Further Insights into the Fourier Transform

Fourier Transform: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$

Inverse Fourier Transform: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$

- **Recall:** for $x(t)$ periodic with period T

  Fourier Series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$

  = "sum of sinewaves"

- Examine Inverse Fourier Transform integral

  Remember: integration = "area under curve"

  ⇒ Discretize inverse FT integral ≈ "area under rectangles"

$$x(t) \approx \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t} \Delta\omega$$

$$\approx \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} X(k\Delta\omega) \; e^{jk\Delta\omega t}$$

= "sum of sinewaves"

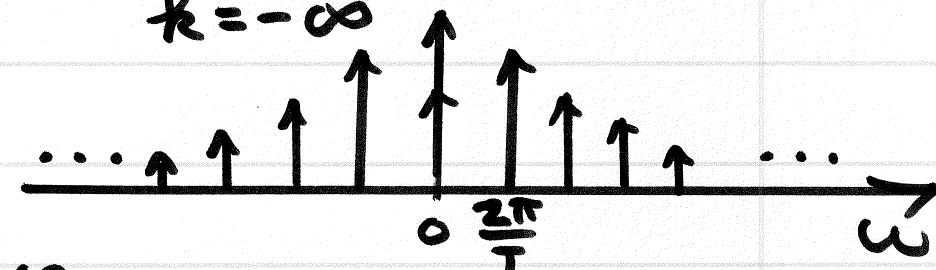
When you let $\Delta\omega \to 0$ to get back to Inverse FT Integral, we see that for

⇒ aperiodic $x(t)$ is an infinite sum of sinewaves infinetesimally close in frequency

⇒ the energy of an aperiodic signal is spread over a continuum of frequencies

⇒ in contrast to periodic signal ⇒ for which energy is at equi-spaced discrete frequencies periodic $x(t)$: infinite energy, finite power

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \xleftrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - k\frac{2\pi}{T}\right)$$

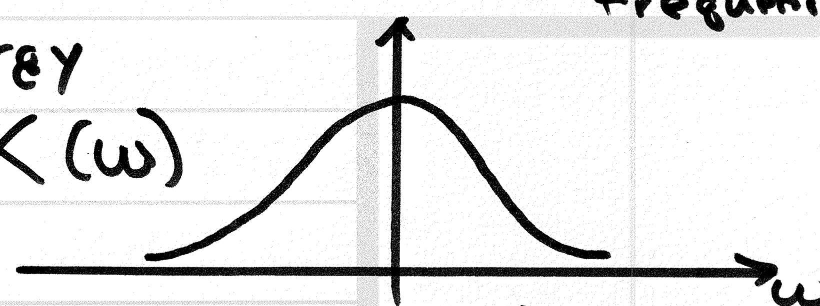


$$\text{avg. power} = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 =$$

sum of powers in all the sinewa
at the harmonic f
frequencies

aperiodic $x(t)$: finite energy

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$



$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

energy spread over a continuum of frequencies

- Expressing the Fourier Transform and Inverse FT in terms of Hz (Hertz = $\frac{\text{cycles}}{\text{sec}}$)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

- <u>change of variables:</u> (substitution)

$$\omega = 2\pi f$$

$$d\omega = 2\pi\, df$$

$$f = \frac{\omega}{2\pi} \qquad \omega\Big]_{-\infty}^{\infty} \longrightarrow f\Big]_{-\infty}^{\infty}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, 2\pi\, df$$

Leads to FT and IFT in terms of Hz

FT: $$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

IFT: $$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

See Table of Fourier Transform Properties and Table of Fourier Transform Pairs expressed in Hz posted at course web site

- most noteworthy: 2π factors are gone in many cases

For example:

Duality:

If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$

Then: $X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$

Multiplication in Time

$$x(t)\,y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) * Y(f)$$

Sinewave:

$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f - f_0)$$

"turned-on" for all time

Energy:

Parseval's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

Some intellectual curiosity results :)

- For a Gaussian pulse with $\sigma = \frac{1}{\sqrt{2\pi}}$

$$x(t) = e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} X(f) = e^{-\pi f^2}$$

- "Its FT is equal to itself"

Of courses, for any other standard deviation, we can apply the time-scaling property

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

to obtain general FT pair

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-\frac{1}{2}(2\pi\sigma)^2 f^2}$$

Another one: "FT equal to itself"

$$\sum_{n=-\infty}^{\infty} \delta(t-n) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \delta(f-k)$$

- Again, use time-scaling property in conjunction with property of Dirac Delta Fn $\delta(ax) = \frac{1}{a}\delta(x)$ to obtain more general result

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_0) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \frac{1}{T_0}\delta\left(f-\frac{k}{T_0}\right)$$