positive
- Consider time-shift:

$$y(t) = x(t-t_0) = e^{-\frac{(t-t_0)^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} X(\omega)\, e^{-j\omega t_0} = Y(\omega)$$

$$= \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2}{2\,1/\sigma^2}}\, e^{-j\omega t_0}$$

$$|Y(\omega)| = X(\omega) \qquad \angle X(\omega) = -\omega t_0$$

The phase is a line with slope $-t_0$ that passes thru the origin



- This would be the same for products of sinc functions, e.g.,

$$x(t) = \frac{\pi}{\omega_1} \frac{\sin(\omega_1 t)}{\pi t} \frac{\sin(\omega_2 t)}{\pi t} \qquad \begin{array}{l} \omega_2 > \omega_1 \\ -\infty < t < \infty \end{array}$$

- This is another case where $X(\omega)$ is non-negative for all ω $\left(\underset{\forall \omega}{X(\omega) \geq 0} \right)$ as well as real-valued