SOLUTION

Name:

Final Exam

ECE301 Signals and Systems **Tuesday, May 6, 2014**

Cover Sheet

Write your name on this page and every page to be safe.
Test Duration: 120 minutes.
Coverage: Comprehensive
Open Book but Closed Notes. Three two-sided handwritten sheets.
Calculators NOT allowed.
This test contains **four** problems, each with multiple parts.
You have to draw your own plots.
You must show all work for each problem to receive full credit.
Good luck! It was great having you in class this semester!
Have a great Summer!

Problem 1.

(a) For the signal $s(t)$ below, plot the Fourier Transform $S(\omega)$ which is purely real-valued:

$$s(t) = \frac{\pi}{2.5}\left\{\frac{\sin(2.5t)}{\pi t}\right\}^2 2\,\cos(5t) \tag{1}$$

(b) An AM signal $r(t)$ is formed from $s(t)$ above as prescribed below, where $k = 0.1$ and $\phi = \frac{\pi}{4}$.

$$r(t) = [1 + k\, s(t)] \cos(35t + \phi) \tag{2}$$

The signal $r(t)$ above is applied to a square-law device, followed by amplification, to form $x(t) = 10r^2(t)$

$$\begin{aligned} x(t) &= 10\,\{[1 + k\, s(t)] \cos(35t + \phi)\}^2 \\ &= 10\,[1 + k\, s(t)]^2 \cos^2(35t + \phi) \end{aligned} \tag{3}$$

Plot the magnitude of the Fourier Transform of $x(t)$ denoted $|X(\omega)|$, IGNORE as negligible any term which is scaled by $k^2 = 0.01$. Recall $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$.

(c) Consider an LTI system with impulse response

$$h(t) = \left\{\frac{\pi}{15}\,\frac{\sin(15t)}{\pi t}\,\frac{\sin(25t)}{\pi t}\right\} \tag{4}$$

Determine and plot the frequency response, $H(\omega)$, the Fourier Transform of $h(t)$.

(d) For the LTI system with this impulse response, determine the output $y(t)$ for the input $x(t)$ above. Plot $|Y(\omega)|$. Again, ignore any term which is amplitude-scaled by $k^2 = 0.01$ as negligible.

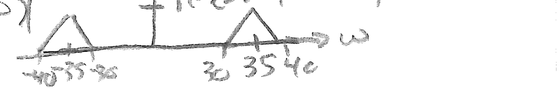
$$y(t) = x(t) * h(t)$$

(e) Would your answer to part (d) change if the phase shift in the sinewave was changed from $\phi = \frac{\pi}{4}$ to $\phi = \frac{\pi}{2}$? Does the value of the phase shift ϕ have any impact on the recovery of $s(t)$ via the method above? Explain your answer.

(f) In the scheme above, a strong DC component is added before the signal is multiplied by the cosine wave. Determine the final output $y(t)$ via the same sequence of steps with the DC component removed: (1) Form $r(t) = 2s(t)\cos(35t + \phi)$, (2) Multiply by cosine at the same frequency $x(t) = r(t)\cos(35t)$, and (3) $y(t) = x(t) * h(t)$. Recall: $2\cos(\theta)\cos(\phi) = \cos(\theta + \phi) + \cos(\theta - \phi)$. How does the output $y(t)$ depend on the value of ϕ? What is the output when the phase is $\phi = \pi/2$?

① $r(t) = 2s(t)\cos(35t + \phi)$

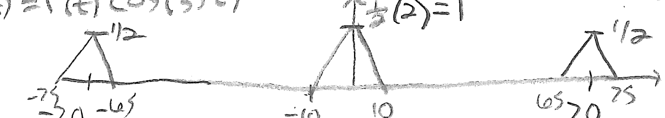
$|R(\omega)|$ $\mathcal{F}\{2(\frac{1}{2})\} = |1\, e^{j\omega\phi/35}| = 1$



② $x(t) = r(t)\cos(35t)$

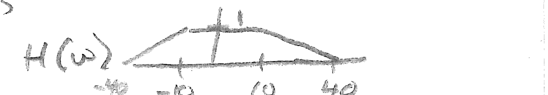
$|X(\omega)|$

$\frac{1}{2}(2) = 1$



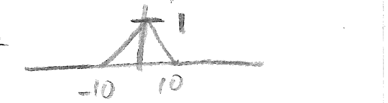
③ $y(t) = x(t) * h(t) \rightarrow Y(\omega) = X(\omega)H(\omega)$

$H(\omega)$



→ Again, $y(t)$ does not depend on phase when looking @ magnitudes same reasoning as e.

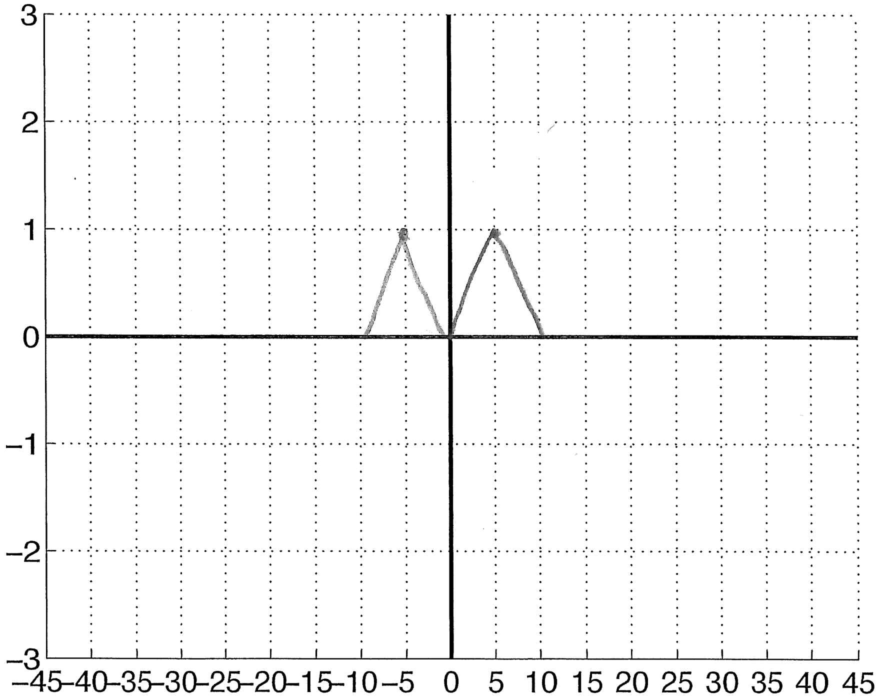
$\therefore |Y(\omega)| =$



$\boxed{\therefore y(t) = s(t)}$

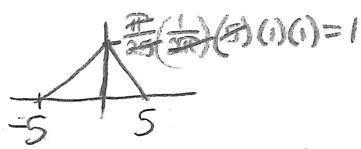
when $\phi = \pi/2$ $\boxed{y(t) = s(t)}$

Show all work and plot for Problem 1 (a.).

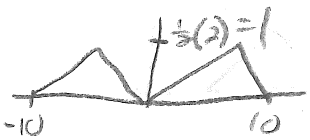


$$s(t) = \frac{\pi}{2.5}\left\{\frac{\sin(2.5t)}{\pi t}\right\}^2 2\cos(5t)$$

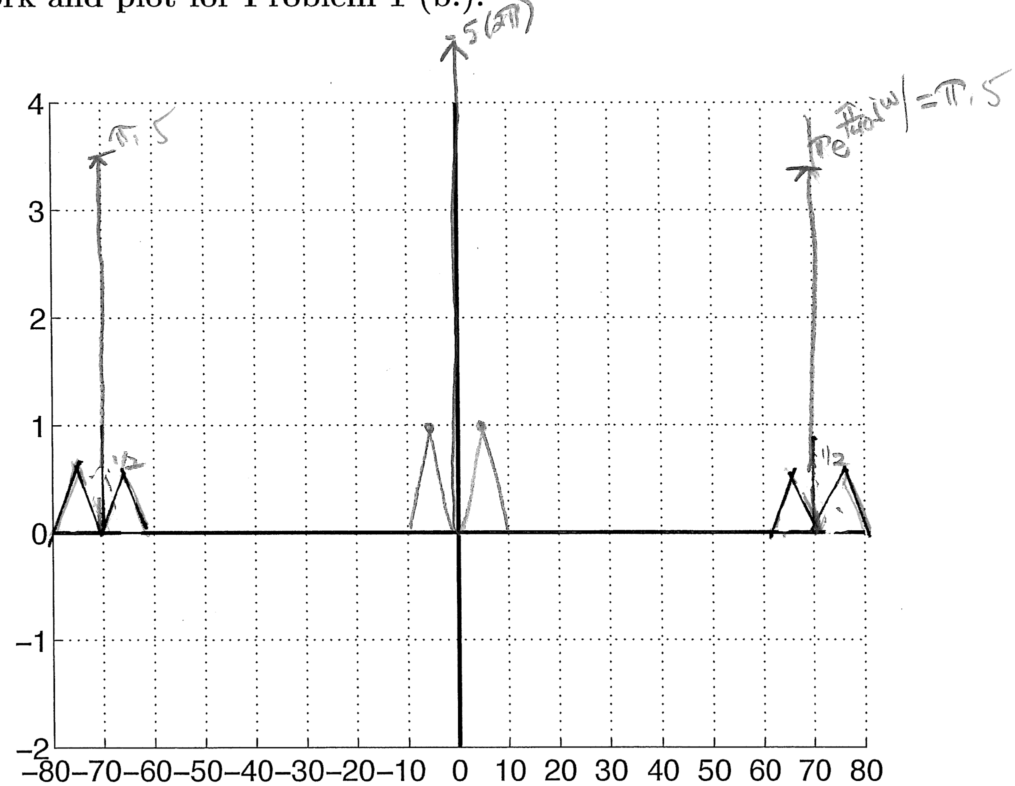
$S(\omega) \Rightarrow$



$\rightarrow$



Show all work and plot for Problem 1 (b.).



$$x(t) = 10[1+ks(t)]^2\cos^2(35t+\phi) \qquad k=0.1 \qquad \phi=\frac{\pi}{4}$$

$$= 10[1+2ks(t)+\underbrace{k^2s^2(t)}_{\text{Ignore}}]\cos^2(35t+\phi)$$

$$= [10+20ks(t)]\cos^2(35t+\phi)$$

$$= [10+2s(t)]\left(\frac{1}{2}+\frac{1}{2}\cos(2(35t+\phi))\right)$$

$$= 5+5\cos(2(35t+\phi))+s(t)+s(t)\cos(2(35t+\phi))$$

$$= 5+5\cos\left(70t+2\left(\frac{\pi}{4}\right)\right)+s(t)+s(t)\cos\left(70t+2\left(\frac{\pi}{4}\right)\right)$$

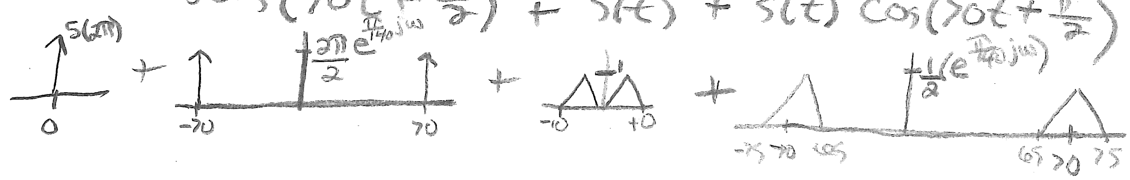
$$= 5+5\cos\left(70t+\frac{\pi}{2}\right)+s(t)+s(t)\cos\left(70t+\frac{\pi}{2}\right)$$

$20k$

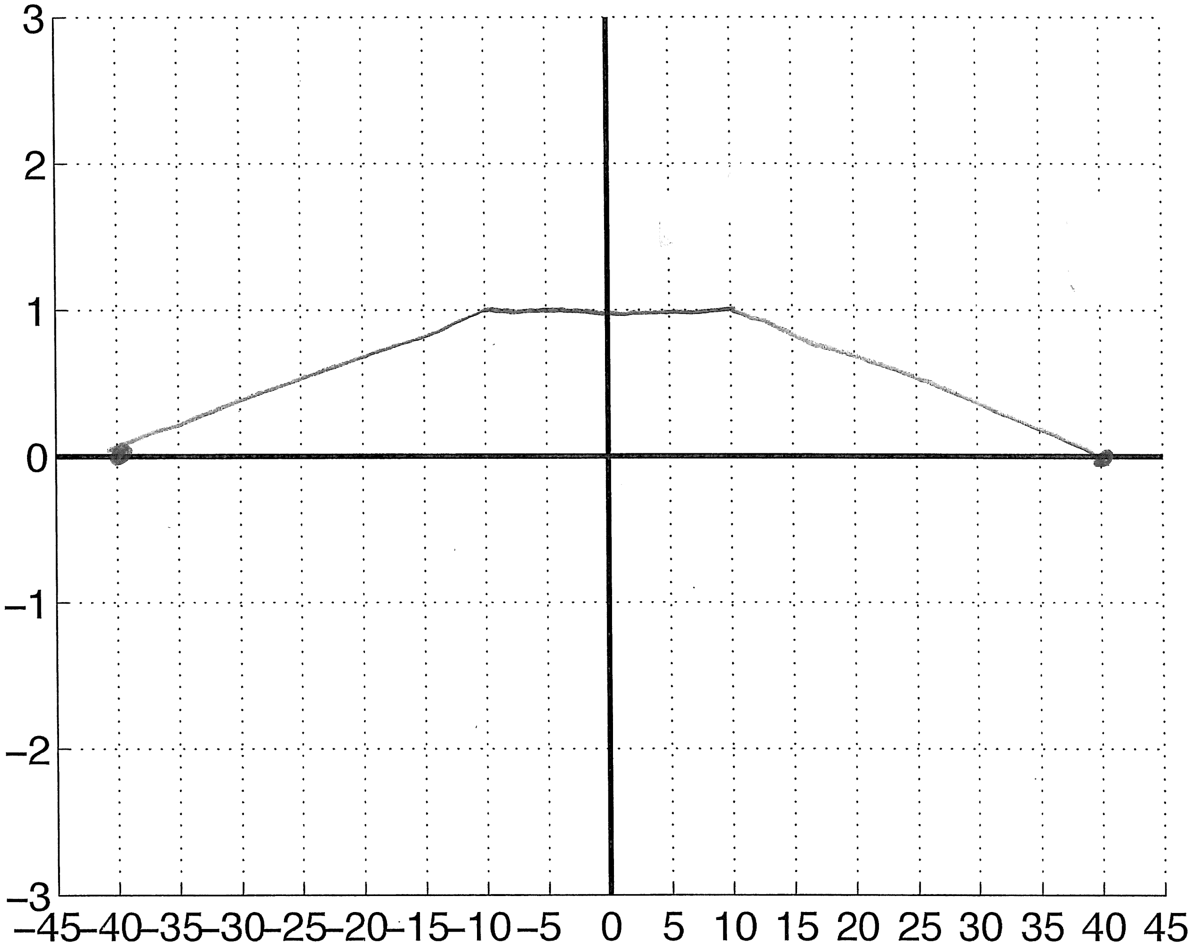
$\frac{1}{20}$

$\cos\left(70\left(t+\frac{\pi}{140}\right)\right)$

$X(\omega) \rightarrow$



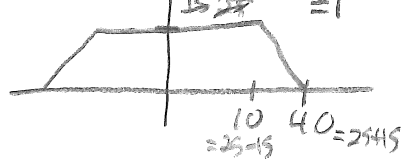
Show all work and plot for Problem 1 (c.).



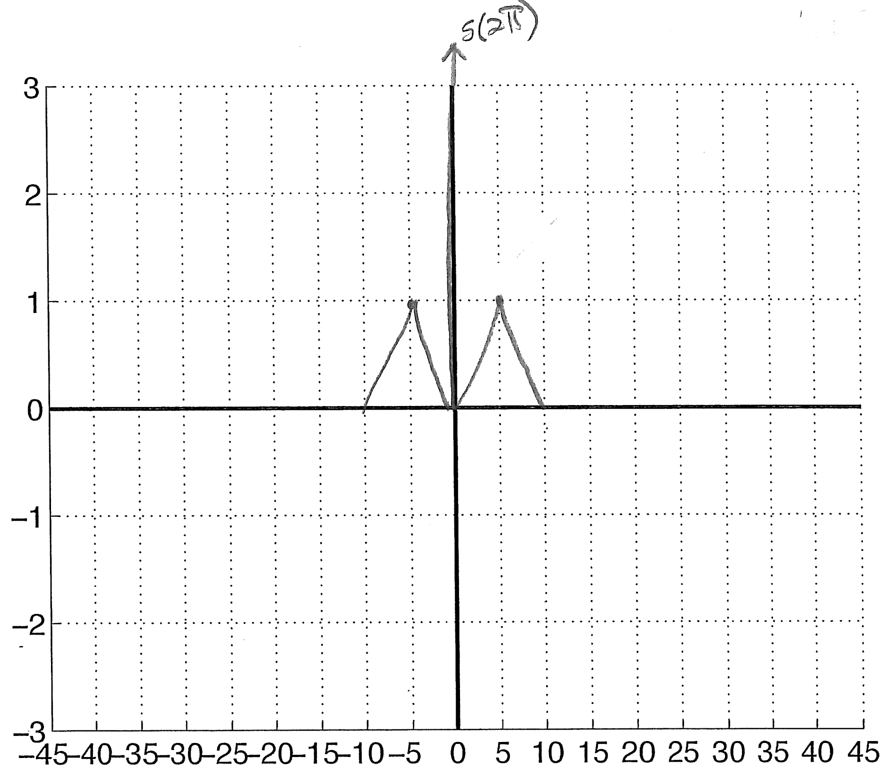
$$h(t) = \mathcal{F}^{-1}\left\{ \frac{\pi}{15} \frac{\sin(15t)}{\pi t} \frac{\sin(25t)}{\pi t} \right\}$$

$$\frac{\pi}{15} \frac{1}{\cancel{2\pi}} (\cancel{30})(1)(1) = 1$$

$=$

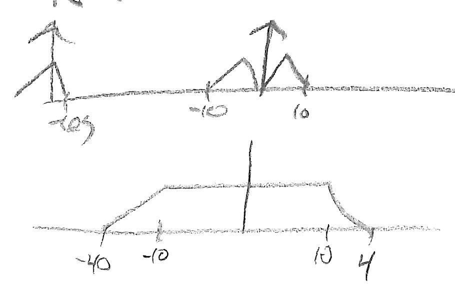


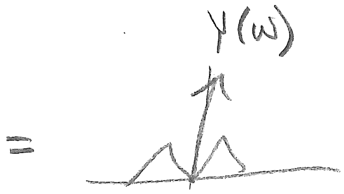
Show all work and plot for Problem 1 (d.).



$y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega)\, H(\omega)$





$\boxed{y(t) = 5 + s(t)}$

e The phase shift does not affect the graph of the magnitude of a signal. This is because in the $\mathcal{F}$, the phaseshift translates to an $e^{-j\omega t_0}$ & $|e^{-j\omega t_0}| = \sqrt{\cos^2(j\omega t_0) + \sin^2(j\omega t_0)}$

∴ phase shift would not change part d. $= \sqrt{1} = 1.$

f See page w/ problemstatement

Problem 2.

The rectangular pulse $x_{in}(t) = \{u(t) - u(t-2)\}$ of duration 2 secs is input to the following integrator

$$x_a(t) = 2\int_{t-2}^{t} x_{in}(\tau)d\tau$$

The output $x_a(t)$ is sampled every $T_s = 1/2$ seconds to form $x[n] = x_a(nT_s)$. The sampling rate is $f_s = 2$ samples/sec. Note: Part (b) can be done independently of Part (a). Show work. Clearly label and write your final answer in the space provided on the next few pages.

(a) Do a stem plot of the DT signal $x[n]$, or you can simply write the numbers that comprise $x[n]$ (indicate with an arrow where the $n = 0$ value is.)

(b) Determine a closed-form expression for the DTFT $X(\omega)$. Show work and clearly label and write your final answer in the space provided on the next few pages. **VIP:** List the values of ω within the range from $-\pi \leq \omega \leq \pi$ for which $X(\omega) = 0$.

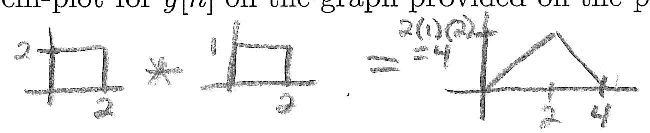
(c) The Discrete-Time (DT) signal $x[n]$, created as described above, is input to the DT system described by the difference equation below:

$$y[n] = -2y[n-1] + x[n] + 8x[n-3]$$

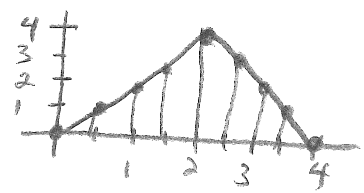
(i) First, determine and plot the impulse response $h[n]$ for this system. Do the stem-plot for $h[n]$ on the graph provided on the next page.

(ii) Determine and plot the output $y[n]$ by convolving the input $x[n]$ defined above with the impulse response $h[n]$. Show all work in the space provided. Do the stem-plot for $y[n]$ on the graph provided on the page after next.

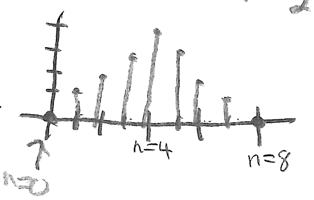
a) $x_a(t) =$



→ sample every 1/2 second



$x[n]$



or $\{0, 1, 2, 3, 4, 3, 2, 1, 0\}$ ↑ $n=0$

duration $= N_1 + N_2 - 1$
$= 7$
$N_1 + N_2 = 8$
$2N = 8$
$N = 4$

b) $x[n] = \{u[n] - u[n-4]\} * \{u[n] - u[n-4]\}$

$X(\omega) = \left\{\dfrac{\sin(\frac{N}{2}\omega)}{\sin(\frac{1}{2}\omega)} e^{-j\frac{(N-1)\omega}{2}}\right\}^2 = \boxed{\left\{\dfrac{\sin(\frac{4}{2}\omega)}{\sin(\frac{1}{2}\omega)} e^{-j(\frac{3}{2})\omega}\right\}^2 = X(\omega)}$

$X(\omega) = 0$ @ $\omega_l = l\frac{2\pi}{N}$ $l \neq 0$, integer

$l=1 \Rightarrow (1)\frac{2\pi}{4} = \frac{\pi}{2}$

$l=2 \Rightarrow (2)\frac{2\pi}{4} = \pi$

$\boxed{\text{When } \omega = -\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \pi \quad X(\omega) = 0}$

Show your work and plots for Problem 2 here.

c) $y[n] = -2y[n-1] + x[n] + 8x[n-3]$

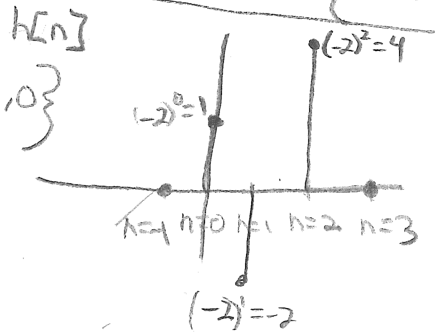
i) $h[n]$, Plot

of the form $y[n] = a\,y[n-1] + x[n] - a^D x[n-D]$ $\quad a = (-2)$, $D = 3$, $(-2)^3 = -8$

$\therefore h[n] = a^n \{u[n] - u[n-D]\}$

$$\boxed{h[n] = (-2)^n \{u[n] - u[n-3]\}}$$

$h[n] = \{\underset{\underset{n=0}{\uparrow}}{1}, -2, 4, 0\}$



$h[n] = (-2)^n u[n] - (-2)^n u[n-3]$

$h[n] = (-2)^n u[n] - (-2)^{n-3}(-2)^3 u[n-3]$

$\downarrow \quad = (-2)^n u[n] - (-8)(-2)^{n-3} u[n-3]$

$H(\omega) = \dfrac{1}{1-ae^{-j\omega}} - (-8)\dfrac{1}{1-ae^{-j\omega}} e^{-j\omega(3)}$

$= \dfrac{1}{1+2e^{-j\omega}} + 8\dfrac{e^{-3j\omega}}{1+2e^{-j\omega}}$

$H(\omega) = \dfrac{1+8e^{-3j\omega}}{1+2e^{-j\omega}}$

ii) $y[n] = x[n] * h[n]$ Table Method

$x[n] = \{\underset{\underset{n=0}{\uparrow}}{0}, 1, 2, 3, 4, 3, 2, 1, 0\}$

$h[n] = \{\underset{\underset{n=0}{\uparrow}}{1}, -2, 4, 0\}$

#colums = length $y[n]$ $= \cancel{6}+3-1 = 8$

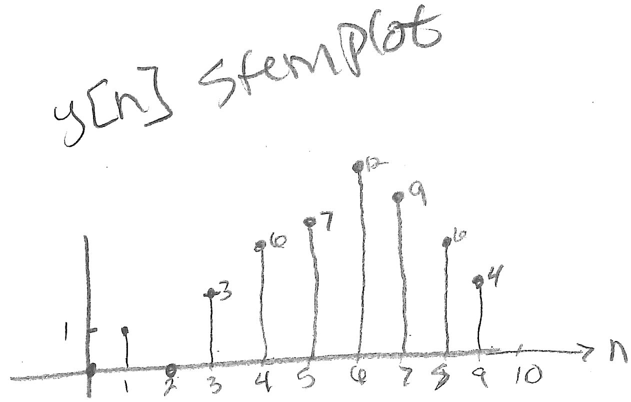
#rows = length $x[n]$ $= \cancel{8}\,6$

n	0	1	2	3	4	5	6	7	8	9
$x[0]h[n-0]$	0	0	0	0	0	0	0	0	0	6
$x[1]h[n-1]$	1·0 0	1·1 1	-2	4	0	0	0	0	0	0
$x[2]h[n-2]$	0	0	2	-4	8	0	0	0	0	0
$x[3]h[n-3]$	0	0	0	3	-6	12	0	0	0	0
$x[4]h[n-4]$	0	0	0	0	4	-8	16	0	0	0
$x[5]h[n-5]$	0	0	0	0	0	3	-6	12	0	0
$x[6]h[n-6]$	0	0	0	0	0	0	2	-4	8	0
$x[7]h[n-7]$	0	0	0	0	0	0	0	1	-2	4
$x[8]h[n-8]$	0	0	0	0	0	0	0	0	0	0
Σ	0	1	0	3	6	7	12	9	6	4

$$\boxed{y[n] = \{\underset{\underset{n=0}{\uparrow}}{0}, 1, 0, 3, 6, 7, 12, 9, 6, 4\}}$$

Stemplot $\rightarrow$

Show your work and plots for Problem 2 here.

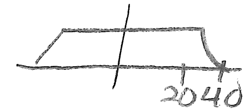


Problem 3 (a). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 60$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{60}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{\frac{60}{2\pi}\right\}\left\{\frac{\sin(\frac{\pi}{3}n)}{\pi n} + \frac{\sin(\frac{2\pi}{3}n)}{\pi n}\right\} \qquad \text{where: } T_s = \frac{2\pi}{60}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s\frac{\pi}{10}\frac{\sin(10t)}{\pi t}\frac{\sin(30t)}{\pi t}$$



Problem 3 (b). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s\frac{\pi}{15}\frac{\sin(15t)}{\pi t}\frac{\sin(25t)}{\pi t}$$

Problem 3 (c). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where: } T_s = \frac{2\pi}{60} \quad \text{and} \quad h(t) = T_s\frac{\pi}{5}\frac{\sin(5t)}{\pi t}\frac{\sin(35t)}{\pi t}$$

Show all your work for Prob. 3, parts (a)-(b)-(c) on next page.

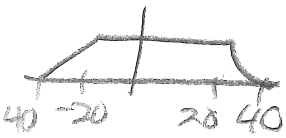
Show your work for Prob. 3, parts (a)-(b)-(c) below.

a) Check these 3 things:

1) $\omega_s > \omega_N$?
$60 > 2(20)$ $60 > 40$? yes ∴ no aliasing ✓

2) $H(\omega)$ flat for bandwidth of $X(\omega)$
$H(\omega)$ is flat $[-20 - 20]$ $\omega_m = 20$ ∴ flat ✓

3) $H(\omega)$ roll to $\emptyset$ @ $\omega_s - \omega_m$ to prevent replicates. ✓
$\omega_s - \omega_m = 60 - 20 = 40$
$H(\omega)$ is $\emptyset$ @ $\omega = 40$



∴ The signal is perfectly reconstructed

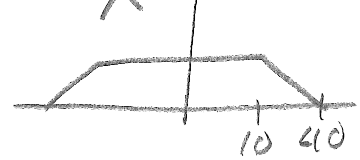
$$x_r(t) = x[n]\Big|_{n=\frac{t}{T_s}} = \left\{\frac{\cancel{60}}{\cancel{2\pi}}\right\}\left[\frac{\sin\left(\frac{\pi}{3}\left(t\,\cancel{\frac{60}{2\pi}}\right)\right)}{\pi\left(\cancel{\frac{60}{2\pi}}\right)t} + \frac{\sin\left(\frac{2\pi}{3}t\,\cancel{\frac{60}{2\pi}}\right)}{\pi\left(\cancel{\frac{60}{2\pi}}\,t\right)}\right]$$

$$\boxed{x_r(t) = \left[\frac{\sin(10t)}{\pi t} + \frac{\sin(20t)}{\pi t}\right]}$$

b) Check 3 things

1) $\omega_s > \omega_N$? $60 > 40$? yes ∴ no aliasing ✓

2) $H(\omega)$ flat for $X(\omega)$ Bandwidth. X
$H(\omega)$ is flat $[-10, 10]$
Bandwidth is $[-20, 20]$
NO



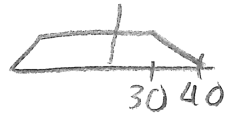
3) $H(\omega)$ is $\emptyset$ @ $\omega_s - \omega_m$? $60 - 20 = 40$ yes ✓

#2 is violated ∴ not a perfect reconstruction
∴ $\boxed{x_r(t) \neq x_a(t)}$

c) Check 3 things: $\omega_m = 20$, $\omega_s = 60$

1) $\omega_s > \omega_N$? yes ∴ no aliasing ✓

2) $H(\omega)$ is flat $[-30, 30]$ flat over $[-20, 20]$ ✓



3) Rolls to $\emptyset$ @ $\omega_s - \omega_m$? ✓

All 3 conditions met ∴

$\boxed{x_r(t) = x_a(t)}$

Problem 3 (d). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 20$ rads/sec. This signal is sampled at a rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{50}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{\frac{50}{2\pi}\right\}\left\{\frac{\sin(\frac{2\pi}{5}n)}{\pi n} + \frac{\sin(\frac{4\pi}{5}n)}{\pi n}\right\} \qquad \text{where:} \quad T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(20t)}{\pi t}$$

Problem 3 (e). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 50$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(35t)}{\pi t}$$

Problem 3 (f). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 60$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h(t - nT_s) \qquad \text{where:} \quad T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s\frac{\sin(15t)}{\pi t}$$

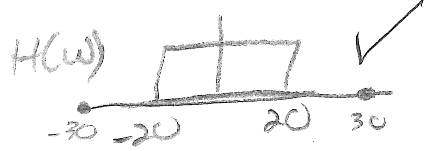
Show all your work for Prob. 3, parts (d)-(e)-(f) on next page.

Show your work for Prob. 3, parts (d)-(e)-(f) below.

d) Check 3 things:

1) $\omega_s > \omega_N$? $50 > 2(20)$
$50 > 40$? yes $\therefore$ no aliasing ✓

2) $H(\omega)$ flat over $X(\omega)$ bandwidth $(-20, 20)$ ✓



3) $H(\omega) = \emptyset$ @ $\omega_s - \omega_m$?
$\omega_s - \omega_n = 50 - 20 = 30$ ✓

1-3 are satisfied $\therefore$ perfect reconstruction.

$\therefore x_r(t) = x_a(t) = x[n]\big|_{n=\frac{t}{T_s}}$ $\quad T_s = \frac{2\pi}{50}$

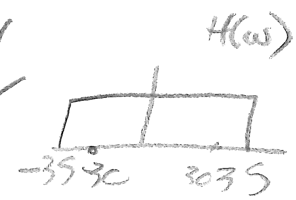
$$x_r(t) = \cancel{\frac{50}{2\pi}}\left(\frac{\sin\left(\frac{\cancel{2\pi}}{5}\left(t\,\frac{50}{\cancel{2\pi}}\right)\right)}{\pi\left(t\,\frac{50}{\cancel{2\pi}}\right)} + \frac{\sin\left(\frac{\cancel{4\pi}}{5}\,t\,\frac{50}{\cancel{2\pi}}\right)}{\pi\,t\,\frac{50}{\cancel{2\pi}}}\right)$$

$$\boxed{x_r(t) = \left(\frac{\sin(10t)}{\pi t} + \frac{\sin(20t)}{\pi t}\right)}$$

e) Check 3 things:

1) $\omega_s > \omega_N$? yes $\therefore$ no aliasing ✓

2) $H(\omega)$ flat over $X(\omega)$ bandwidth $(-20, 20)$ ✓



3) $H(\omega) = \emptyset$ @ $\omega_s - \omega_m$? No ✗
$50 - 20 = 30$

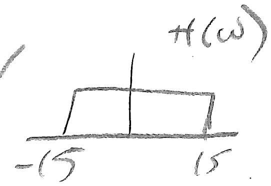
3 is violated $\therefore$ not perfect reconstruction

$\therefore$ $\boxed{x_r(t) \neq x_a(t)}$

f) Check 3 things:

1) $\omega_s > \omega_N$? $60 > 2(20)$ yes $\therefore$ no aliasing ✓

2) $H(\omega)$ flat over $X(\omega)$ bandwidth $(-20, 20)$ NO ✗



3) $H(\omega) = \emptyset$ @ $\omega_s - \omega_m$? yes ✓
$60 - 20 = 40$

2 is violated $\therefore$ not perfect reconstruction

$\therefore$ $\boxed{x_r(t) \neq x_a(t)}$

Show all your work for Prob. 3, parts (g)-(h) on this page.

g) $x[n] = \frac{30}{2\pi}\left\{ \frac{\sin(\frac{2\pi}{3}n)}{\pi n} + \frac{\sin(\frac{4\pi}{3}n)}{\pi n} \right\}$ $\quad T_s = \frac{2\pi}{30}$

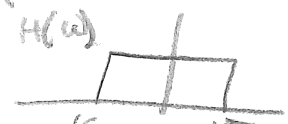
Check 3 Things:

1) $\omega_s > \omega_N$? $30 > 2(20)$? No $\therefore$ Aliasing X

2) $H(\omega)$ flat over $X(\omega)$ bandwidth
No

3) $H(\omega) = \emptyset$ @ $\omega_s - \omega_m$?
$30 - 20 = 10$ No

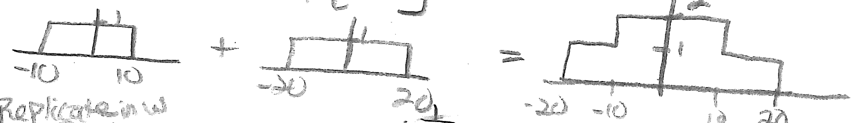
H(ω)



$x[n]\big|_{n=\frac{t}{T_s}} = \cancel{\frac{30}{2\pi}}\left\{ \frac{\sin(\frac{2\pi}{3}t\frac{30}{2\pi})}{\pi(\cancel{\frac{30}{2\pi}}t)} + \frac{\sin(\frac{4\pi}{3}t\frac{30}{2\pi})}{\pi(\cancel{\frac{30}{2\pi}}t)} \right\}$

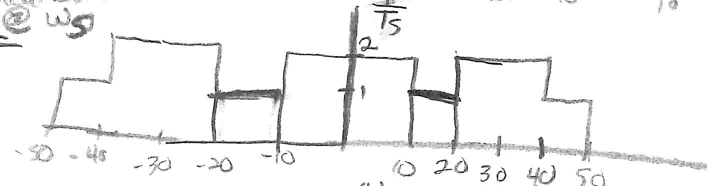
① $x_a(t) = \left[\frac{\sin(10t)}{\pi t} + \frac{\sin(20t)}{\pi t} \right]$

② $X_a(\omega) =$



③ Sample $x_a(t)$ = Replicate in ω @ ω_s
• Scale by F_s
$T_s =$

=



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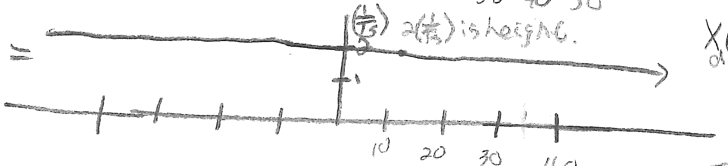
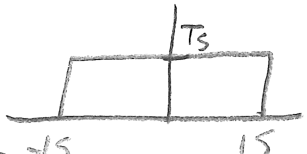
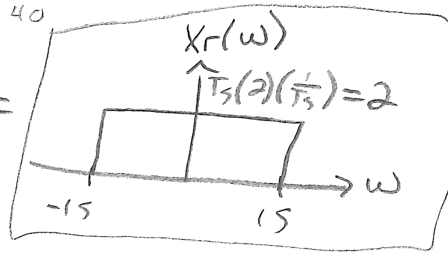


④ multiply by $H(\omega)$



multiply

=



⑤ $\therefore x_r(t) = 2\,\frac{\sin(15t)}{\pi t}$

Problem 3 (g). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at a rate $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{30}$ sec, yielding the following discrete-time sequence:

$$x[n] = x_a(nT_s) = \left\{\frac{30}{2\pi}\right\}\left\{\frac{\sin(\frac{2\pi}{3}n)}{\pi n} + \frac{\sin(\frac{4\pi}{3}n)}{\pi n}\right\} \qquad \text{where: } \ T_s = \frac{2\pi}{30}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h\left(t - nT_s\right) \qquad \text{where: } \ T_s = \frac{2\pi}{30} \ \text{ and } \ h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Problem 3 (h). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 20$ rads/sec. This signal is sampled at the same rate $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{30}$ sec, but at a different starting point. This yields the Discrete-Time $x[n]$ signal below:

$$x_\epsilon[n] = x_a(nT_s + 0.5T_s) = \left\{\frac{30}{2\pi}\right\}\left\{\frac{\sin(\frac{2\pi}{3}(n+0.5))}{\pi(n+0.5)} + \frac{\sin(\frac{4\pi}{3}(n+0.5))}{\pi(n+0.5)}\right\} \qquad \text{where: } \ T_s = \frac{2\pi}{30}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n]h\left(t - (n+0.5)T_s\right) \qquad \text{where: } \ T_s = \frac{2\pi}{30} \ \text{ and } \ h(t) = T_s\frac{\sin(15t)}{\pi t}$$

Show your work for Prob. 3, parts (g)-(h) on next page.

Show all your work for Prob. 3, parts (g)-(h) on this page.

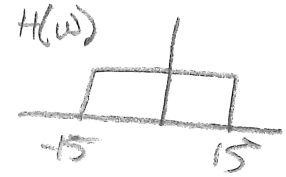
h) $x_\varepsilon[n] = \left\{\frac{30}{2\pi}\right\}\left\{\frac{\sin\left(\frac{2\pi}{3}(n+.5)\right)}{\pi(n+.5)} + \frac{\sin\left(\frac{4\pi}{3}(n+.5)\right)}{\pi(n+.5)}\right\}$ $T_s = \frac{2\pi}{30}$

Check 3 things!

1) $\omega_s > \omega_n$? $30 > 2(20)$? No ∴ Aliasing

2) $H(\omega)$ flat over $X(\omega)$ bandwidth? $(-20, 20)$ No

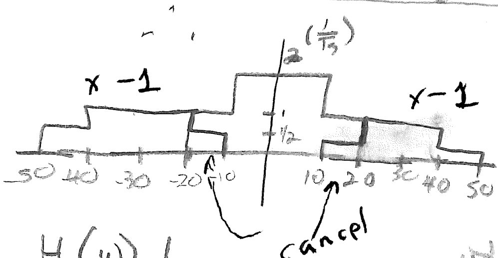
3) $H(\omega) = \emptyset$ @ $\omega_s - \omega_m$? $30-20=10$ No.

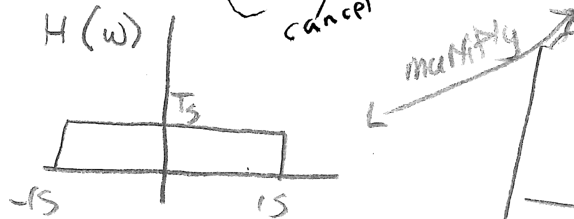


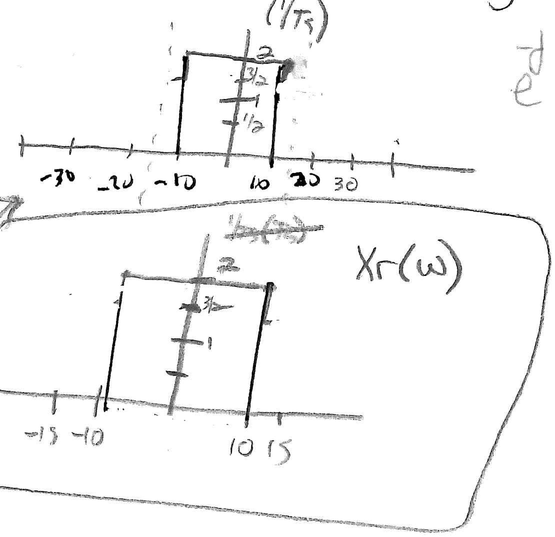
$x[n]\Big|_{n=\frac{t}{T_s}} = \frac{30}{2\pi}\left(\frac{\sin\frac{2\pi}{3}\left(t\frac{30}{2\pi}+.5\right)}{\pi\left(t\frac{30}{2\pi}+.5\right)} + \frac{\sin\left(\frac{4\pi}{3}\left(t\frac{30}{2\pi}+.5\right)\right)}{\pi\left(t\frac{30}{2\pi}+.5\right)}\right)$

$x_a(t) = \frac{30}{2\pi}\left(\frac{\sin\left(10t+\frac{\pi}{3}\right)}{15t+\frac{\pi}{2}} + \frac{\sin\left(20t+\frac{2\pi}{3}\right)}{15t+\frac{\pi}{2}}\right)$

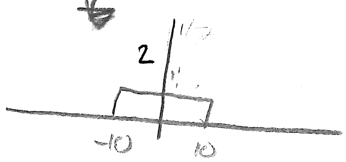
ε will only change "heights" of replicates by $-1 = e^{-j2\pi 0.5}$







$X_r(\omega) =$



$$\boxed{x_r(t) = 2\frac{\sin(10t)}{\pi t}}$$

ADDENDUM

In contrast to multiplying $x_a(t)$ by $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ to pick off values, i.e, sample, of $x_a(t)$ every $x_a(t)$ every T_s seconds, consider multiplying by $p(t-\tau) = \sum_{n=-\infty}^{\infty} \delta(t - \tau - nT_s)$ where $\tau = \epsilon T_s$ and $-1/2 < \epsilon < \frac{1}{2}$

__Since:__

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{\mathcal{F}} \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

__Then:__

$$p(t-\tau) \xleftrightarrow{\mathcal{F}} e^{-j\omega\tau} \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

Using Sifting property of Dirac Delta function

$$p(t-\tau) \xleftrightarrow{\mathcal{F}} \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk2\pi\frac{\tau}{T_s}} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

__Thus:__

$$x_a(t)p(t-\tau) \xleftrightarrow{\mathcal{F}} X_s(\omega) \overset{1/2\pi}{=} X_a(\omega) * \mathcal{F}\{p(t-\tau)\}$$

$\tau = \epsilon T_s$

$-\frac{1}{2} < \epsilon < \frac{1}{2}$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk2\pi\frac{\tau}{T_s}} X_a\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{-jk2\pi\epsilon} X_a\left(\omega - k\frac{2\pi}{T_s}\right)$$

If $\omega_s > 2\omega_M$, no overlap of replicas and filter out only $k=0$ term which is original signal $\Rightarrow$ perfect reconstruction

If $\omega_s < 2\omega_M$, replicas have different phase weightings and reconstructed signal will depend on τ