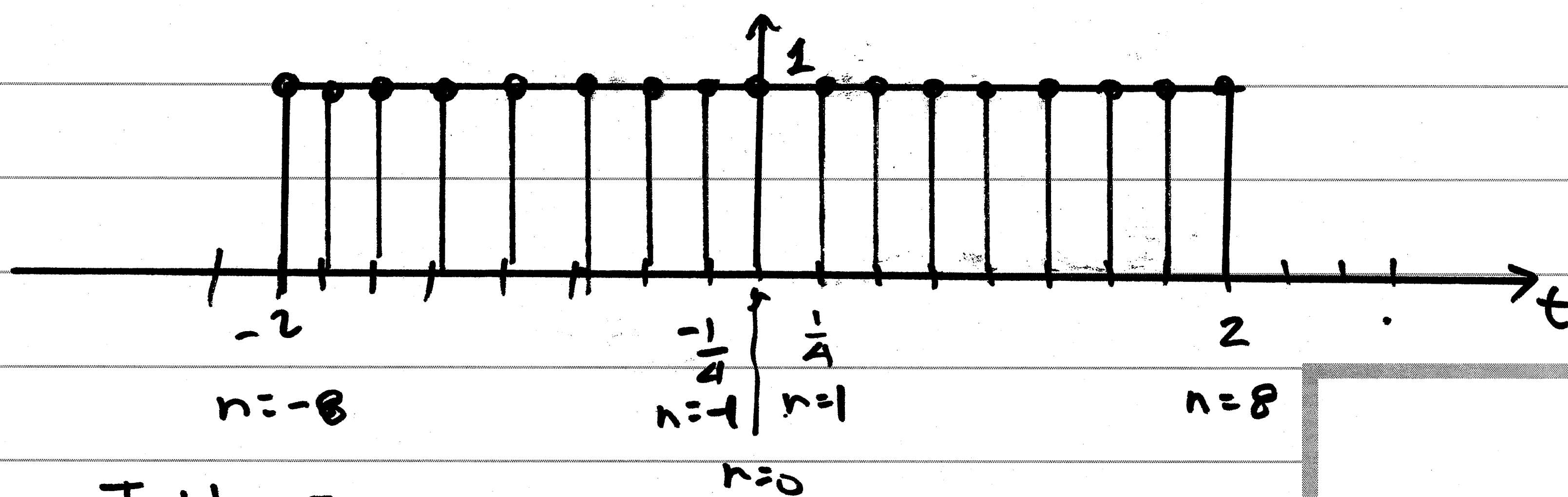
2a $\quad x_a(t) = \underbrace{\text{rect}\left(\frac{t}{4}\right)}_{\text{turned on for 4 secs}} \qquad F_s = \frac{1}{T_s} = \frac{1}{.25} = 4\ \frac{\text{samples}}{\text{sec}}$

height: 1 ~~V~~ volt

Number of nonzero samples: $\quad 4 \text{ secs} \times 4\ \frac{\text{samples}}{\text{sec}} = 16$ samples

Since "turned on" at both ends = 17 samples



From Table 5.2

$$X(\omega) = \frac{\sin\left(\left(8+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)} = \frac{\sin\left(\frac{17}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

2b $\quad x_a(t) = t\, x_a^{(2a)}(t)$

$$x_a[n] = nT_s\, x_a^{(2a)}(nT_s) = T_s\, n\, x^{(2a)}[n]$$

Table 5.1 $\quad n\,x[n] \xleftrightarrow{\text{DTFT}} j\dfrac{d}{d\omega}X(\omega)$ $\qquad T_s = .25$

$F_s = 4$

Thus: $X_a(\omega) = (.25)\, j \dfrac{d}{d\omega}\left\{\dfrac{\sin\left(\frac{17}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right\}$

Sidenote: if $x[n] = x_a(nT_s) = x_a\left(\frac{n}{F_s}\right)$ $\quad F_s = \frac{1}{T_s}$

then: $X(\omega) = X_s(F_s \omega)$

where: $X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$

OR: $X(\omega) = \sum_{k=-\infty}^{\infty} X_d(\omega - k2\pi)$ where: $X_d(\omega) = F_s X_a(F_s \omega)$

Note: if $\omega_s > 2\omega_M$, no aliasing, then:

$$X(\omega) = F_s X_a(F_s \omega) \quad \text{for } -\pi < \omega < \pi$$

For this problem: $x_a(t) = \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} X_a(\omega) = \dfrac{\sin\left(\frac{T}{2}\omega\right)}{\left(\frac{1}{2}\omega\right)}$

There is $\underline{\underline{\text{no}}}$ max frequency for which $X_a(\omega) = 0$ for $\omega > \omega_M$

Thus, there is aliasing!

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4 (a) $x(t) = e^{-2|t|}$ $\quad y(t) = x(t-5)$

$$Y(\omega) = e^{-j5\omega} X(\omega)$$

$$|Y(\omega)|^2 = Y(\omega)\, Y^*(\omega) = |X(\omega)|^2 e^{-j5\omega} e^{+j5\omega}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} x^2(t)\, dt$$

$$= 2\int_0^{\infty} \left(e^{-2t}\right)^2 dt = 2\int_0^{\infty} e^{-4t} dt = \frac{2}{-4} e^{-4t}\Big]_0^{\infty}$$

since $x(-t) = x(t)$

and $|t|$ for $t>0 = t$

$$= -\frac{1}{2}\{0-1\} = \underbrace{\frac{1}{2}}_{\text{ans}}$$

4b

$$y(t) = e^{j5t} x(t) \qquad |y(t)|^2 = x^2(t)\, e^{j5t} e^{-j5t} = x^2(t)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{\infty}^{\infty} x^2(t)\, dt = \underbrace{\frac{1}{2}}_{\text{ans}}$$

note: $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(\omega-5)|^2 d\omega$

$\underline{4c}$ First, note: $y(t) = x(at)$ a = real-valued >0

$$E_y = \int_{-\infty}^{\infty} y^2(t)\,dt = \int_{-\infty}^{\infty} x^2(at)\,dt$$

change of variables:

$\tau = at$ $d\tau = a\,dt$

$t = \frac{\tau}{a}$ $dt = \frac{1}{a}\,d\tau$

limits still $-\infty < \tau < \infty$

$$= \int_{-\infty}^{\infty} x^2(\tau)\frac{1}{a}\,d\tau = \frac{1}{a}E_x$$

$$E_y = \frac{1}{a}E_x$$

So, answer is: $\frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

Check: $$\int_{-\infty}^{\infty} y^2(t)\,dt = \int_{-\infty}^{\infty}\left(e^{-2|2t|}\right)^2 dt = 2\int_0^{\infty} e^{-8t}\,dt$$

$$= 2\,\frac{1}{-8}\,e^{-8t}\Big]_0^{\infty} = -\frac{1}{4}(0-1) = \frac{1}{4}$$ ✓

4d $y(t) = t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = j\frac{d}{d\omega}X(\omega)$

$$\int_{-\infty}^{\infty} y(t)\,dt = Y(\omega)\Big|_{\omega=0} = j\frac{d}{d\omega}\{X(\omega)\}\Big|_{\omega=0}$$

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$$ in this case, $a=2$

$$\frac{d}{d\omega}\left\{\frac{2a}{\omega^2+a^2}\right\}\Bigg|_{\omega=0} = 2a\frac{(-2\omega)}{(\omega^2+a^2)^2}\Bigg|_{\omega=0} = \underbrace{0}_{\text{ans}}$$

4e $y(t) = \frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = j\omega X(\omega)$

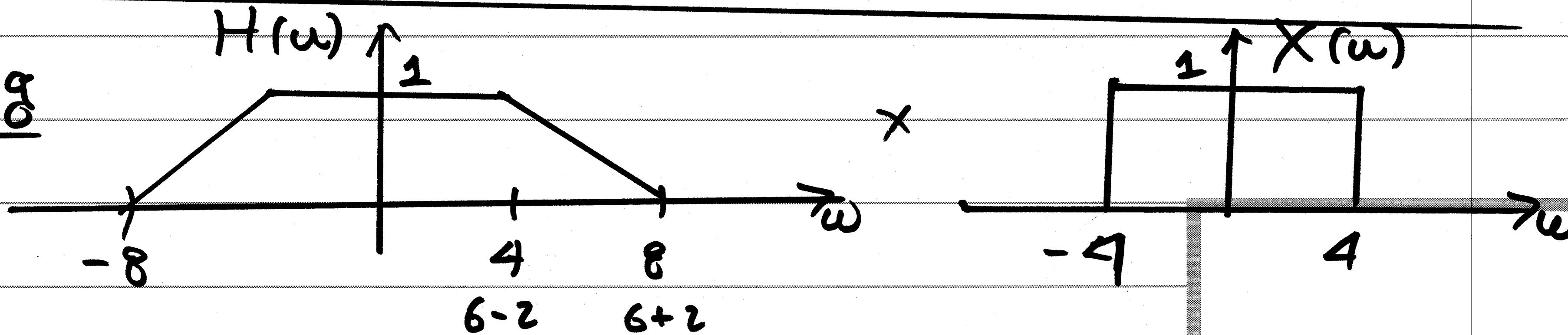
$$\int_{-\infty}^{\infty} y(t)\,dt = Y(\omega)\Big|_{\omega=0} = j(0)X(0) = \underbrace{0}_{\text{ans}}$$

4f $y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$

$$Y(\omega) = \frac{2(2)}{\omega^2 + 2^2} \; \frac{2(3)}{\omega^2 + 3^2} = \frac{24}{(\omega^2+4)(\omega^2+9)}$$

$$\int_{-\infty}^{\infty} y(t)\,dt = Y(\omega)\Big|_{\omega=0} = \frac{24}{(4)(9)} = \frac{24}{36} = \frac{2}{3}$$

4g



$$Y(\omega) = H(\omega)X(\omega) = X(\omega)$$

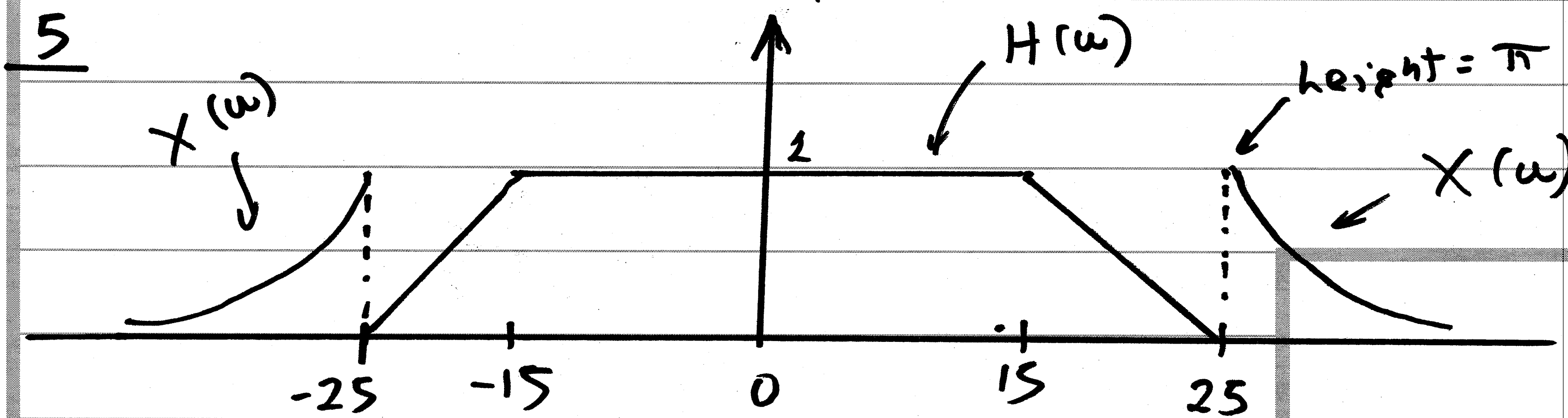
$$y(t) = x(t) = \frac{\sin(4t)}{\pi t}$$

4a $E_Y = \int_{-\infty}^{\infty} y^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$

in this case: $Y(\omega) = X(\omega)$

Thus: $E_Y = \frac{1}{2\pi} \underbrace{(8)}_{\text{width}} \underbrace{(1)^2}_{\text{height squared}} = \frac{8}{2\pi} = \frac{4}{\pi}$

5



Since $Y(\omega) = H(\omega)X(\omega) = 0 \Rightarrow y(t) = 0$

Note:

$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2} \Rightarrow \frac{2t}{t^2+a^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$

Thus: $\frac{1}{t^2+1} \overset{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}$

$a=1$