SOLUTION

Name: ECE301 Signals and Systems

Final Exam
Thursday, May 6, 2016

Cover Sheet

Write your name on this page and every page to be safe.
Test Duration: 120 minutes.
Coverage: Comprehensive
Open Book but Closed Notes. Four two-sided handwritten sheets.
Calculators NOT allowed.
This test contains **four** problems, each with multiple parts.
You have to draw your own plots.
You must show all work for each problem to receive full credit.
Good luck! It was great having you in class this semester!
Have a great Summer!

Problem 1. Using the Z-Transform, determine the impulse response, $h[n]$, of the LTI system defined by the Difference Equation below. There is no partial fraction expansion needed as there is only a single pole.

$$y[n] = \frac{1}{2}y[n-1] + x[n] - \frac{1}{16}x[n-4]$$

You must use the Z-Transform to solve this problem, i.e., use basic Z-Transform properties and a basic Z-Transform pair to ultimately determine the impulse response $h[n]$.

$$Y(z) = \frac{1}{2}z^{-1}Y(z) + X(z) - \frac{1}{16}z^{-4}X(z) \qquad x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z)$$

$$Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)\left(1 - \frac{1}{16}z^{-4}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{16}z^{-4}}{1 - \frac{1}{2}z^{-1}} = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{1}{16}z^{-4}}{1 - \frac{1}{2}z^{-1}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$a = \frac{1}{2} \qquad\qquad a = \frac{1}{2} \quad (D = 4)$$

$$\frac{z}{z-a} \overset{Z}{\longleftrightarrow} a^n u[n]$$

$$h[n] = a^n u[n] - a^D a^{n-D} u[n-D]$$

$$= \left(\frac{1}{2}\right)^n u[n] - \cancel{\left(\frac{1}{2}\right)^4}\left(\frac{1}{2}\right)^{n-4} u[n-4]$$

$$\boxed{h[n] = \left(\frac{1}{2}\right)^n \left\{u[n] - u[n-4]\right]}$$

Problem 2.

The rectangular pulse $x_{in}(t) = 4\{u(t) - u(t-3)\}$ of duration 3 sec with amplitude 4 is input to the following integrator

$$x_a(t) = \int_{t-1}^{t} x_{in}(\tau)d\tau$$

The output $x_a(t)$ is sampled every $T_s = 1/4$ seconds to form $x[n] = x_a(nT_s)$. The sampling rate is $f_s = 4$ samples/sec. Show work. Clearly label and write your final answer in the space provided on the next few pages.

(a) You can simply write the numbers that comprise $x[n]$ in sequence form (indicate with an arrow where the $n = 0$ value is) OR do a stem plot of the DT signal $x[n]$.

(b) The Discrete-Time (DT) signal $x[n]$, created as described above, is input to the DT system described by the difference equation below:

$$y[n] = -x[n] + 2x[n-1] - x[n-2]$$

(i) First, determine the impulse response $h[n]$ for this system.

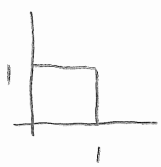
(ii) Determine the output $y[n]$ by convolving the input $x[n]$ defined above with the impulse response $h[n]$. Show all work in the space provided. You can simply write the numbers that comprise $y[n]$ in sequence form (indicate with an arrow where the $n = 0$ value is) OR do a stem plot of the DT signal $y[n]$.

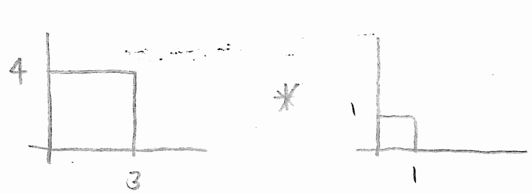
a)

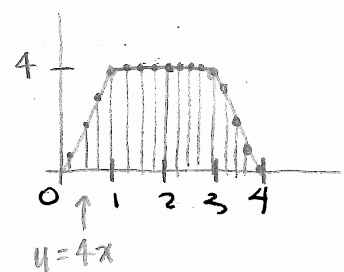
$x_a(t) = x_{in}(t) * h_a(t)$ $\quad h_a(t) = \int_{t-1}^{t} \delta(\tau)\,d\tau \rightarrow$ area = 1 for $t-1 < \tau < t$, $0 < \tau < 1$

$h_a(t) = u(t) - u(t-1)$

$x_a(t) = 4\{u(t) - u(t-3)\} * \{u(t) - u(t-1)\}$







$a = 1 \quad b = 3$

$x[n] = x_a\left(\frac{n}{4}\right) = \{\underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 2, 1\}$

Show your work and plots for Problem 2 here.

b) $y[n] = -x[n] + 2x[n-1] - x[n-2]$

$h[n] = -\delta[n] + 2\delta[n-1] - \delta[n-2]$

$$\boxed{h[n] = \{\underset{\substack{\uparrow \\ n=0}}{-1}, 2, -1\}}$$

$y[n] = x[n] * h[n]$

	n=0																		
(-1)	0	-1	-2	-3	-4	-4	-4	-4	-4	4	4	4	4	-3	-2	-1	0	0	0
(2)	0	0	~~1~~ 2	~~2~~ 4	~~3~~ 6	~~4~~ 8	~~4~~ 8							~~4~~ 8	~~3~~ 6	~~2~~ 4	~~1~~ 2	0	0
(-1)	0	0	0	-1	-2	-3	-4	.	.	.	.	.	.	-4	-4	-3	-2	-1	0
	0	-1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	-1	0

$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{0}, -1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1\}$$

Problem 3 (a). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 30$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 30$ rads/sec. This signal is sampled at a rate $\omega_s = 80$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{80}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{ \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n T_s} + \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n T_s} + \frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n T_s} \right\} \quad \text{where: } T_s = \frac{2\pi}{80}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where: } T_s = \frac{2\pi}{80} \text{ and } h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \frac{\sin(40t)}{\pi t}$$

Problem 3 (b). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 30$ rads/sec. This signal is sampled at the same rate $\omega_s = 80$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where: } T_s = \frac{2\pi}{80} \text{ and } h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(30t)}{\pi t} + \frac{\sin(50t)}{\pi t} \right\}$$

Problem 3 (c). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 30$ rads/sec. This signal is sampled at the same rate $\omega_s = 80$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where: } T_s = \frac{2\pi}{80} \text{ and } h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(35t)}{\pi t} + \frac{\sin(45t)}{\pi t} \right\}$$

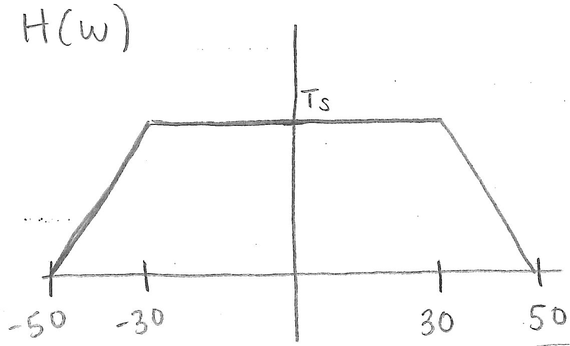
Show all your work for Prob. 3, parts (a)-(b)-(c) on next page.

$w_M = 30 \quad w_s = 80 \quad T_s = \frac{2\pi}{80}$

Show your work for Prob. 3, parts (a)-(b)-(c) below.

a) $h(t) = T_s \frac{\pi}{10} \cdot \frac{\sin(10t)}{\pi t} \frac{\sin(40t)}{\pi t}$ $\qquad w_1 = 10$, $w_2 = 40$

$H(w)$



✓ • $w_s > 2w_M$

✓ • $H(w) = 0$ for $|w| > 50$

✓ • $H(w) = T_s$ for $|w| < w_M$

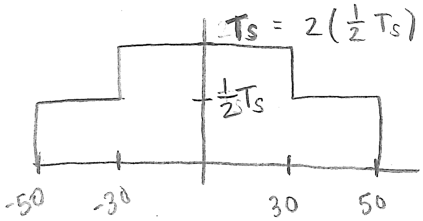
$\rightarrow$ so $x_r(t) = x_a(t) = x[n]\Big|_{n=\frac{t}{T_s}}$

$n = \frac{40}{\pi} t$

$$\boxed{x_r(t) = \frac{\sin(10t)}{\pi t} + \frac{\sin(20t)}{\pi t} + \frac{\sin(30t)}{\pi t}}$$

b) $h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(30t)}{\pi t} + \frac{\sin(50t)}{\pi t} \right\}$

$H(w) =$



✓ • $w_s > 2w_M$

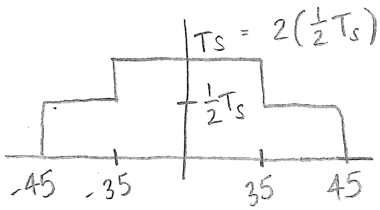
✓ • $H(w) = 0$ for $|w| > 50$ $(80-30)$

✓ • $H(w) = T_s$ for $|w| < 30$

$\rightarrow x_r(t) = x_a(t)$

c) $h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(35t)}{\pi t} + \frac{\sin(45t)}{\pi t} \right\}$

$H(w) =$



✓ • $w_s > 2w_M$

✓ • $H(w) = 0$ for $|w| > 50$ $(80-30)$

✓ • $H(w) = T_s$ for $|w| < 30$

$\rightarrow x_r(t) = x_a(t)$

Problem 3 (d). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 30$ rads/sec. This signal is sampled at a rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{50}$ sec, yielding the following discrete-time sequence:

$$x[n] = x_a(nT_s) = \left\{ \overset{10}{\frac{\sin\left(\frac{2\pi}{5}n\right)}{\pi n T_s}} + \overset{20}{\frac{\sin\left(\frac{4\pi}{5}n\right)}{\pi n T_s}} + \overset{30}{\frac{\sin\left(\frac{6\pi}{5}n\right)}{\pi n T_s}} \right\} \qquad \text{where: } T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(25t)}{\pi t}$$

Problem 3 (e). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = \overset{3}{\cancel{4}}0$ rads/sec. This signal is sampled at the same rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{50}$ sec, but offset by $Ts/2$ from $t = 0$, i.e., at the points $t = nT_s + T_s/2$. This yields the Discrete-Time $x[n]$ signal below:

$$x[n] = x_a(nT_s) = \left\{ \frac{\sin\left(\frac{2\pi}{5}(n+0.5)\right)}{\pi(n+0.5)T_s} + \frac{\sin\left(\frac{4\pi}{5}(n+0.5)\right)}{\pi(n+0.5)T_s} + \frac{\sin\left(\frac{6\pi}{5}(n+0.5)\right)}{\pi(n+0.5)T_s} \right\} \qquad \text{where: } T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. *Hint:* before you do a lot of work, look at the interpolating lowpass filter being used below.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n]h(t - (n+0.5)T_s) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(25t)}{\pi t}$$

Show your work for Prob. 3, parts (d)-(e) on next page.

Show all your work for Prob. 3, parts (d)-(e) on this page.

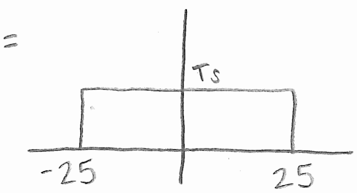
d) $\omega_M = 30$ $\omega_s = 50$ $T_s = \frac{2\pi}{50}$

X • $\omega_s > 2\omega_M$

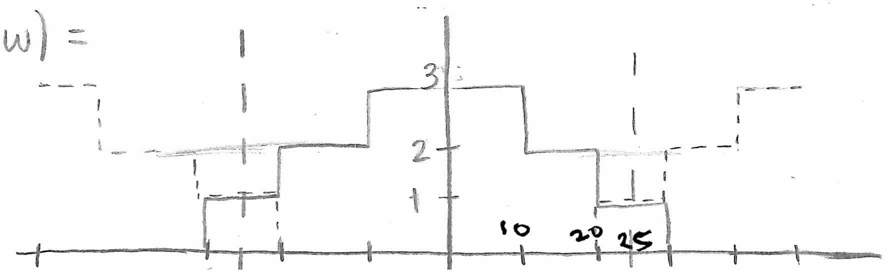
X • $H(\omega) = 0$ for $|\omega| > 20$

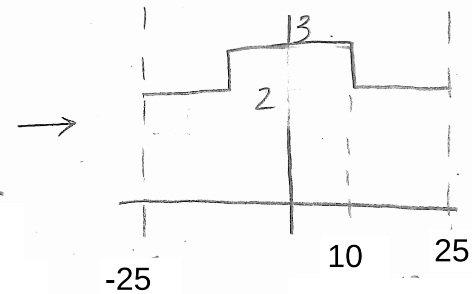
X • $H(\omega) = T_s$ for $|\omega| < 30$

$h(t) = T_s \frac{\sin(25t)}{\pi t}$ $H(\omega) =$



$X(\omega) =$





$\Rightarrow \frac{\omega_s}{2} = 25$

$X_r(\omega) = 2\,\mathrm{rect}\left(\frac{\omega}{50}\right) + \mathrm{rect}\left(\frac{\omega}{20}\right)$

$$\boxed{x_r(t) = \frac{2\sin(25t)}{\pi t} + \frac{\sin(10t)}{\pi t}}$$

Note: $\omega_d = \omega_a T_s \Rightarrow \omega_a = \frac{\omega_d}{T_s}$ $\frac{1}{T_s} = \frac{50}{2\pi} = \frac{25}{\pi}$

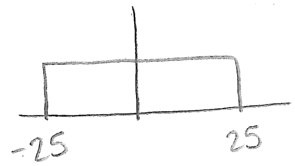
Show all your work for Prob. 3, parts (d)-(e) on this page.

e) $w_m = 30 \quad w_s = 50$ $\quad$ * $\frac{T_s}{2}$ offset * $\rightarrow$ multiplied by $e^{-j\pi k}$

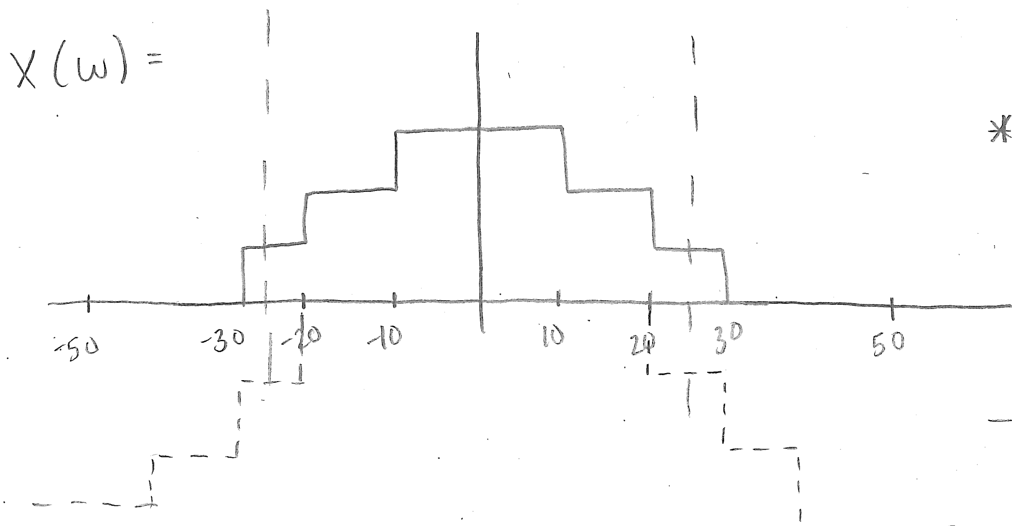
$k = 0 \rightarrow 1$

$k = \pm 1 \rightarrow -1$

$H(w) =$

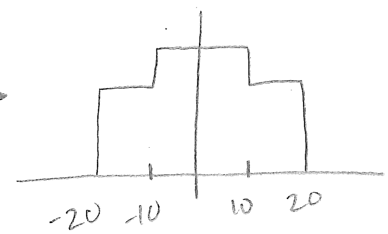


$X(w) =$



* $X(w) = 0$ for $20 < w < 30$

$\rightarrow$



$$\boxed{x_r(t) = 2\frac{\sin(20t)}{\pi t} + \frac{\sin(10t)}{\pi t}}$$

Problem 4. For each part: show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

Problem 4 (a). Determine the Fourier Transform $X(\omega)$ of the signal defined below

$$\begin{aligned} x(t) &= 1+\cos(\pi t), \quad \text{for } |t|<1 &(1)\\ &= 0, \quad \text{for } |t|>1 &(2)\\ & &(3) \end{aligned}$$

You don't need to plot $X(\omega)$, just determine a closed-form expression for $X(\omega)$. Note that $x(t)$ is an even-symmetric function of time so that its Fourier Transform $X(\omega)$ is purely real-valued. There should be no $j=\sqrt{-1}$ anywhere in your final answer.

a) $x(t) = \left(1+\cos(\pi t)\right)\text{rect}\left(\frac{t}{2}\right)$

$= \text{rect}\left(\frac{t}{2}\right) + \cos(\pi t)\,\text{rect}\left(\frac{t}{2}\right)$

$X(\omega) = \dfrac{2\sin(\omega)}{\omega} + \dfrac{\sin(\omega-\pi)}{\omega-\pi} + \dfrac{\sin(\omega+\pi)}{\omega+\pi}$

Problem 4 (b). Compute the energy $E = \int_{-\infty}^{\infty} x^2(t)dt$ of the signal below.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$x(t) = \frac{4}{t^2+4} \quad (4)$$

Show all work and clearly indicate your final answer.

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a+\omega^2} \qquad X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

$$\frac{4}{4+t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-2|-\omega|} \longrightarrow \text{even, so } \omega = -\omega$$

$$E = \frac{1}{2\pi}\int_0^{\infty} (4\pi^2) e^{-4\omega} d\omega$$

$$\frac{Ex}{2} = 2\pi \int_0^{\infty} e^{-4\omega} d\omega = 2\pi \left[-\frac{1}{4}e^{-4\omega}\right]_0^{\infty}$$

$$= 2\pi\left[0 - \left(-\frac{1}{4}\right)\right]$$

$$E_x = \boxed{\pi}$$

Problem 4 (c). You are given that the Fourier Transform of a Gaussian pulse $x(t) = e^{\frac{-t^2}{2}}$ is $X(\omega) = \sqrt{2\pi}\, e^{\frac{-\omega^2}{2}}$. That is,

$$x(t) = e^{\frac{-t^2}{2}} \longleftrightarrow X(\omega) = \alpha\, e^{\frac{-\omega^2}{2}}$$

This says that, except for the scalar $\alpha = \sqrt{2\pi}$, the functional form of the Fourier Transform is the same as in the time-domain function. Is the same true for the $y(t) = te^{\frac{-t^2}{2}}$? That is, is the following true:

$$y(t) = t\, e^{\frac{-t^2}{2}} \overset{??}{\longleftrightarrow} X(\omega) = \beta\, \omega e^{\frac{-\omega^2}{2}}$$

Determine if the above is true and, if so, determine the numerical value of β. Show all work and clearly indicate which Fourier Transform properties you are using.

property: $t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j \frac{d}{d\omega} X(\omega)$

$$t\left(e^{-\frac{t^2}{2}}\right) \overset{\mathcal{F}}{\longleftrightarrow} \alpha j\left(-\frac{\cancel{2}\omega}{\cancel{2}} e^{\frac{-\omega^2}{2}}\right)$$

$$\boxed{\beta = -j\alpha} = -j\sqrt{2\pi}$$

Problem 4 (d). Given the real-valued signal

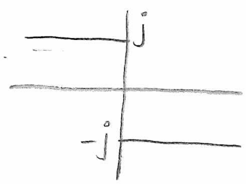
$$x(t) = t\, e^{\frac{-t^2}{2}}$$

a complex-valued signal is formed as

$$y(t) = x(t) + j\hat{x}(t) \quad \text{where:} \quad \hat{x}(t) = x(t) * \frac{1}{\pi t}$$

Determine and plot $Y(\omega)$, the Fourier Transform, of $y(t)$. Show work and sketch the magnitude $|Y(\omega)|$. Explicitly indicate the frequency at which $Y(\omega)$ achieves its maximum value.

$$\frac{1}{\pi t} \xleftrightarrow{\mathcal{F}} j\,\mathrm{sgn}(\omega) \rightarrow$$



$$Y(\omega) = X(\omega) + j\hat{X}(\omega)$$

$$= X(\omega) + jX(\omega)\,j\,\mathrm{sgn}(\omega)$$

$$= X(\omega)(1 - \mathrm{sgn}(\omega))$$

$\hookrightarrow$ 2 for $\omega > 0$
0 for $\omega < 0$ (neg. freq. content removed)

$$X(\omega) = -j\omega e^{-\omega^2/2}$$

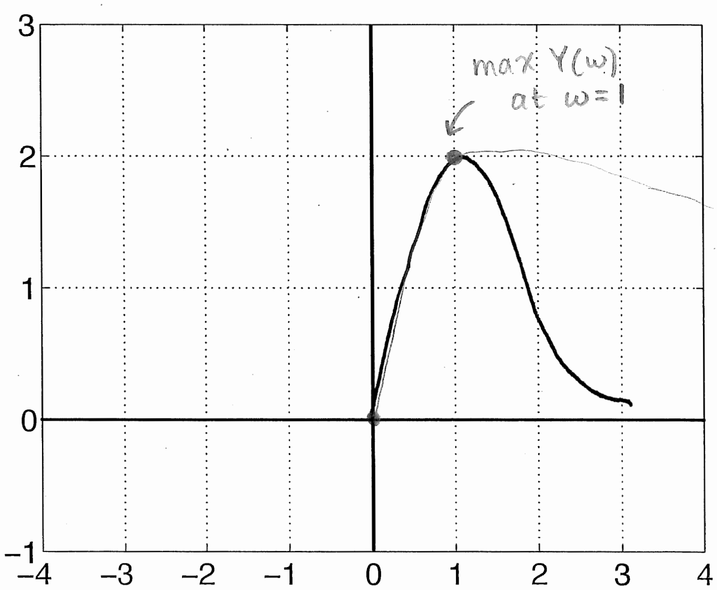
$$Y(\omega) = -2j\omega e^{-\frac{\omega^2}{2}} u(\omega)$$

$$\frac{d}{d\omega} = -2je^{-\frac{\omega^2}{2}} + 2j\omega e^{-\frac{\omega^2}{2}} = 0$$

$$\frac{2j\omega e^{-\frac{\omega^2}{2}}}{2je^{-\omega^2/2}} = \frac{2je^{-\frac{\omega^2}{2}}}{2je^{-\omega^2/2}}$$

$$\boxed{\omega = 1}$$

$$|Y(0)| = 0$$



$$|Y(1)| = 2e^{-\frac{1}{2}}$$

Problem 4 (e). Determine the Fourier Transform $X(\omega)$ of the signal

$$x(t) = \frac{1}{\pi} ln(t)$$

where $ln(t)$ is the natural logarithm of t such that

$$\frac{d}{dt}x(t) = \frac{1}{\pi t}$$

You don't need to plot $X(\omega)$. You just need to determine a closed-form expression for $X(\omega)$.

$$\frac{d}{dt}x(t) \xrightarrow{F} j\omega X(\omega) = -j\,\text{sgn}(\omega) \xleftarrow{F} \frac{1}{\pi t}$$

$$X(\omega) = \frac{-j\,\text{sgn}(\omega)}{j\omega}$$

$$\boxed{X(\omega) = -\frac{\text{sgn}(\omega)}{\omega}}$$

Problem 4 (f). Determine the value of the integral below, where a is a real-valued constant with $a > 0$. Does the answer depend on the constant a? Show all work and explain your answer.

$$\int_{-\infty}^{\infty} \frac{2a}{\omega^2 + a^2} d\omega = ??$$

$$\int_{-\infty}^{\infty} X(\omega)\, d\omega = 2\pi x(0)$$

$$\frac{2a}{\omega^2+a^2} \xrightarrow{F} e^{-a|t|}$$

$$2\pi e^{-a|0|} = \boxed{2\pi}$$

No, answer does not depend on the value of a, as it will be multiplied by 0 regardless. For any a, integral $= 2\pi$