- consider: $x(t) = x(t+T)$ for all $t$
- Fourier series expansion for periodic signal with period T:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, s_k(t)$$

$$s_k(t) = e^{j 2\pi \frac{k}{T} t}$$

"Fourier Series coefficients"
= complex amplitudes of the sinewaves

orthogonal sinewaves forming a complete basis for all periodic signals with period T

$\frac{1}{T}$ = fundamental frequency in Hz

$\frac{k}{T}$ = $k$-th harmonic frequency ($k$ = integer)

- $s_k(t)$ is periodic with period $\frac{T}{k}$ and is thus periodic with period $T$ ($k$ = integer)

- $s_k(t)$ and $s_\ell(t)$ are orthogonal:

$$\int_{-T/2}^{T/2} s_k(t)\, s_\ell^*(t)\, dt = 0 \quad \text{if} \quad k \neq \ell$$

Proof: $$\int_{-T/2}^{T/2} e^{j2\pi \frac{k}{T} t} e^{-j2\pi \frac{\ell}{T} t} dt = \int_{-T/2}^{T/2} e^{j2\pi \frac{(k-\ell)}{T} t} dt$$

$$= \frac{1}{j2\pi \frac{(k-\ell)}{T}} e^{j2\pi \frac{(k-\ell)}{T} t} \Big]_{-T/2}^{T/2}$$

$$= \frac{T}{j2\pi(k-\ell)} \left\{ e^{j2\pi \frac{(k-\ell)}{T} \frac{T}{2}} - e^{-j2\pi \frac{(k-\ell)}{T} \frac{T}{2}} \right\}$$

$$= \frac{T}{j2\pi(k-\ell)} \left\{ \left(e^{j\pi}\right)^{k-\ell} - \left(e^{-j\pi}\right)^{k-\ell} \right\} = 0$$

since $e^{j\pi} = e^{-j\pi} = -1$

More generally: $$\int_{t_0}^{t_0+T} s_k(t)\, s_\ell^*(t)\, dt = T\,\delta[k-\ell]$$

$t_0$ arbitrary

e.g. $$\int_0^T s_k(t)\, s_\ell^*(t)\, dt = \begin{cases} T, & \text{if } k=\ell \\ 0, & \text{if } k\neq\ell \end{cases} = T\,\delta[k-\ell]$$

- As a result, the FS coefficients (complex amplitudes) may be found as:

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, s_k^*(t)\, dt = \frac{1}{T}\int_{t_0}^{t_0+T}\left(\sum_{\ell=-\infty}^{\infty} a_\ell\, s_\ell(t)\right) s_k^*(t)\, dt$$

$$= \sum_{\ell=-\infty}^{\infty} a_\ell \frac{1}{T}\int_{t_0}^{t_0+T} s_\ell(t)\, s_k^*(t)\, dt = \sum_{\ell=-\infty}^{\infty} a_\ell \frac{1}{T}\cdot T\,\delta[k-\ell] = a_k$$

- in particular: $$\boxed{a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi\frac{k}{T}t}\, dt}$$

Complex amplitude for $k$-th harmonic sinewave computed as:

$$s_k(t) = e^{jk\frac{2\pi}{T}t} = e^{j2\pi\frac{k}{T}t}$$

$$a_k = \frac{\int_{-T/2}^{T/2} x(t)\, s_k^*(t)\, dt}{\underbrace{\int_{-T/2}^{T/2} s_k(t)\, s_k^*(t)\, dt}_{\int_{-T/2}^{T/2} |s_k(t)|^2 dt = T}} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi\frac{k}{T}t}\, dt$$

Just like vectors in 3-D space:

$\underline{x}, \underline{y}, \underline{z}$, orthogonal basis

$$\underline{w} = a\underline{x} + b\underline{y} + c\underline{z}$$

$$\underline{x}^T\underline{w} = a\,\underline{x}^T\underline{x} + b\,\cancelto{0}{\underline{x}^T\underline{y}} + c\,\cancelto{0}{\underline{x}^T\underline{z}}$$

$$a = \frac{\underline{x}^T\underline{w}}{\underline{x}^T\underline{x}} \qquad \underline{x}^T\underline{w} = \sum_{i=1}^{3} w_i x_i$$

**Sidenote:**

Recall energy of a signal: $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt$

For more general case of a complex-valued signal:

NOTATION

$$x(t) = x_I(t) + j\,x_Q(t)$$

in-phase    quadrature-phase

$$\text{OR:} \quad = x_r(t) + j\,x_i(t)$$

real part = $\mathrm{Re}\{x(t)\}$
$= \frac{1}{2}\{x(t) + x^*(t)\}$

imaginary part $\mathrm{Im}\{x(t)\}$
$= \frac{1}{2j}\{x(t) - x^*(t)\}$

$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} x_r^2(t)\,dt + \int_{-\infty}^{\infty} x_i^2(t)\,dt$$

$=$ energy of real part $+$ energy of imaginary part

- Energy for sinewave is typically computed as energy per period:

$$E_k = \int_{-T/2}^{T/2} |s_k(t)|^2 \, dt$$

$$\boxed{|s_k(t)|^2 = s_k(t)\, s_k^*(t)}$$

$$= \int_{-T/2}^{T/2} e^{j2\pi \frac{k}{T} t} \, e^{-j2\pi \frac{k}{T} t} \, dt = \int_{-T/2}^{T/2} e^{0} \, dt = T$$

- With a complex-amplitude:

energy of $a_k s_k(t)$ is: $T|a_k|^2$ $\Big\}$ = energy of $a_k e^{j2\pi \frac{k}{T} t}$

= energy in $k$-th harmonic term in FS expansion with frequency $\frac{k}{T}$ in Hz

Critical Property ⇒ Parseval's Theorem

- Energy per period of periodic signal
  $\equiv$ sum of energy (per period) in each of the harmonic sinewaves in the FS expansion of $x(t)$

• Mathematically:

$$\boxed{\int_{-T/2}^{T/2} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2 T}$$

• Proof derives from orthogonality of the sinewaves:

$$\int_{-T/2}^{T/2} x(t)\,x^*(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} \sum_k a_k s_k(t) \sum_l a_l^* s_l^*(t)\,dt$$

$$= \sum_k \sum_l a_k a_l^* \cdot \underbrace{\int_{-T/2}^{T/2} s_k(t)\, s_l^*(t)\,dt}_{\cdot\, T\,\delta[k-l]} = \sum_{k=-\infty}^{\infty} |a_k|^2 T$$

Q.E.D.

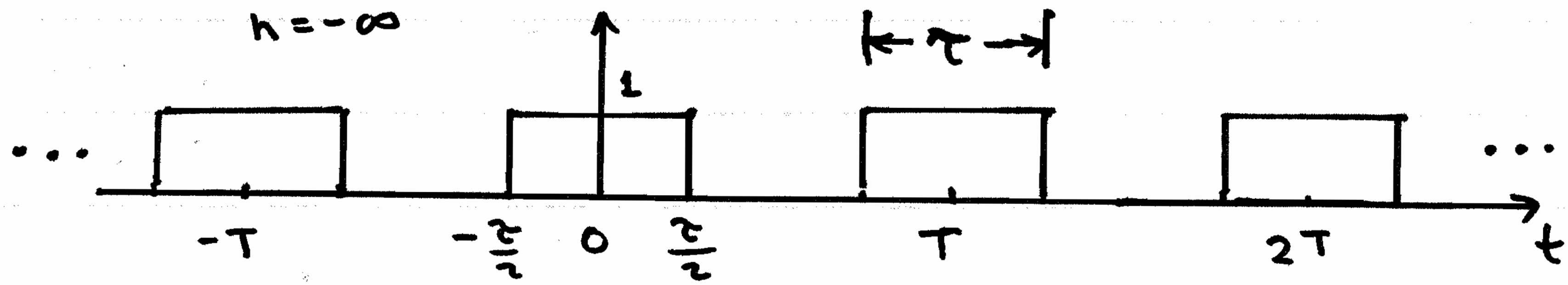
---

• This is our first (big) step towards examining how the energy of a signal is distributed as a function of frequency!

- Some basic Fourier Series!

1. Periodic train of rectangular pulses:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-nT}{\tau}\right)$$




$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi \frac{k}{T} t}\, dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-j2\pi \frac{k}{T} t}\, dt$$

$$= \frac{1}{T} \cdot \frac{1}{-j2\pi \frac{k}{T}} \left. e^{-j2\pi \frac{k}{T} t} \right]_{-\tau/2}^{\tau/2} = \frac{1}{k\pi} \frac{-1}{2j} \left\{ e^{-j\frac{2\pi k \tau}{2T}} - e^{j\frac{2\pi k \tau}{2T}} \right\}$$

$$a_k = \frac{\sin\left(k\pi \frac{\tau}{T}\right)}{k\pi} \qquad -\infty < k < \infty$$

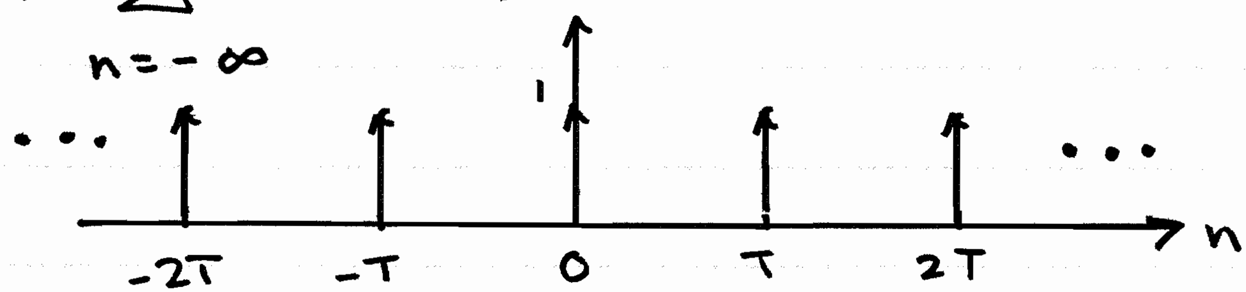
at $k=0 \Rightarrow \frac{\tau}{T}$

See Example 3.5 on pg. 193-195

2. Periodic Train of Dirac Delta Functions

=> very important for our theoretical analysis of sampling theory later in Chap. 7

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$




$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j 2\pi \frac{k}{T} t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-j 2\pi \frac{k}{T} t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{0}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T} \quad \forall k$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j 2\pi \frac{k}{T} t}$$

# Some Key Properties of Fourier Series

- Starting point: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+j 2\pi \frac{k}{T} t}$

- **Time-Shift Property:**

$$y(t) = x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi \frac{k}{T}(t-t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k e^{-j 2\pi k \frac{t_0}{T}}\right)}_{\text{new FS coeffs for } y(t) = x(t-t_0)} e^{j 2\pi \frac{k}{T} t}$$

- **Time-Scaling:**

$$y(t) = x(at) = \sum_{k=-\infty}^{\infty} a_k e^{+j 2\pi \frac{k}{T} at} = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi k t}{T/a}}$$

$\Rightarrow$ FS coefficients are the same

$\Rightarrow$ new period $= T/a$

- <u>Differentiation</u>

$$y(t) = \frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k\, j\frac{2\pi k}{T}\right)}_{\substack{\text{new FS coeffs} \\ \text{for } y(t) = \frac{dx(t)}{dt}}} e^{j\frac{2\pi k}{T}t}$$

- For additional properties of FS, see Table 3.1 on pg. 206

- <u>Basic Property:</u> $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k}{T}t}$ $\quad$ $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi k}{T}t}$

same period

$$z(t) = A\,x(t) + B\,y(t)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(A a_k + B b_k\right)}_{\substack{\text{new FS coeffs} \\ \text{for } z(t)}} e^{j\frac{2\pi k}{T}t}$$

## Expressing Fourier Series Expansion in Terms of Real-Valued Sinewaves for x(t) real-valued

- For $x(t)$ real-valued, $a_{-k} = a_k^*$
- In polar form: $a_k = |a_k| e^{j\angle a_k}$, $a_k^* = |a_k| e^{-j\angle a_k}$
- Thus:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi\frac{k}{T}t} = a_0 + \sum_{k=1}^{\infty}\left(a_k e^{j2\pi\frac{k}{T}t} + a_{-k} e^{-j\frac{k}{T}2\pi t}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} |a_k| \left(e^{j\left(2\pi\frac{k}{T}t + \angle a_k\right)} + e^{-j\left(2\pi\frac{k}{T}t + \angle a_k\right)}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2|a_k| \cos\left(2\pi\frac{k}{T}t + \angle a_k\right)$$