Fourier Transform

- Consider sinewave input to LTI System

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ? = H(\omega_0)e^{j\omega_0 t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega_0 (t-\tau)}\, d\tau$$

$$= \left\{ \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau}\, d\tau \right\} e^{j\omega_0 t} = H(\omega_0) e^{j\omega_0 t}$$

where $H(\omega)$ is the Fourier Transform of the impulse response defined as

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

(τ was just a dummy variable of integration)

The FT of a signal $x(t)$ is defined the same way, just interpreted differently:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- $X(\omega)$ reflects how the energy of a signal is distributed as a function of frequency
  - we will prove Parseval's Theorem later

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

  - other useful info. about the signal is gleaned from $X(\omega)$ which is generally complex-valued for every frequency ω

- Notation and Inverse Fourier Transform

$$X(\omega) = \mathcal{F}\{x(t)\}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$x(t)$

Time Domain

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$X(\omega)$

Frequency Domain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

- VIP: this mapping is one-to-one $\Rightarrow$ uniqueness

Notation Issue:

Recall Laplace Transform: $X(s) = \int_0^{\infty} x(t) e^{-st} dt$

- ~~One~~-Sided Laplace Transform useful for dealing with initial conditions
- Two-Sided Laplace Transform defined as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

- THUS, many textbooks make connection between Laplace and Fourier Transforms as

$$X(j\omega) = X(s)\Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- That is, the Fourier Transform is the Laplace Transform evaluated along the imaginary axis in the s-plane $\Rightarrow s = \sigma + j\omega$

- We will <u>not</u> use this notation
  - it's awkward, confusing, and cumbersome
- Consider basic Fourier Transform in Example 4.4 on pg. 293 for rectangular pulse

$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$= u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)$$

- The book writes: $X(j\omega) = \underbrace{\frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}}_{\text{no } j \text{ here}}$
  - does that make sense?
- Also, we will define the Fourier Transform of signals, like sinewaves turned-on forever, for which the Laplace Transform is not even defined

- $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = |X(\omega)|\, e^{j\angle X(\omega)}$$

  is generally complex-valued at every frequency ω

- The phase of $X(\omega)$, $\angle X(\omega)$, allows us to distinguish between the FT of $x(t)$ and the FT of $x(t-t_0)$

  $\boxed{\text{energy distribution as a function of } \omega \text{ is same for both}}$

- The phase of $X(\omega)$ also allows us to distinguish between the FT of $\cos(\omega_0 t)$ and the FT of $\sin(\omega_0 t)$

  $\boxed{\text{both have all their energy concentrated at } \omega_0 \text{ and } -\omega_0}$

- Remember the uniqueness of the Fourier Transform
  1-to-1 mapping between time domain and frequency domain

Example 4.1 on pg. 290 of text

$$x(t) = e^{-at} u(t)$$

$$X(\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = \int_0^\infty e^{-(a+j\omega)t} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big]_0^\infty = \frac{-1}{a+j\omega}(0-1)$$

- have our first FT pair:

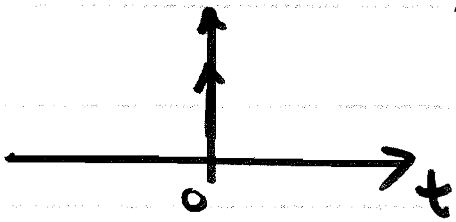
$$x(t) = e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} = X(\omega)$$

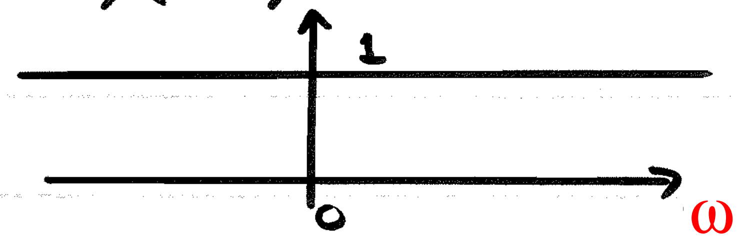
$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \quad ; \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

See Plots in Fig. 4.5 on pg. 291

- Example 4.3

$$x(t) = \delta(t) \xleftrightarrow{\mathcal{F}} X(\omega) = 1 \quad \forall \omega$$





Proof: $$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{0} dt = 1$$

- An impulse (Dirac Delta) has equal energy at any and all frequencies
- Intuition: narrow in time-domain

  $\Rightarrow$ wide in frequency domain (take up a lot of bandwidth)

- this is the extreme case

- What about $x(t) = 1 \;\; \forall t$ $\Rightarrow$ DC $\omega = 0$
- conjecture $X(\omega) \propto \delta(\omega)$ since all energy is concentrated at $\omega = 0$ $\Rightarrow$ no other frequency content
- validate by using inverse Fourier Transform

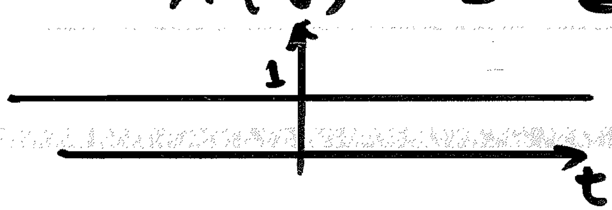
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

- Try $X(\omega) = \alpha \, \delta(\omega)$: $\displaystyle \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha \, \delta(\omega) e^{j\omega t} \, d\omega$

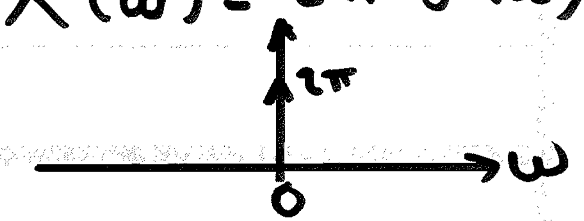
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha \, \delta(\omega) e^{j0} \, d\omega = \frac{\alpha}{2\pi}$$

- Choosing $\alpha = 2\pi$, we get

$$x(t) = 1 \;\xleftrightarrow{\mathcal{F}}\; X(\omega) = 2\pi \, \delta(\omega)$$



$$\xleftrightarrow{\mathcal{F}}$$



- **Sinewaves**

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

for all t

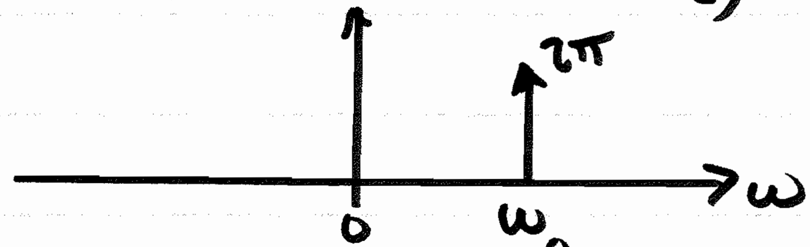
- only frequency present is ω_0 $\Rightarrow$ all energy concentrated at ω_0 $\Rightarrow$ conjecture $X(\omega) = 2\pi\,\delta(\omega - \omega_0)$
- check inverse Fourier Transform:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(2\pi\,\delta(\omega-\omega_0)\right) e^{j\omega t}\,d\omega$$

$$= \frac{2\pi}{2\pi}\int_{\infty}^{\infty}\delta(\omega-\omega_0)\, e^{j\omega_0 t}\,d\omega$$

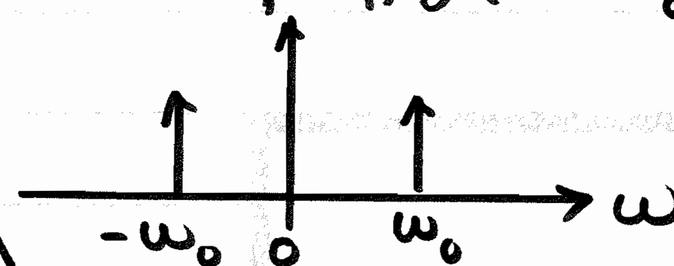
$$= \left\{\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\,d\omega\right\} e^{j\omega_0 t} = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = 2\pi\,\delta(\omega - \omega_0)$$



- Using Euler's formulas plus the fact that the Fourier Transform is a linear operator, we have

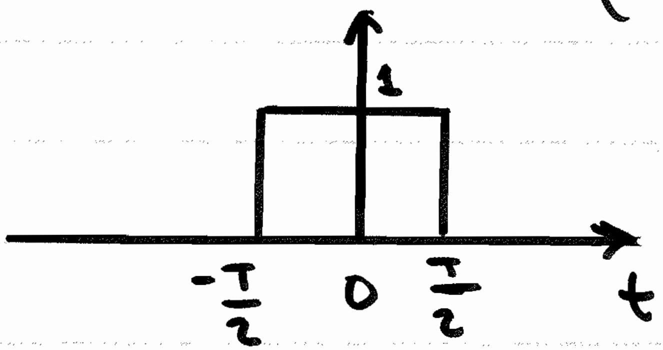
$$\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)$$

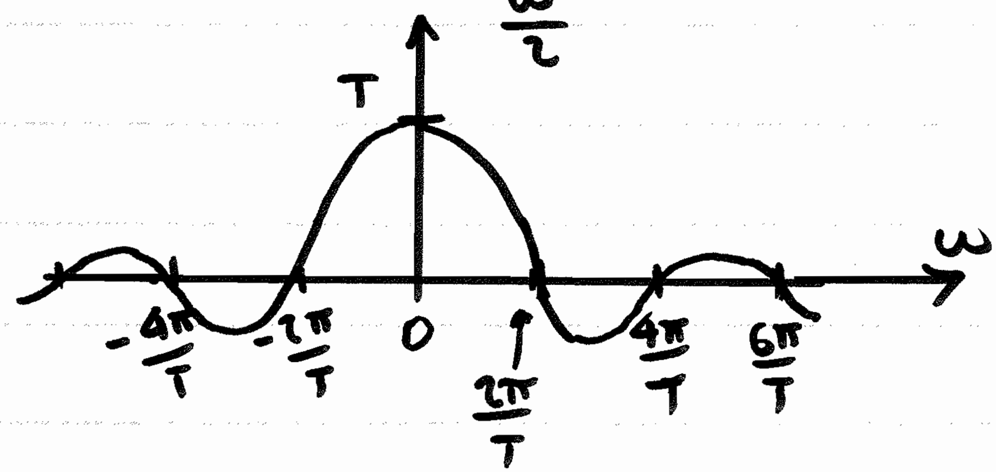


$$\sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

Example 4.4

$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$





Proof on next page:

- for value at $\omega=0$, use L'Hospital's Rule:

$$X(\omega)\Big|_{\omega=0} = \frac{\frac{T}{2}\cos\left(T\frac{\omega}{2}\right)\Big|_{\omega=0}}{\frac{1}{2}} = T$$

- zero crossings at: $T\frac{\omega}{2} = m\pi$, $\sin(m\pi) = 0$ for m integer

- $X(\omega) \to 0$ as $\omega \to \infty$ or $\omega \to -\infty$

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- Proof: $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\, e^{-j\omega t}\Big]_{-T/2}^{T/2}$$

$$= \frac{-1}{j\omega}\left\{ e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}} \right\} \frac{2}{2}$$

$$= \frac{2}{\omega}\left\{ \frac{1}{2j}\, e^{j\omega\frac{T}{2}} - \frac{1}{2j}\, e^{-j\omega\frac{T}{2}} \right\}$$

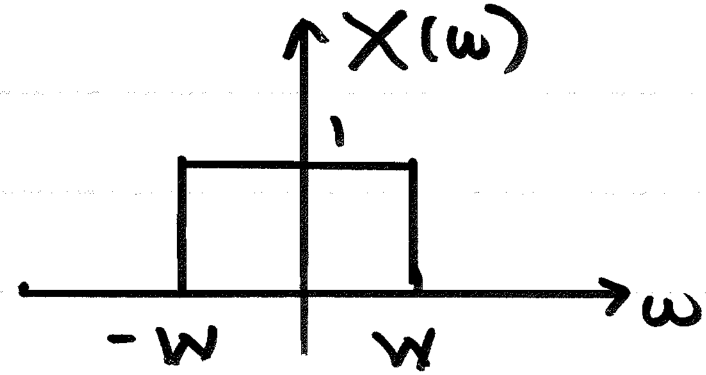
$$= \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

purely real-valued because $x(t)$ is symmetric

$$x(-t) = x(t)$$

Example 4.5 What if rectangle shape is in frequency domain? Lowpass Filter

$$x(t) = ? \xleftrightarrow{\mathcal{F}}$$



Use inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

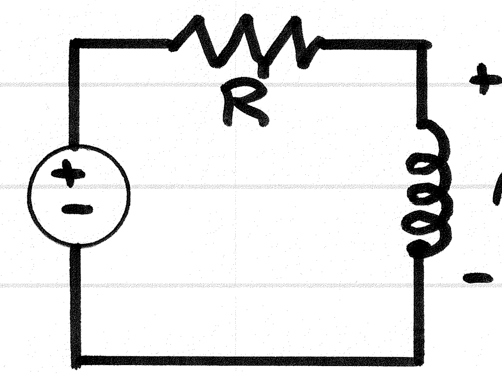
$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi j t}\left\{ \left[e^{j\omega t} \right]_{-W}^{W} \right\}$$

$$= \frac{1}{\pi t}\,\frac{1}{2j}\left\{ e^{jWt} - e^{-jWt} \right\} = \frac{\sin(Wt)}{\pi t}$$

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega}{2W}\right)$$

Relationship of Fourier Transform to AC Steady State Circuit Analysis in ECE 201

Simple Example



$$A e^{j\omega_o t} \qquad v_L = \frac{j\omega_o L}{R + j\omega_o L} A e^{j\omega_o t}$$

$$= A \frac{\omega_o L}{\sqrt{R^2 + \omega_o^2 L^2}} e^{j\left(\omega_o t + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)}$$

$$H(\omega) = \mathcal{F}\{h(t)\} = \frac{j\omega L}{R + j\omega L} = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega_o L}{R}\right)\right)}$$

$$= |H(\omega)| e^{j\angle H(\omega)}$$