Solution

EE301 Signals and Systems — Spring 2011
Final Exam — Saturday, May 7, 2011

Cover Sheet

Test Duration: 120 minutes.
Coverage: Comprehensive
Open Book but Closed Notes.
Three handwritten 8.5 in. x 11 in. crib sheets
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. Ignore the numbers at the bottom. For each plot, the abscissa goes from -2π to $+2\pi$ with tic marks every $\pi/8$. There is a dashed vertical line at $\omega = -\pi$ and another dashed vertical line at $\omega = +\pi$. You only have to plot over $-\pi < \omega < \pi$.*

Problem 1. Consider a finite-length sinewave of length N, as described below,

$$x[n] = e^{j\omega_o n}\{u[n] - u[n-N]\} \tag{1}$$

the input to an LTI filter with impulse response $h[n]$, so that the output is

$$y[n] = x[n] * h[n]$$

$h[n]$ is an arbitrary filter of length L, where $L < N$; $h[n]$ is only nonzero for $n = 0, 1, ..., L-1$. Since all you are required to do is to find the limits on the convolution summation for different intervals, it may be helpful to think of $h[n]$ as discrete-time rectangle: $h[n] = \{u[n]-u[n-L]\}$. **You can use space on the next page to work out the answers.** Remember $N > L$.

(a) The region $0 \leq n \leq L-1$ corresponds to *partial overlap.* The convolution sum may be expressed as below, where we have chosen to flip and shift the finite-length sinewave $x[n]$ as per the graphical method for doing convolution.

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{partial overlap:}\ 0 \leq n \leq L-1 \tag{2}$$

Determine the upper and lower limits in the convolution sum above for $0 \leq n \leq L-1$:

lower limit = 0	and	upper limit = n

(b) The region $L \leq n \leq N-1$ corresponds to *full overlap.* The convolution sum is:

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{full overlap:}\ L \leq n \leq N-1 \tag{3}$$

(i) Determine the upper & lower limits in the convolution sum for $L \leq n \leq N-1$.

lower limit = 0	and	upper limit = L-1

(ii) Substituting $x[n]$ in Eqn (1) into Eqn (3), show that for this range $y[n]$ simplifies to:

$$y[n] = H(\omega_o)e^{j\omega_o n} \quad \text{for} \quad L \leq n \leq N-1 \tag{4}$$

which is identical to the output that is obtained with an infinite length sinewave. To get the points, you must show all work and explain all details.

(c) The region $N \leq n \leq N+L-2$ corresponds to *partial overlap.*

$$y[n] = \sum_{k=??}^{??} h[k]x[n-k] \quad \textit{partial overlap:}\ N \leq n \leq N+L-2 \tag{5}$$

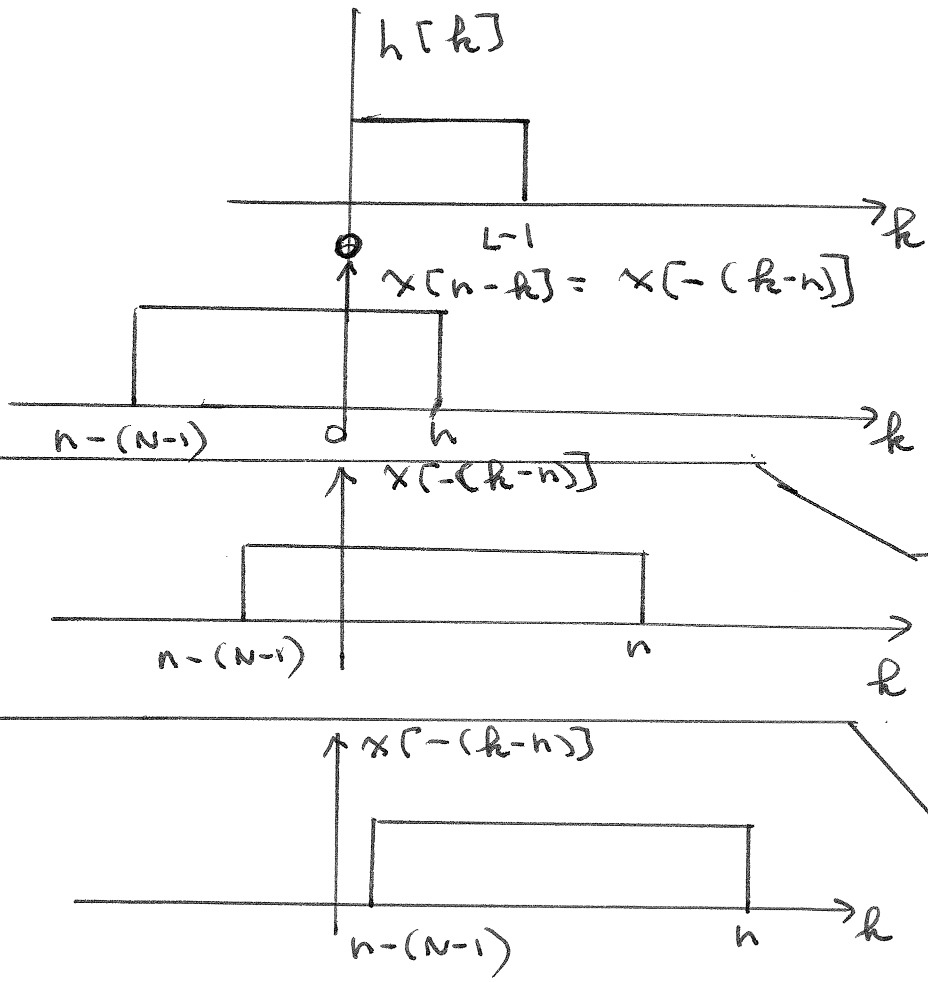
Determine the upper & lower limits in the convolution sum for $N \leq n \leq N+L-2$:

lower limit = n - (N-1)	and	upper limit = L-1

Show work for Problem 1 below, particularly part (b)-(ii).

Can assume arbitrary shape for both $x[n]$ and $h[n]$ in order to determine limits for convolution sum. So, for illustrative purposes, assume rectangles for both. Even more so, draw them as if they were continuous (avoid stem plots)

$\boxed{\text{given } N > L}$



for $0 \le n \le L-1$: partial overlap

limits on convolution

$$y[n] = \sum_{k=0}^{n} h[k]\, x[n-k]$$

for $L \le n \le N-1$

$\Rightarrow$ full overlap

limits on convolution

$$y[n] = \sum_{k=0}^{L-1} h[k]\, x[n-k]$$

for $N \le n \le N+L-2$

$\Rightarrow$ partial overlap

$$y[n] = \sum_{n-(N-1)}^{L-1} h[k]\, x[n-k]$$

(b)-(ii): full overlap region

$$y[n] = \sum_{k=0}^{L-1} h[k]\, e^{j\omega_0(n-k)}$$

$$= \left\{ \sum_{k=0}^{L-1} h[k]\, e^{-jk\omega_0} \right\} e^{j\omega_0 n} = H(\omega_0)\, e^{j\omega_0 n}$$

Problem 2. The signal below is defined for all parts of this problem.

$$x(t) = \frac{\sin(10t)}{\pi t} \tag{6}$$

For each sub-part, a different $y(t)$ is defined in terms of $x(t)$.

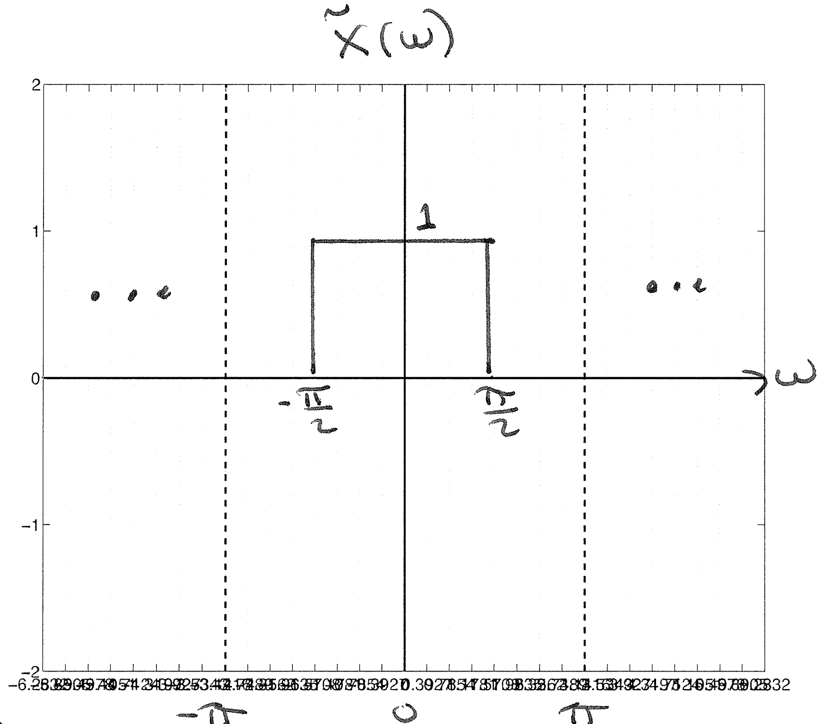
Problem 2 (a). For $x(t) = \dfrac{\sin(10t)}{\pi t}$,

(i) Compute the energy $E_x = \int_{-\infty}^{\infty} x^2(t)dt$. Show work below; write answer in this box:

$\boxed{E_x = 10/\pi}$

(ii) What is the Nyquist sampling rate for $x(t)$? Write answer in this box: $\boxed{\omega_s = 20}$

(iii) Define $\tilde{x}(t) = T_s x(t)$. Plot the DTFT of $\tilde{x}[n] = \tilde{x}(nT_s)$, where $T_s = \frac{2\pi}{40}$. Plot $\tilde{X}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided below.



from Parseval's Theorem:

$$E_x = \frac{1}{2\pi}\int_{-10}^{10} (1)^2 \, d\omega = \frac{2(10)}{2\pi} = \frac{10}{\pi}$$

$$T_s = \frac{2\pi}{\omega_s} \qquad \omega_s = 40$$

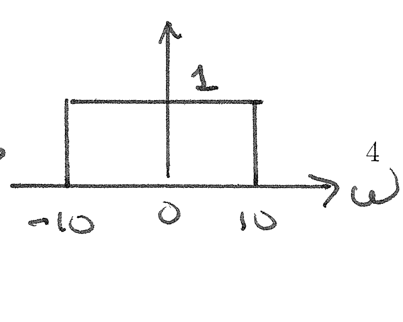
$\omega_s = 40 > 2(10)$

no aliasing!

$\omega_M = 10$ mapped to

$$\omega_M T_s = 10\,\frac{2\pi}{40} = \frac{\pi}{2}$$

$$\frac{\sin(10t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow}$$



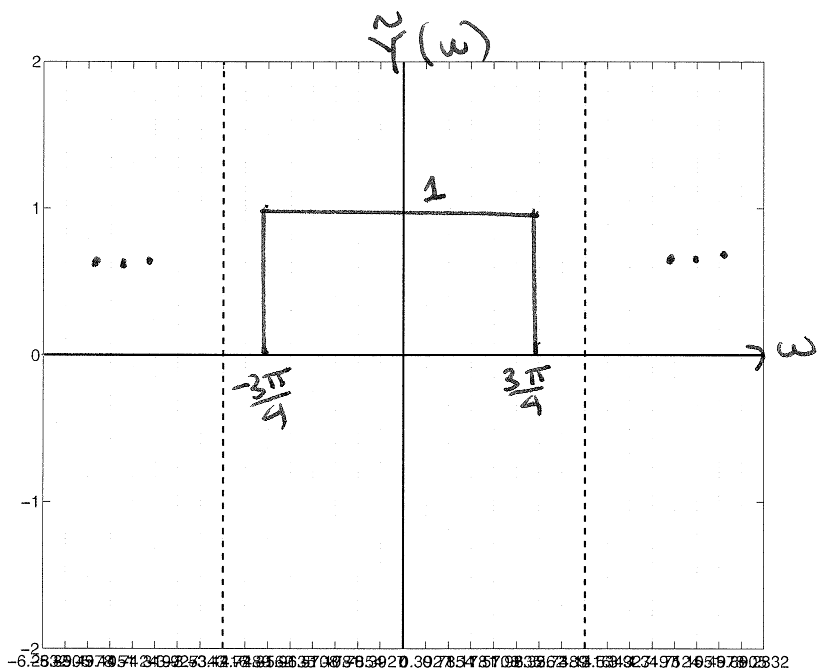
Problem 2 (b). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

$$y(t) = x(t-2)$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} y^2(t)dt$. Explain answer; write answer in this box: $\boxed{E_y = 10/\pi}$

(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = 20}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the magnitude DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{80/3} = \frac{3\pi}{40}$. Plot the magnitude of $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided below.



$Y(\omega) = e^{-j2\omega} X(\omega) \Rightarrow |Y(\omega)| = |X(\omega)|$

$\Rightarrow |Y(\omega)|^2 = |X(\omega)|^2 \Rightarrow$ so same energy as $x(t)$

Different sampling rate: $\omega_s = \frac{80}{3} = 26.667 > 2(10)$

$\Rightarrow$ no aliasing!

$\omega_M = 10$ mapped to $10\,T_s = 10\,\frac{3\pi}{40} = \frac{3\pi}{4}$

Problem 2 (c). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

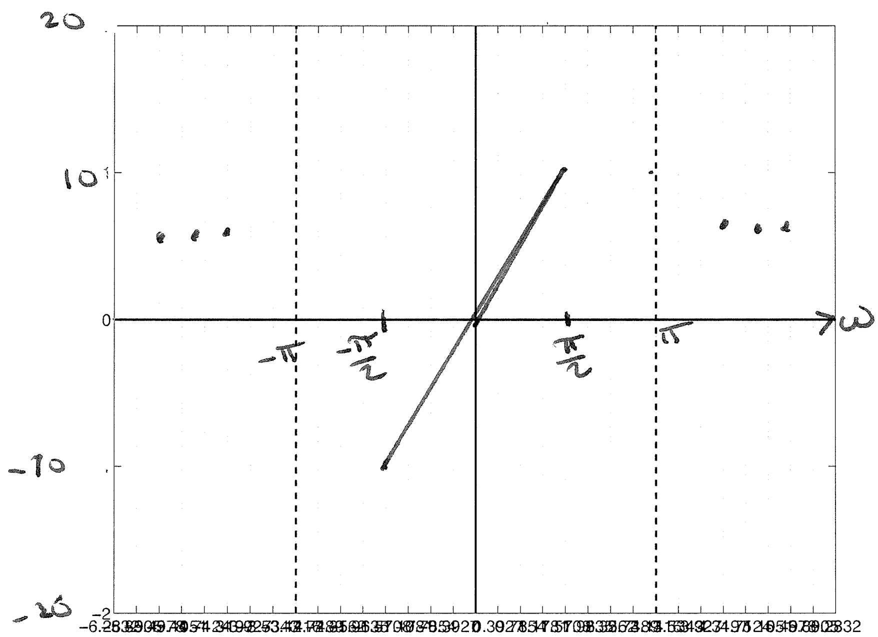
$$y(t) = -j\frac{d}{dt}x(t)$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$. Show work below; write answer in this box:

$\boxed{E_y = 1000/3\pi}$

(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = 20}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{40}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided.



no aliasing

So, $\hat{Y}(\omega) = \frac{1}{T_s} X(F_s\omega)$

$= F_s X(F_s\omega)$

$= F_s F_s \omega$

$\tilde{Y}(\omega) = T_s \hat{Y}(\omega)$

$= F_s\omega = \omega / T_s$

$= \frac{40}{2\pi}\omega$

Since

$\frac{d\,x(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega) \Rightarrow Y(\omega) = -j(j\omega)X(\omega)$

$= \omega\ \mathrm{rect}\left(\frac{\omega}{20}\right)$

doesn't change Nyquist rate $\Rightarrow 2(10) = 20$

$E_y = 2\frac{1}{2\pi}\int_0^{10}\omega^2\, d\omega = \frac{1}{\pi}\frac{\omega^3}{3} = \frac{1000}{3\pi}$

$\omega_s = 40 > 2(10)$ no aliasing!

$\omega_M = 10$ mapped to $10T_s = 10\frac{2\pi}{40} = \frac{\pi}{2}$

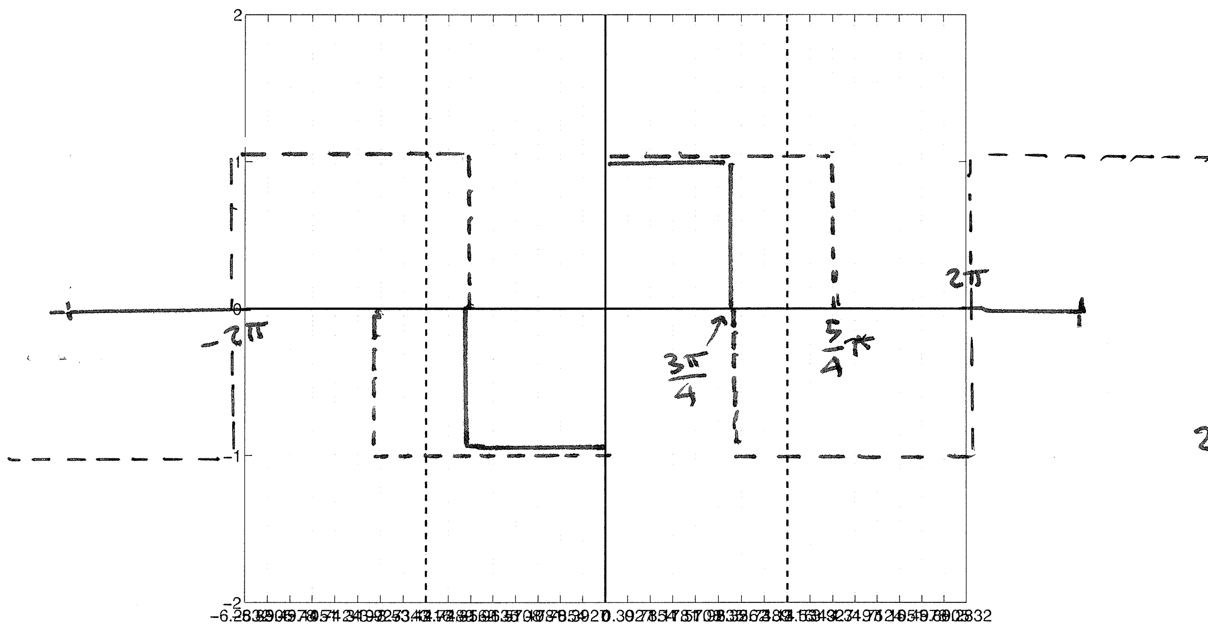
Problem 2 (d). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

$$y(t) = j2\pi t x^2(t)$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$. Show work below; write answer in this box: $\boxed{E_y = 20/\pi}$

(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = \quad 40}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{32}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided.



$2\pi - \frac{5\pi}{4}$

$= \frac{3\pi}{4}$

At least 2 ways to solve this problem. Simpler way:

$y(t) = 2j\pi t \dfrac{\sin(10t)}{\pi t} \dfrac{\sin(10t)}{\pi t}$

$= \dfrac{\sin(10t)}{\pi t} 2j \sin(10t) = \dfrac{\sin(10t)}{\pi t}\{e^{j10t} - e^{-j10t}\}$

$\overset{\mathcal{F}}{\longleftrightarrow} -\mathrm{rect}\left(\dfrac{\omega+10}{20}\right) + \mathrm{rect}\left(\dfrac{\omega-10}{20}\right)$

$E_y = \dfrac{1}{2\pi}\displaystyle\int_{-20}^{20} (1)^2 d\omega = \dfrac{40}{2\pi} = \dfrac{20}{\pi}$

$\omega_M = 20$

Nyquist rate $= 2(20) = 40$

$\omega_s = 32 < 40 \Rightarrow$ aliasing

ω_M mapped to $20T_s = 20\dfrac{2\pi}{32} = \dfrac{5}{4}\pi > \pi$

Problem 2 (e). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

$$y(t) = x(t) * h(t)$$

where

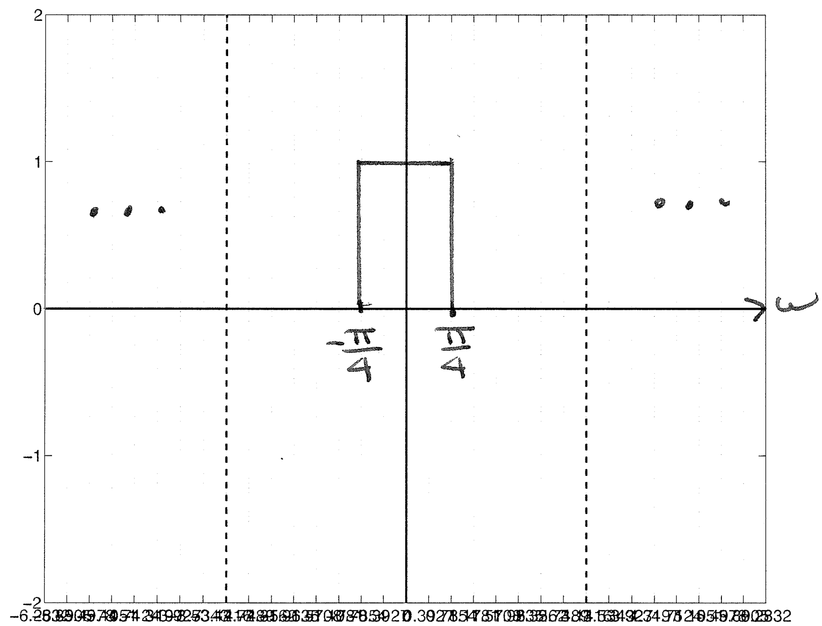
$$h(t) = \frac{\sin(5t)}{\pi t}$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$. Show work below; write answer in this box:

$\boxed{E_y = 5/\pi}$

(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = 10}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{40}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided.



$Y(\omega) = X(\omega)\,H(\omega) = \text{rect}\left(\frac{\omega}{20}\right)\text{rect}\left(\frac{\omega}{10}\right) = \text{rect}\left(\frac{\omega}{10}\right)$

$\omega_s = 40 > 10$

$\omega_M = 5$ Nyquist rate $= 2(5) = 10$

no aliasing

$\omega_M = 5$ mapped to $5T_s = \frac{5 \cdot 2\pi}{40} = \frac{\pi}{4}$

$E_y = \frac{1}{2\pi}\int_{-5}^{5} (1)^2 d\omega = \frac{10}{2\pi} = \frac{5}{\pi}$

Problem 2 (f). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

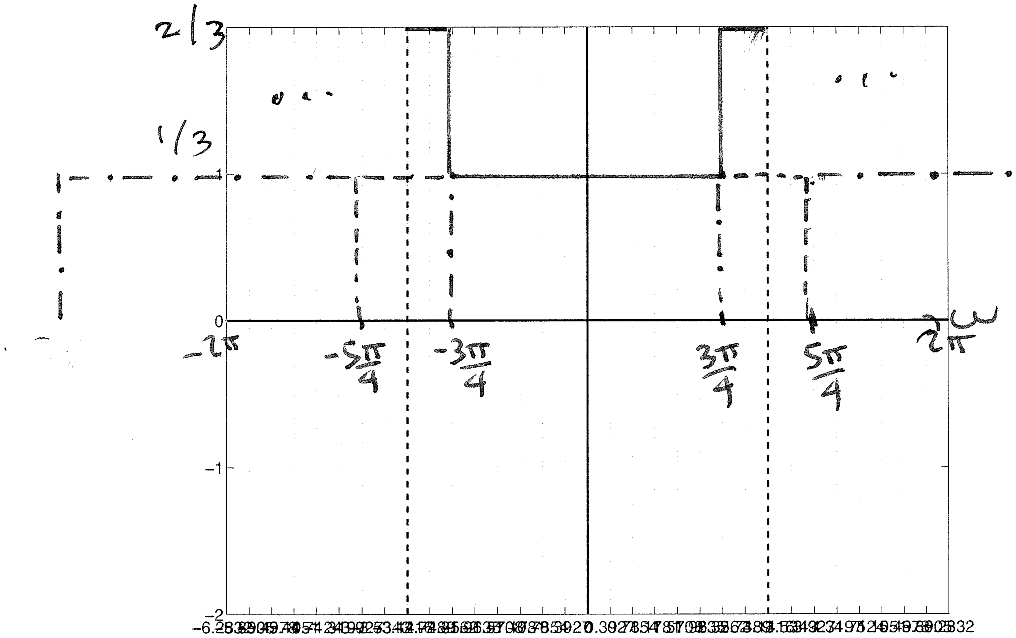
$$y(t) = x(3t)$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$. Show work below; write answer in this box:

$\boxed{E_y = 10/3\pi}$

(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = 60}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{48}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided.



$x(3t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{3} X\left(\frac{\omega}{3}\right)$ expansion by a factor of 3

$$E_y = \frac{1}{2\pi}\int_{-30}^{30} \left(\frac{1}{3}\right)^2 d\omega = \frac{1}{9}\,\frac{60}{2\pi} = \frac{10}{3\pi}$$

$\omega_M = 30$

Nyquist rate $= 2(30)$
$= 60$

$\omega_M = 30$ mapped to $30T_s = 30\,\frac{2\pi}{48}$
$= 5\frac{\pi}{4}$

$\omega_s = 48 < 60$

Aliasing

Problem 2 (g). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

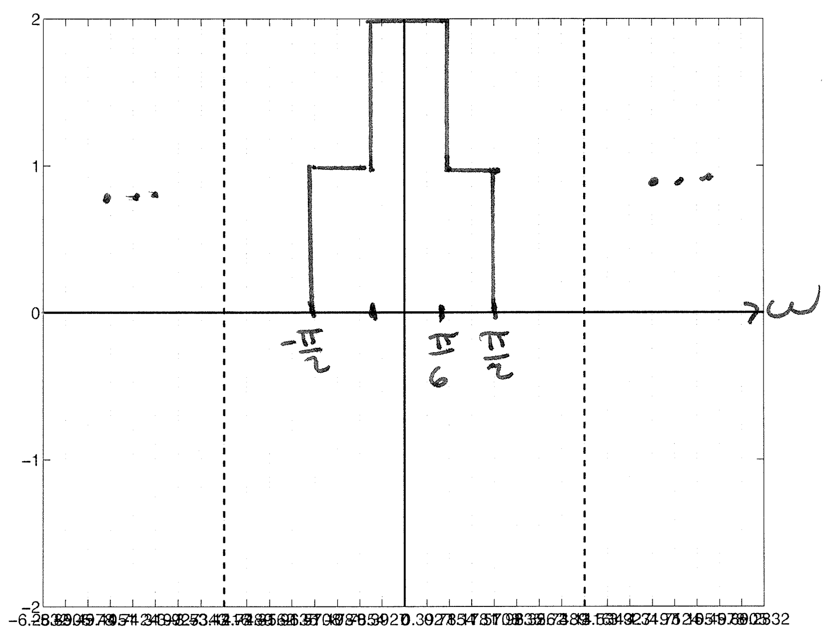
$$y(t) = 2x(t)\cos(5t)$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$. Show work below; write answer in this box: $\boxed{E_y = 30/\pi}$

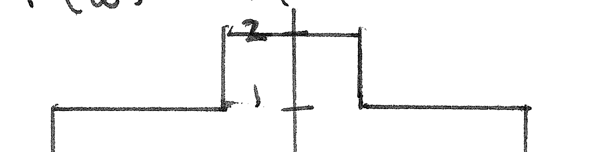
(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = \quad 30}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{60}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided.

$\tilde{Y}(\omega)$



$Y(\omega) = X(\omega - 5) + X(\omega + 5)$ $\quad X(\omega) = \text{rect}\left(\frac{\omega}{20}\right)$



$\omega_M = 15 \Rightarrow$ Nyquist rate $= 30$

$\omega_s = 60 > 30$ No aliasing

$E_y =$

$$E_y = \frac{1}{2\pi}\int_{-5}^{5}(2)^2 d\omega + \frac{1}{2\pi}(2)\int_{5}^{15}(1)^2 d\omega$$

$$= \frac{4 \cdot 10}{2\pi} + \frac{1}{\pi}10 = \frac{10}{\pi}(2+1)$$

$\omega_M = 15$ mapped to

$15 T_s = 15 \frac{2\pi}{60} = \frac{\pi}{2}$

$\omega_M = 5$ mapped to $\pi/6$

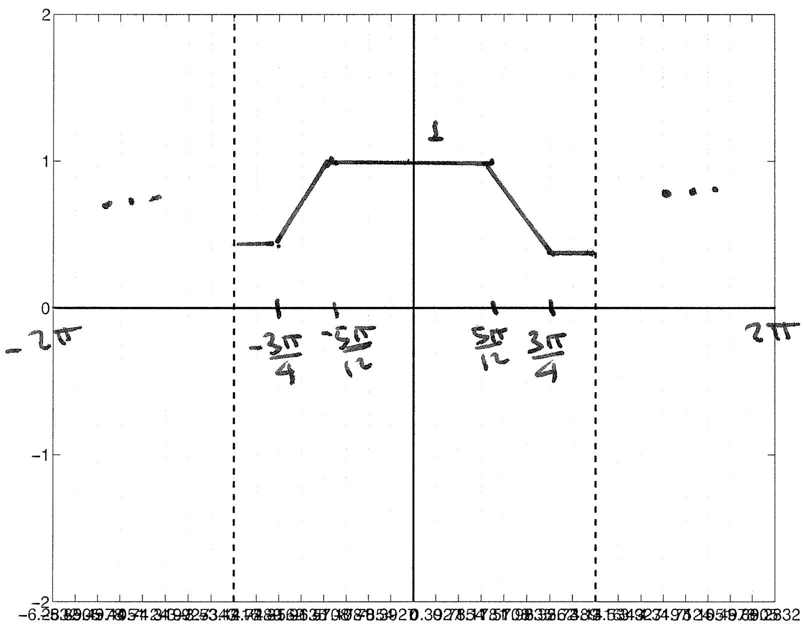
Problem 2 (h). Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

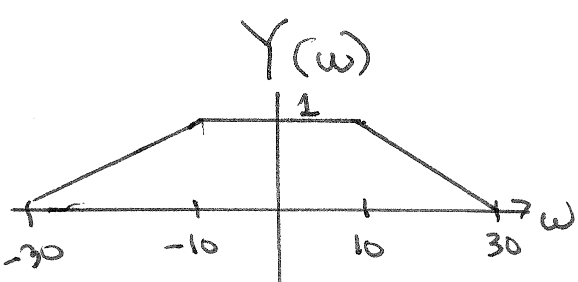
$$y(t) = x(t)\,\frac{\pi}{10}\left\{\frac{\sin(20t)}{\pi t}\right\} = \frac{\pi}{10}\,\frac{\sin(10t)}{\pi t}\,\frac{\sin(20t)}{\pi t}$$

(i) Compute the energy $E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$. Show work below; write answer in this box: $\boxed{E_y = 50/3\pi}$

(ii) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = \quad}$

(iii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{48}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided.





$\omega_M = 30$ Nyquist rate $= 2(30) = 60$

$\omega_s = 48 < 60 \Rightarrow$ aliasing

$\omega_M = 30$ mapped to $30\,\frac{2\pi}{48} = \frac{5\pi}{4}$

$\omega_M = 10$ mapped to $\frac{1}{3}\,\frac{5\pi}{4} = \frac{5\pi}{12}$

$$E_y = \frac{1}{2\pi}\int_{-10}^{10} (1)^2\, d\omega + \frac{1}{2\pi}(2)\int_{10}^{30} (\omega - 30)^2 \left(\frac{-1}{20}\right)^2 d\omega$$

$$= \frac{10}{\pi} + \frac{1}{\pi}\left(-\frac{1}{20}\right)^2 \left.\frac{(\omega-30)^3}{3}\right]_{10}^{30} = \frac{10}{\pi} + \frac{1}{\pi}\left(\frac{1}{20}\right)^2\left\{\frac{20^3}{3}\right\} = \frac{1}{\pi}\left\{10 + \frac{20}{3}\right\} = \frac{50}{3\pi}$$

Problem 3. This problem is about Vestigial Sideband (VSB) Modulation where we transmit a small part of the negative portion of the spectrum along with the positive frequency portion of the spectrum.

(a) Consider the following complex-valued filter.

$$h[n] = 16\, e^{j\frac{7\pi}{16}n} \left\{ \frac{\sin\left(\frac{2\pi}{16}n\right)}{\pi n} \frac{\sin\left(\frac{7\pi}{16}n\right)}{\pi n} \right\}$$

Plot the DTFT of $h[n]$, $H(\omega)$, over $-\pi < \omega < \pi$.

(b) For illustrative purposes, consider the following simple input signal

$$x[n] = \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n}$$

Plot the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$.

(c) The signal in part (b) is run through the filter in part (a) to produce the output $y[n]$

$$y[n] = x[n] * h[n]$$

Plot the DTFT of $y[n]$, $Y(\omega)$, over $-\pi < \omega < \pi$.

(d) The real-part of the complex-valued signal $y[n]$ may be expressed as

$$y_R[n] = Re\{y[n]\} = \frac{1}{2}\{y[n] + y^*[n]\}$$

Plot the DTFT of the complex-conjugate signal $y^*[n]$ over $-\pi < \omega < \pi$.

(e) Sum your respective answers to parts (c) and (d), and divide by 2, to form the DTFT of $y_R[n]$, denoted $Y_R(\omega)$. Plot $Y_R(\omega)$ over $-\pi < \omega < \pi$.

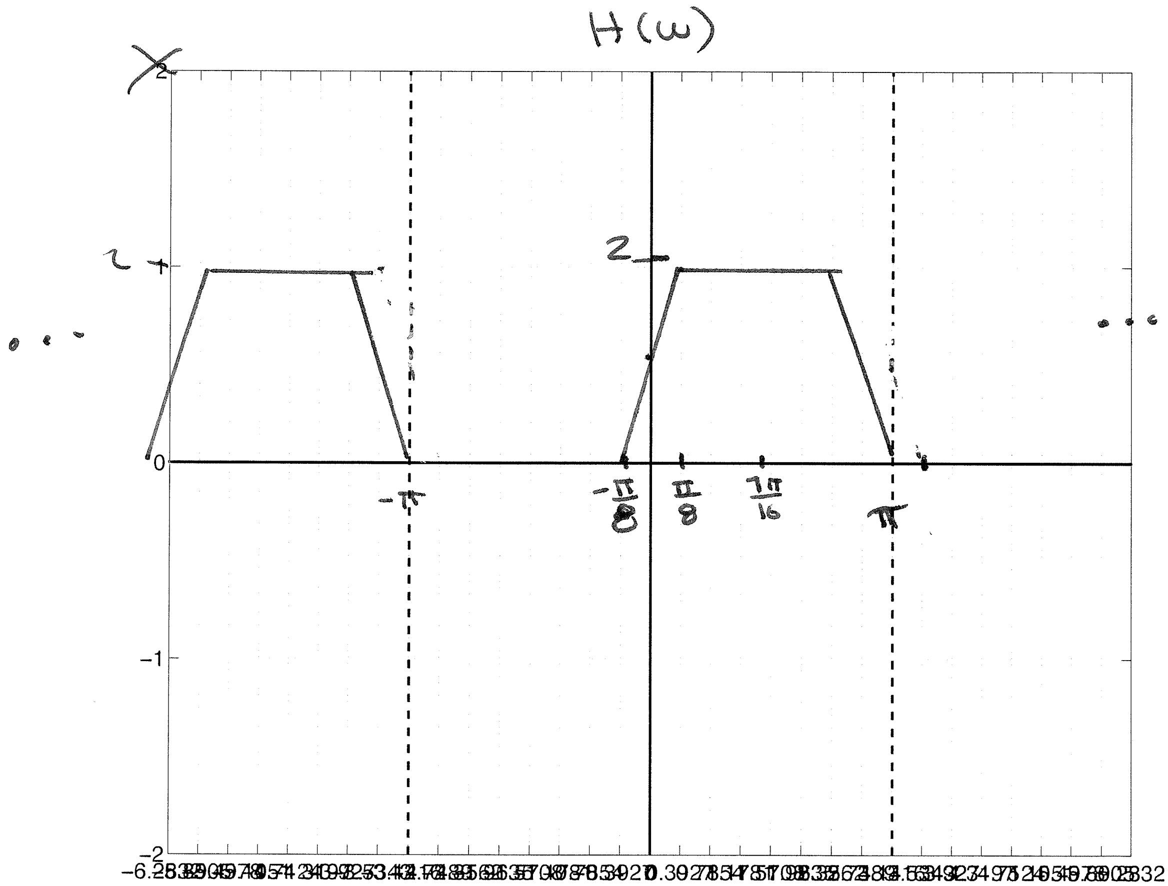
(f) Is your answer to part (e) equal to the DTFT of the original signal $x[n]$? That is, is $y_R[n] = x[n]$? Why or why not? Explain your answer directly below.

$$\left.\begin{aligned} \frac{7\pi}{16} + \frac{2\pi}{16} &= \frac{9\pi}{16} \\ \frac{7\pi}{16} - \frac{2\pi}{16} &= \frac{5\pi}{16} \end{aligned}\right\} \quad \frac{9\pi}{16} - \frac{5\pi}{16} = \frac{4\pi}{16} = \frac{\pi}{4}$$

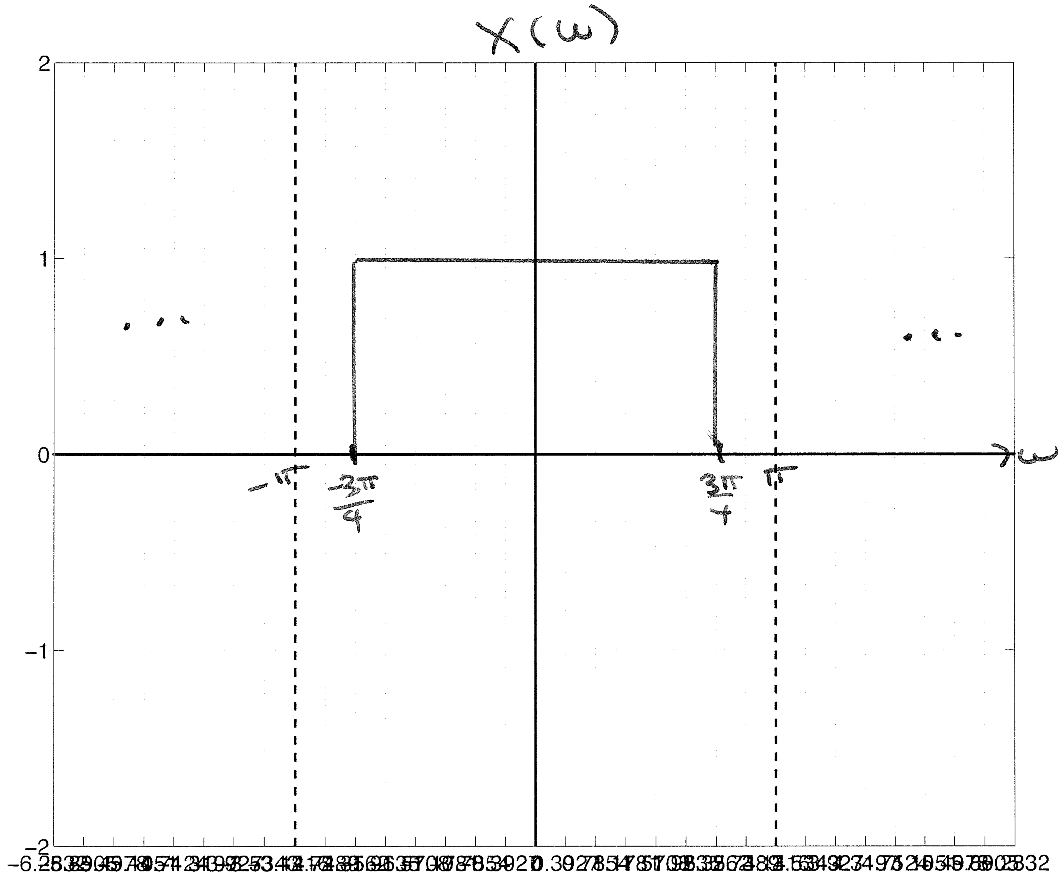
width of linearly sloping region

$$h[n] = h_{LP}[n]\, e^{j\frac{7\pi}{16}n}$$

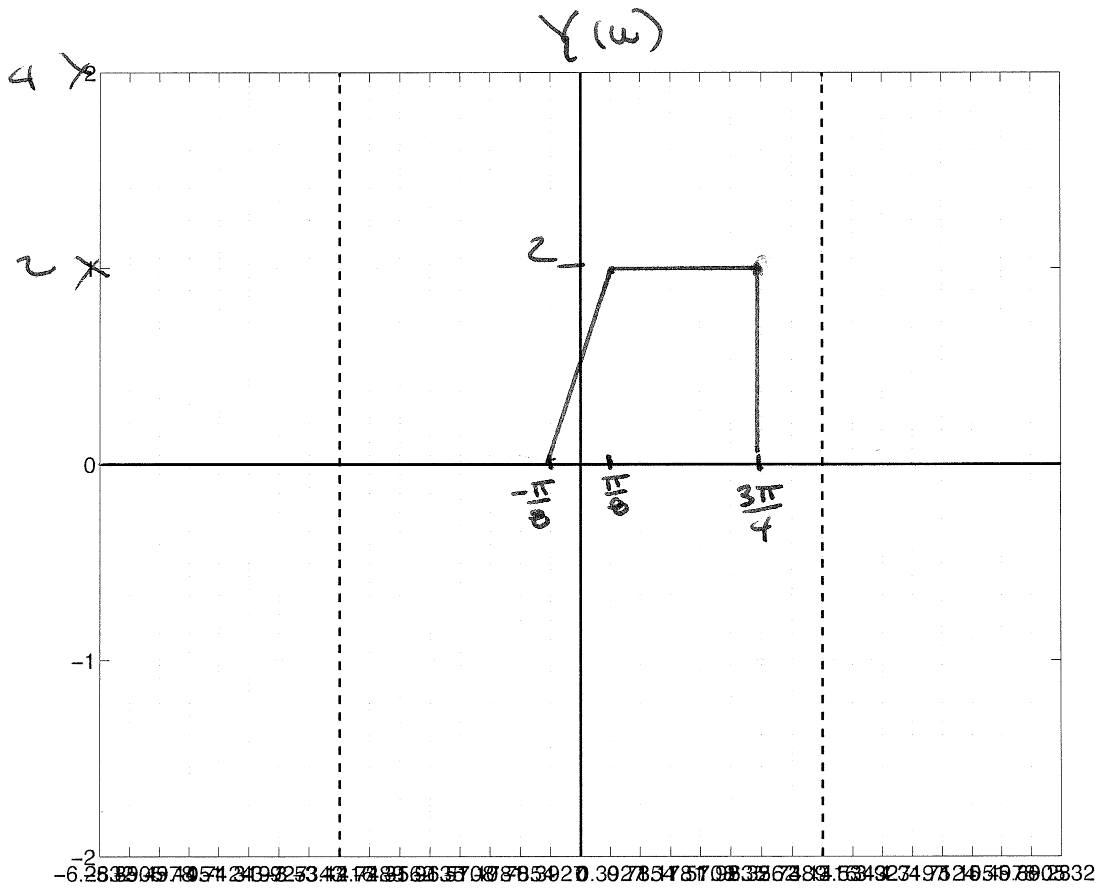
Plot your answer to Problem 3 (a) here. Show work above.



Plot your answer to Problem 3 (b) here. Show work above.



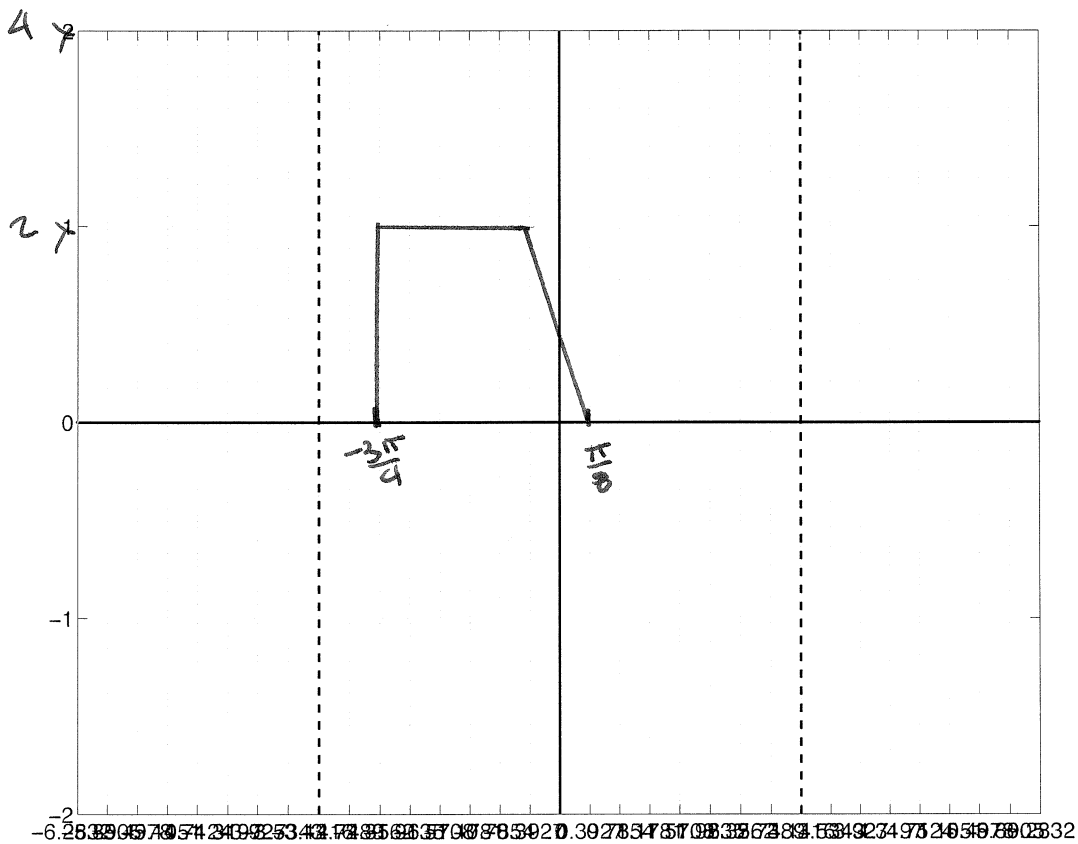
Plot your answer to Problem 3 (c) here. Show work above.



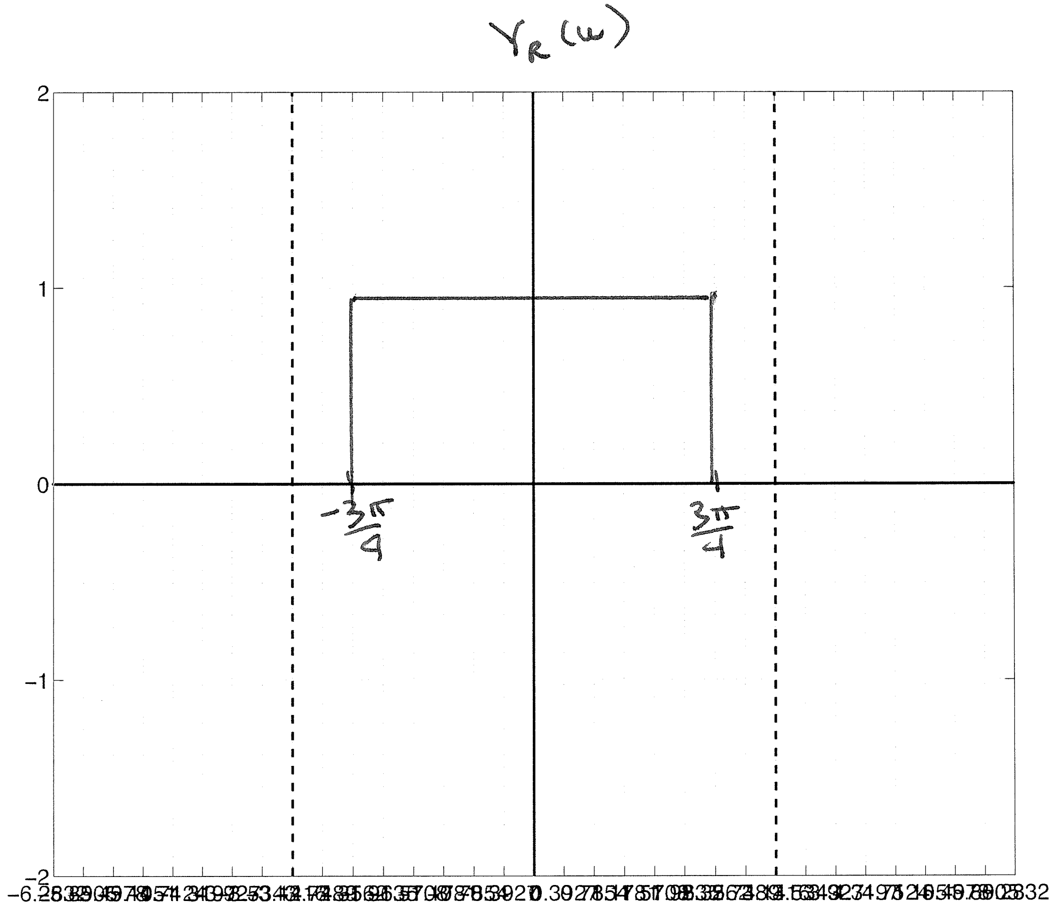
$$\frac{1}{2}\left\{y[n] + y^*[n]\right\} \xleftrightarrow{DTFT} \frac{1}{2}Y(\omega) + \frac{1}{2}Y^*(-\omega$$

$$y^*[n] \xleftrightarrow{DTFT} Y^*(-\omega)$$

Plot your answer to Problem 3 (d) here. Show work above.



Plot your answer to Problem 3 (e) here. Show work above.



(f) Yes!
same