SOLUTION

Name: SOLUTION

Final Exam

ECE301 Signals and Systems

Friday, May 3, 2013

Cover Sheet

Write your name on this page and every page to be safe.

Test Duration: 120 minutes.

Coverage: Comprehensive

Open Book but Closed Notes. Four two-sided handwritten sheets.

Calculators NOT allowed.

This test contains **four** problems, each with multiple parts.

All plots should be done on the corresponding graphs provided.

You must show all work for each problem to receive full credit.

Good luck! It was great having you in class this semester!

Have a great summer!

Problem 1.

The **finite-duration**, decaying exponential signal $x_a(t) = 4e^{-ln(4)t}\{u(t) - u(t-2.5)\}$ is sampled every $T_s = 1$ second to form $x[n] = x_a(nT_s)$. The Fourier Transform of $x_a(t)$ is $X_a(\omega)$ is not strictly band-limited so there will always be some amount of aliasing. We know that the DTFT of $x[n]$ is related to the CTFT $X_a(\omega)$ according to the expressions below, where $F_s = 1$ and $\omega_s = 2\pi$, since $T_s = 1$ sec:

$$X(\omega) = X_s(F_s\omega) \quad \text{where:} \quad X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

(a) Determine a closed-form expression for the DTFT $X(\omega)$. Show work and write your final answer in the space below. *Hint:* Trick question :) $ln(4)$ equal to natural logarithm of 4 and $e^{ln(x)} = x$. *Note:* Closed-form means no summation in final answer.

(b) Defined above, $x[n] = x_a(nT_s)$ is formed by sampling $x_a(t) = 4e^{-ln(4)t}\{u(t)-u(t-2.5)\}$ every $T_s = 1$ second. The Discrete-Time (DT) signal $x[n]$ is then input to the DT system described by the difference equation below:

$$y[n] = 4y[n-1] + x[n] - 64x[n-3]$$

(i) First, determine and plot the impulse response $h[n]$ for this system. Do the stem-plot for $h[n]$ on the graph provided on the next page.

(ii) Determine and plot the output $y[n]$ by convolving the input $x[n]$ defined above with the impulse response $h[n]$. Show all work in the space provided. Do the stem-plot for $y[n]$ on the graph provided on the page after next.

$$x[n] = x_a(nT_s) = x_a(n) = 4\left(e^{-\ln(4)}\right)^n \left(u[n] - u[n-3]\right)$$

$$= 4\left(\frac{1}{e^{\ln(4)}}\right)^n \{u[n] - u[n-3]\} = 4\left(\frac{1}{4}\right)^n \{u[n] - u[n-3]\}$$

$$X(\omega) = 4\sum_{n=0}^{2}\left(\frac{1}{4}\right)^n e^{-j\omega n} = 4\sum_{n=0}^{2}\left(\left(\frac{1}{4}\right)e^{-j\omega}\right)^n$$

$$= 4\,\frac{1 - \left(\frac{1}{4}\right)^3 e^{-j3\omega}}{1 - \left(\frac{1}{4}\right)e^{-j\omega}}$$

answer to (a)

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{4}, 1, \tfrac{1}{4}\}$$

$$y[n] = a\,y[n-1] + x[n] - a^{D}\,x[n-D]$$

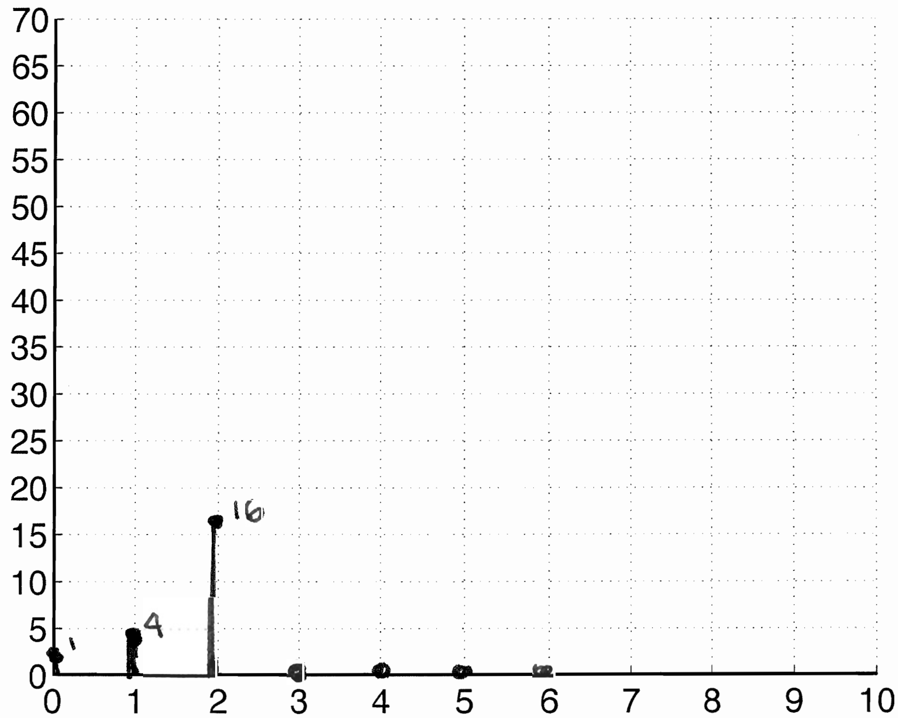
This Difference Eqn is special Form studied in class.

For this problem, $a=4$ and $D=3$

$$h[n] = a^n\{u[n]-u[n-D]\} = 4^n\{u[n]-u[n-3]\}$$

$$=\{\underset{\substack{\uparrow \\ n=0}}{1}, 4, 16\}$$

Plot your answer for $h[n]$ to Problem 1 (b-i) here. Show work above.



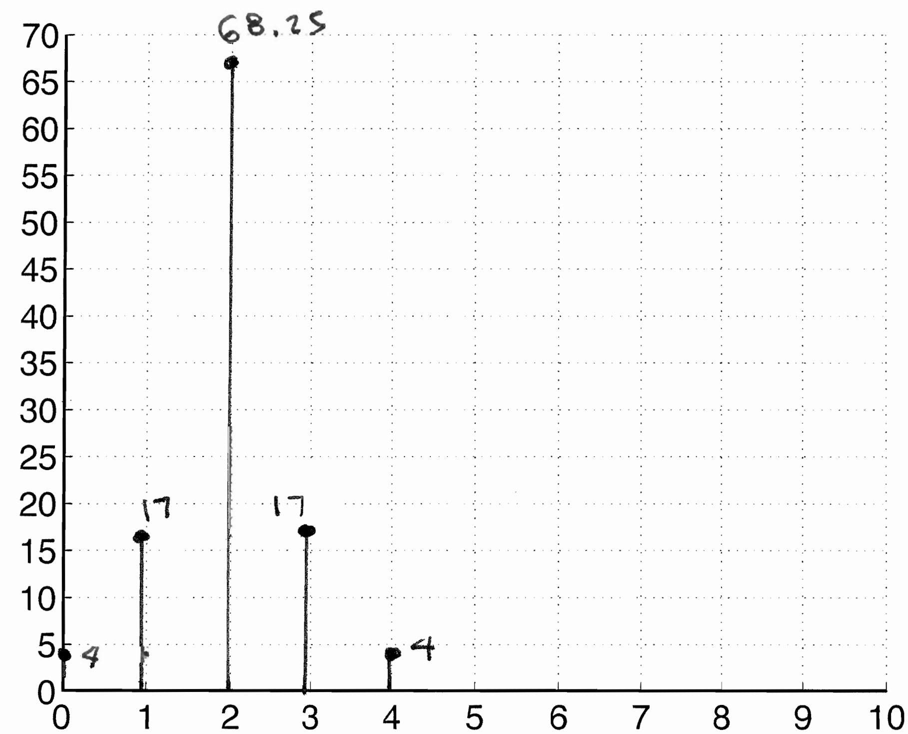
$$y[n] = \{\underset{\uparrow}{4}, 1, \tfrac{1}{4}\} * \{\underset{\uparrow}{1}, 4, 16\}$$

$$= \{1, 4, 16\} * \{4, 1, \tfrac{1}{4}\}$$

$$= \begin{array}{c|c|c|c|c}
4 & 1 & 1/4 & & \\
 & 16 & 4 & 1 & \\
 & & 64 & 16 & 4 \\
\hline
4 & 17 & 68\,1/4 & 20 & 4
\end{array}
\qquad
\begin{aligned}
&= h[0]\,x[n] \\
&+ h[1]\,x[n-1] \\
&+ h[2]\,x[n-2]
\end{aligned}$$

$$y[n] = \{\underset{\substack{\uparrow \\ n=0}}{4}, 17, 68.25, 17, 4\}$$

Plot your answer for $y[n]$ to Problem 1 (b-ii) here. Show work above.



Problem 2. Consider a causal LTI System characterized by the difference equation below. You will find the impulse response, and then the respective outputs for two different inputs.

$$\text{System:} \qquad y[n] = y[n-1] + x[n] - x[n-4]$$

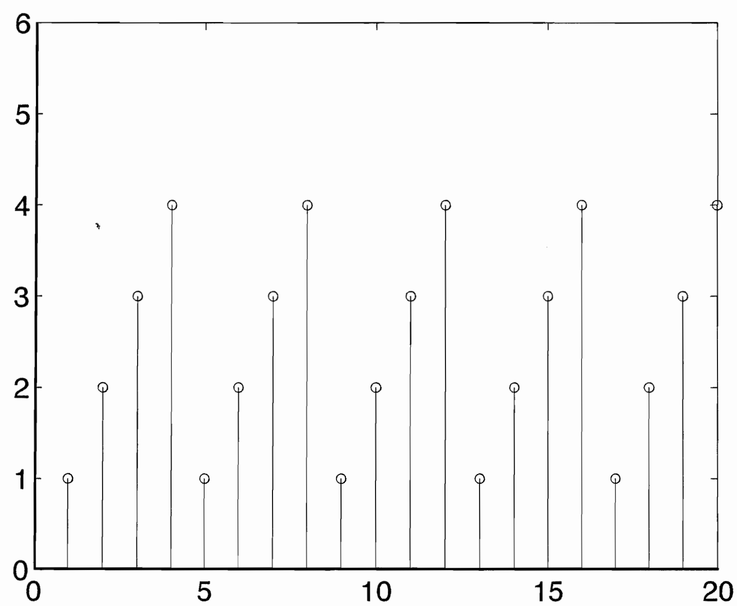
(a) Determine and plot the impulse response $h[n]$ of this system in the graph provided on the sheets attached.

(b) Plot the magnitude of $H(\omega)$, the DTFT of $h[n]$ over $-\pi < \omega < \pi$. Be sure to clearly point any frequencies where $H(\omega)$=0 over $-\pi < \omega < \pi$.

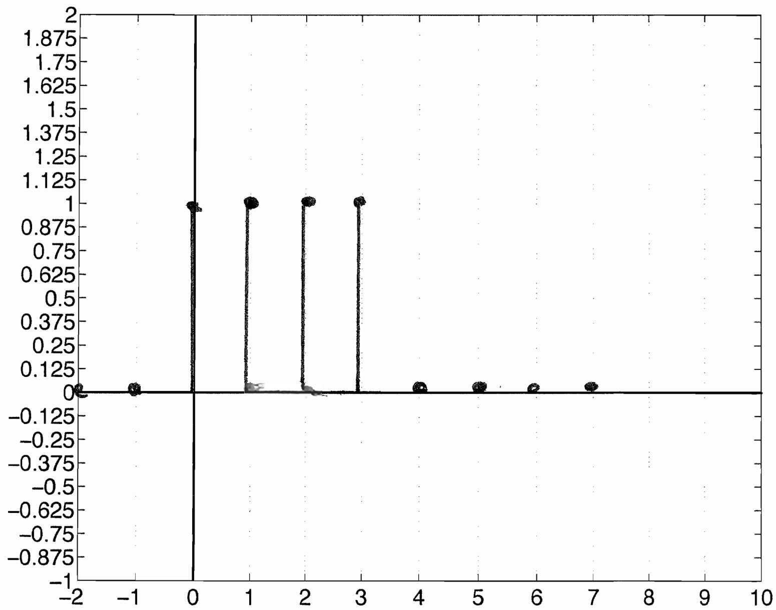
(c) Determine the output $y[n]$ when the input is the finite-length sequence below. Plot $y[n]$ in the space provided on the sheets attached.

$$x[n] = \{u[n] - u[n-6]\}$$

(d) Next, Let the input to this system be the periodic signal $x[n]$ plotted below. **The plot below only shows a couple periods of $x[n]$ but it is periodic for all time.** Determine the output $y[n]$. Plot several periods of $y[n]$ as indicated in the graph on the sheets attached.



Plot your answer $h[n]$ to Problem 2, part (a) on this page.



$$y[n] = y[n-1] + x[n] - x[n-4]$$

special case of: $y[n] = a\,y[n-1] + x[n] - a^D x[n-D]$

with $a = 1$ and $D = 4$

$$h[n] = a^n \{u[n] - u[n-D]\} = (1)^n \{u[n] - u[n-4]\}$$

$$= u[n] - u[n-4]$$

From class, we know:

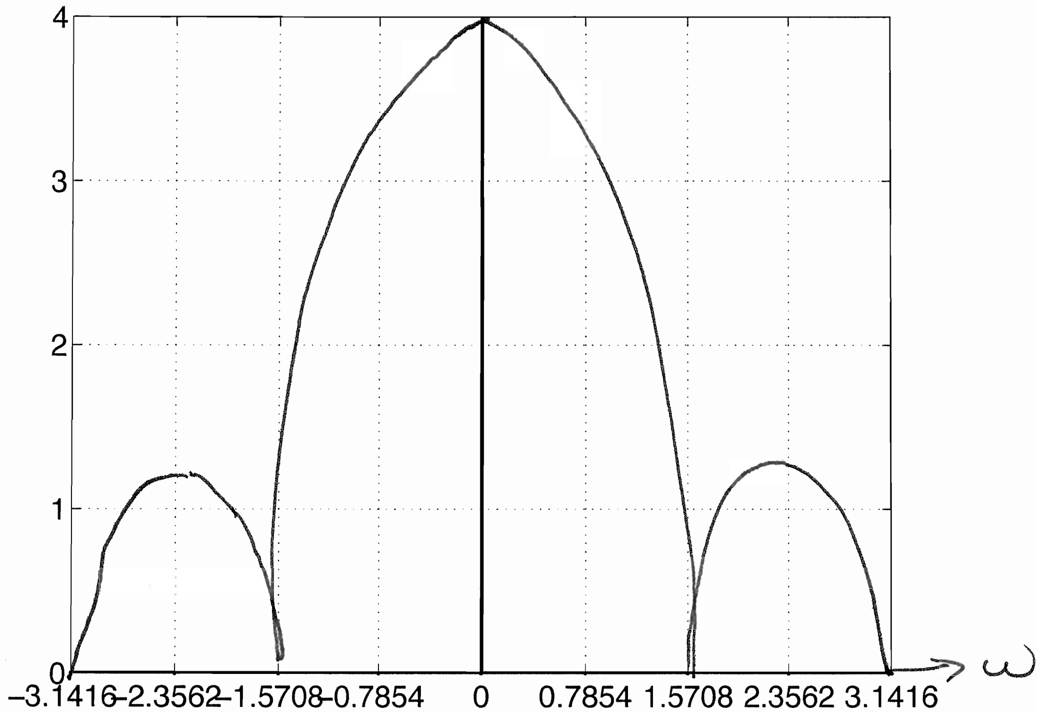
$$u[n] - u[n-N] \xleftrightarrow{DTFT} e^{-j\frac{(N-1)}{2}\omega} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

zero crossings occur at $\omega_\ell = \ell \frac{2\pi}{N}$ $\ell \neq 0$, integer

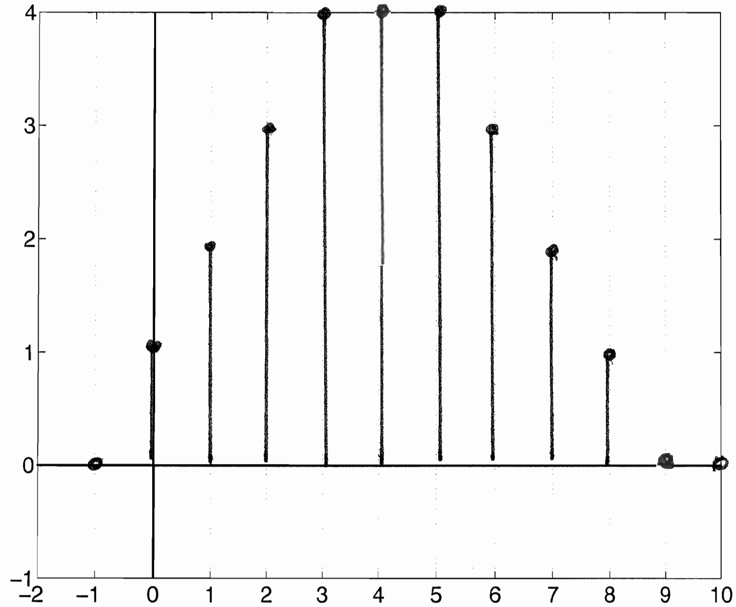
For $N=4$: $H(0) = 4$ $H\left(\ell \frac{2\pi}{4}\right) = 0$ for $\ell = \pm 1, \pm 2$

$$H(\omega) = e^{-j\frac{3}{2}\omega} \frac{\sin(2\omega)}{\sin\left(\frac{1}{2}\omega\right)}$$

Plot your answer $|H(\omega)|$ to Problem 2 (b) here. Show work below.



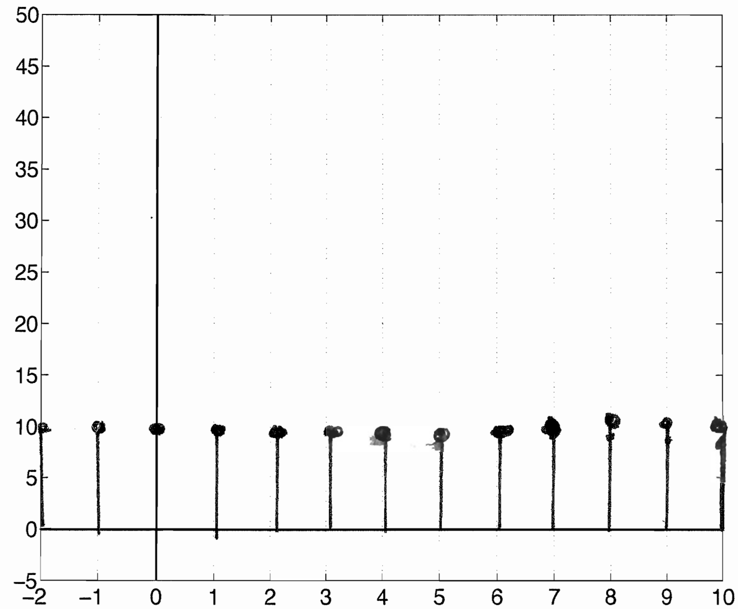
Show your work and plot the output $y[n]$ for Problem 2, part (c) below.



$$y[n] = \{u[n] - u[n-4]\} * \{u[n] - u[n-6]\}$$

n	0	1	2	3	4	5	6	7	8
	1	1	1	1	1	1			
		1	1	1	1	1	1		
			1	1	1	1	1	1	
				1	1	1	1	1	1
$y[n] =$	1	2	3	4	4	4	3	2	1

Show your work and plot several periods of $y[n]$ for Prob. 2, part (d) below.



2 (d) $x[n]$ is periodic with period $N = 4$

Thus, $x[n] = \sum_{k=0}^{3} a_k e^{jk\frac{2\pi}{4}n}$ $\Rightarrow$ sum of sinewaves

$$y[n] = \sum_{k=0}^{3} a_k \underbrace{H\left(k\frac{2\pi}{4}\right)}_{=0 \text{ for } k=1,2,3 \text{ from part (b)}} e^{jk\frac{2\pi}{4}n}$$

Thus, only DC term with $k=0$ passes

$$y[n] = a_0 H(0) = a_0 4$$

$$a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}(1+2+3+4)$$

$k=0$

$$y[n] = 4a_0 = 1+2+3+4 = 10 \quad \forall n$$

Problem 3.

(a) For parts (a) and (b), causal LTI System 1 is characterized by the following difference equation below. Determine the impulse response $h_1[n]$ of System 1 and write the closed-form expression below.

$$\text{System 1:} \quad y[n] = \frac{2}{3}y[n-1] + \frac{2}{3}x[n] - x[n-1]$$

(b) Determine and plot $|Y(\omega)|$, the magnitude of the DTFT of the output $y[n]$ of System 1 when the input is the infinite-length sinewaves below.

$$x[n] = \frac{1}{2\pi}\left\{(-1)^n + 2e^{j\frac{\pi}{4}n} + 3e^{j\frac{\pi}{2}n} + 4e^{j\frac{3\pi}{4}n}\right\}$$

(c) For parts (c) and part (d), causal LTI System 2 is characterized by the difference equation below. Determine the impulse response $h_2[n]$ of System 2 and write the expression in the space provided on the sheets attached.

$$\text{System 2:} \quad y[n] = \frac{1}{2}y[n-1] + \frac{1}{2}x[n] - x[n-1]$$

(d) Determine and plot $|Y(\omega)|$, the magnitude of the DTFT of the output $y[n]$ of System 2 when the input is

$$x[n] = \{u[n] - u[n-4]\}$$

(e) Consider that System 1 and System 2 are put in SERIES such that the output of System 1 is the input to System 2. Determine the impulse response $h[n]$ for the overall series combination write the closed-form expression in the space provided.

From class, Systems 1 & 2 are both all-pass filters

$$y[n] = a\,y[n-1] + a\,x[n] - x[n-1]$$

$$Y(\omega)\left(1 - ae^{-j\omega}\right) = X(\omega)\left(a - e^{-j\omega}\right)$$

$$H(\omega) = \frac{a - e^{-j\omega}}{1 - ae^{-j\omega}} = -e^{-j\omega}\frac{\left(1 - ae^{j\omega}\right)}{1 - ae^{-j\omega}}$$

$$= -e^{-j\omega}\frac{C(\omega)}{C^*(\omega)} = -e^{-j\omega}\frac{|C(\omega)|\,e^{j\angle C(\omega)}}{|C(\omega)|\,e^{-j\angle C(\omega)}}$$

$$= -e^{-j\omega}\,e^{j2\angle C(\omega)}$$

$$\Rightarrow |H(\omega)| = 1 \quad \forall\, \omega$$

Prob. 3(a). Write a closed-form expression for $h_1[n]$, impulse response for System 1.

From class: $y[n] = a\,y[n-1] + x[n]$

$\Rightarrow h[n] = a^n u[n]$

Thus, for $y[n] = a\,y[n-1] + a\,x[n]$

$\Rightarrow h[n] = a\,a^n u[n] = a^{n+1} u[n]$

Also, for $y[n] = a\,y[n-1] - x[n-1]$

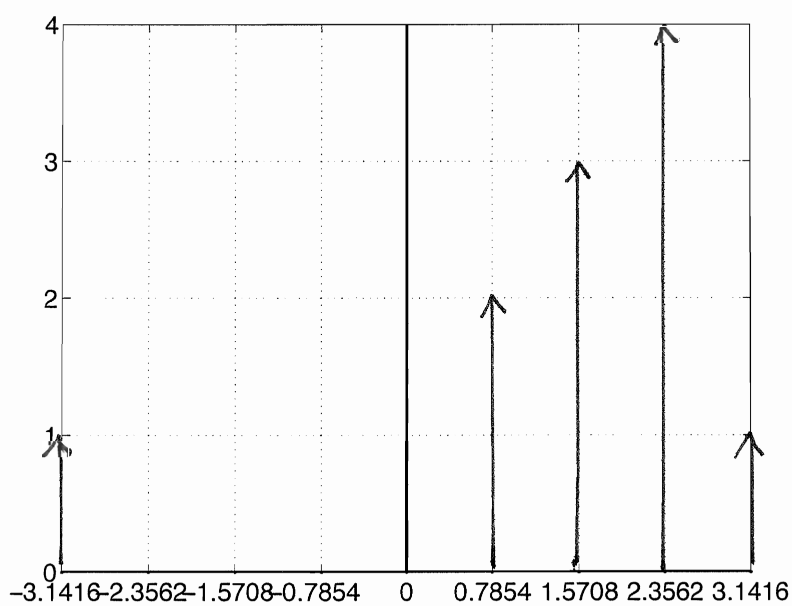
$\Rightarrow h[n] = -a^{n-1} u[n-1]$

Thus, $y[n] = a\,y[n-1] + a\,x[n] - x[n-1]$

$\Rightarrow h[n] = a^{n+1} u[n] - a^{n-1} u[n-1]$

System 1: $h_1[n] = \left(\frac{2}{3}\right)^{n+1} u[n] - \left(\frac{2}{3}\right)^{n-1} u[n-1]$

Prob. 3(b). Plot your output magnitude $|Y(\omega)|$ below. Show work above.



$(-1)^n = e^{j\pi n} = e^{-j\pi n}$

$|H(\omega)| = 1$ so $|Y(\omega)| = |X(\omega)|$

$e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\delta(\omega - \omega_0)$ for $-\pi < \omega < \pi$

assuming $-\pi \le \omega_0 \le \pi$

Alternative / more straightforward to determine impulse response for Prob. 3 parts (a) and (c):

$$y[n] = a\,y[n-1] + a\,x[n] - x[n-1]$$

$$Y(\omega)\left(1 - ae^{-j\omega}\right) = X(\omega)\left(a - e^{-j\omega}\right)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{a - e^{-j\omega}}{1 - ae^{-j\omega}}$$

$$= \frac{a}{1 - ae^{-j\omega}} - \frac{e^{-j\omega}}{1 - ae^{-j\omega}}$$

Basic DTFT pair derived in class, and in textbook and listed in Table 5.2

$$a^n u[n] \overset{DTFT}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

Using linearity and time-shift property:

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(\omega)$$

$$H(\omega) = a\,a^n u[n] - a^{n-1} u[n-1]$$

$$= a^{n+1} u[n] - a^{n-1} u[n-1]$$

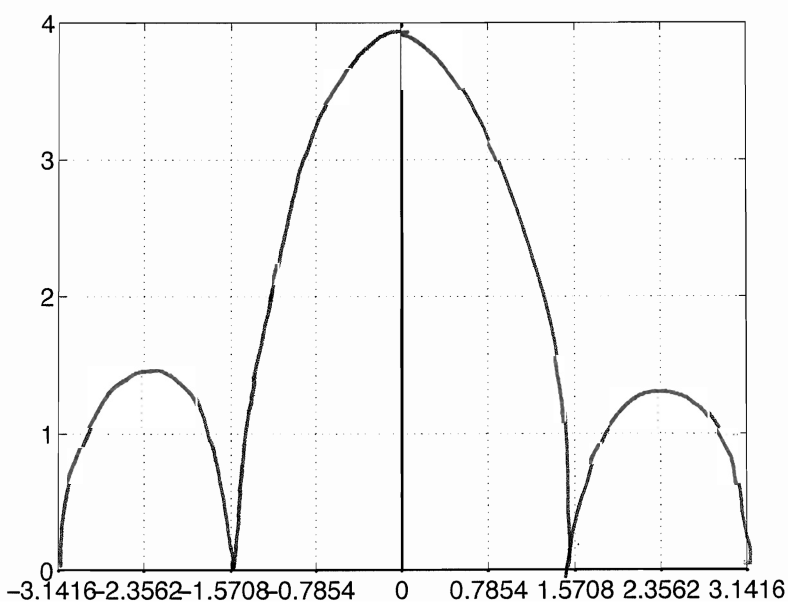
Prob. 3(c). Write a closed-form expression for $h_2[n]$, impulse response for System 2.

$$h_2[n] = \left(\frac{1}{2}\right)^{n+1} u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

For (d): $|Y(\omega)| = |H(\omega)|\,|X(\omega)| = |X(\omega)|$

$$= \left| \frac{\sin\left(\frac{4}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} \right|$$

Prob. 3(d). Plot your output magnitude $|Y(\omega)|$ below. Show work above.



Prob. 3(e). Closed-form expression for $h[n]$, impulse response for series combination of Systems 1 & 2. Show all work.

Consider $h[n] = a^{n+1} u[n] - a^{n-1} u[n-1]$

$$= a\, a^n u[n] - \frac{1}{a} a^n u[n] + \frac{1}{a} \delta[n]$$

$$= \left(a - \frac{1}{a}\right) a^n u[n] + \frac{1}{a} \delta[n]$$

$$= \left(\frac{a^2 - 1}{a}\right) a^n u[n] + \frac{1}{a} \delta[n]$$

$$= \frac{1}{a} \delta[n] - \frac{(1-a^2)}{a} a^n u[n]$$

So: $h_1[n] = c_0 \delta[n] - c_1 a_1^n u[n]$

$h_2[n] = d_0 \delta[n] - d_1 a_2^n u[n]$

$h_1[n] * h_2[n] =$

$$\left(c_0 \delta[n] - c_1 a_1^n u[n]\right) * \left(d_0 \delta[n] - d_1 a_2^n u[n]\right)$$

$$= c_0 d_0 \delta[n] - c_1 d_0 a_1^n u[n] - c_0 d_1 a_2^n u[n]$$
$$+ c_1 d_1 a_1^n u[n] * a_2^n u[n]$$

From Hmwk. $\quad \alpha^n u[n] * \beta^n u[n] = \left(\dfrac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right) u[n]$

$$a_1^n u[n] * a_2^n u[n] = \left(\frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2}\right) u[n]$$

<u>substitute:</u> $\quad a_1 = \frac{2}{3} \qquad a_2 = \frac{1}{2}$

$c_0 = \frac{1}{a_1} \quad c_1 = \frac{(1-a_1^2)}{a_1} \qquad d_0 = \frac{1}{a_2} \quad d_1 = \frac{1-a_2^2}{a_2}$

Problem 4 (a). Consider an analog signal with maximum frequency $\omega_M = 20$ rads/sec. The sampling rate is chosen to be $\omega_s = 60$ rads/sec., where $\omega_s = 2\pi/T_s$. A reconstructed signal is formed from the samples according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show all work in the space below.

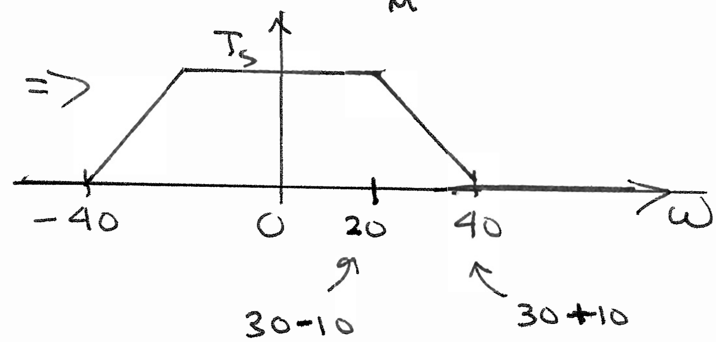
$$x_r(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s^2} \frac{\sin(\frac{\pi}{6}n)}{\pi n} \frac{\sin(\frac{\pi}{2}n)}{\pi n} h(t - nT_s) \quad \text{where: } T_s = \frac{2\pi}{60} \text{ and } h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \frac{\sin(30t)}{\pi t}$$

Nyquist rate $= 2\omega_M = 2(20) = 40$

$\omega_s = 60 > 40 \Rightarrow$ no aliasing

Also, since we oversampled, the LPF can roll off to zero from $\omega_M = 20$ to $\omega_s - \omega_M = 60 - 20 = 40$

$H(\omega) \Rightarrow$



Thus, we will achieve perfect reconstruction

Since: $x[n] = x_a(t)\Big|_{t=nT_s} = \dfrac{\sin(\frac{\pi}{6}n)}{\pi n T_s} \dfrac{\sin(\frac{\pi}{2}n)}{\pi n T_s}$

Then: $x_a(t) = x[n]\Big|_{n=\frac{t}{T_s}}$

$$= \frac{\sin\left(\frac{\pi}{6}\frac{t}{T_s}\right)}{\pi t} \frac{\sin\left(\frac{\pi}{2}\frac{t}{T_s}\right)}{\pi t}$$

$T_s = \dfrac{2\pi}{60} = \dfrac{\pi}{30}$

$$= \frac{\sin\left(\frac{\pi}{6}\frac{30}{\pi}\right)}{\pi t} \frac{\sin\left(\frac{\pi}{2}\frac{30}{\pi}\right)}{\pi t}$$

$$= \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t}$$

Problem 4 (b). Consider an analog signal with maximum frequency $\omega_M = 20$ rads/sec. The sampling rate is chosen to be $\omega_s = 40$ rads/sec., where $\omega_s = 2\pi/T_s$. A reconstructed signal is formed from the samples according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show all work in the space below.

$$x_r(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s^2} \frac{\sin(\frac{\pi}{4}n)}{\pi n} \frac{\sin(\frac{3\pi}{4}n)}{\pi n} h\left(t - nT_s\right) \quad \text{where: } T_s = \frac{2\pi}{40} \text{ and } h(t) = T_s \frac{\sin(20t)}{\pi t}$$

Sampled at Nyquist rate

Thus, need ideal LPF with cut-off at $\frac{\omega_s}{2} = 20$

Thus, we achieve perfect reconstruction

$$x_a(t) = x[n]\Big|_{n=\frac{t}{T_s}} = t\,\frac{40}{2\pi} = \frac{20}{\pi}t$$

$$x_a(t) = \frac{\sin\left(\frac{\pi}{4}\frac{20}{\pi}t\right)}{\pi t} \frac{\sin\left(\frac{3\pi}{4}\frac{20}{\pi}t\right)}{\pi t}$$

$$= \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t}$$

Problem 4 (c). Consider an analog signal with maximum frequency $\omega_M = 20$ rads/sec. The sampling rate is chosen to be $\omega_s = 30$ rads/sec., where $\omega_s = 2\pi/T_s$. A reconstructed signal is formed from the samples according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show all work in the space below.

$$x_r(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s^2} \frac{\sin(\frac{\pi}{3}n)}{\pi n} \frac{\sin(\pi n)}{\pi n} h(t - nT_s) \quad \text{where } T_s = \frac{2\pi}{30} \quad \text{and } h(t) = T_s \frac{\sin(15t)}{\pi t}$$

$$\omega_s = 30 < 2(\omega_M) = 2(20) = 40$$

$\Rightarrow$ Aliasing!

Several ways to solve this problem.

Easiest way: $\dfrac{\sin(\pi n)}{\pi n} = \delta[n]$

$$x_r(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s^2} \frac{\sin(\frac{\pi}{3}n)}{\pi n} \delta[n]\, h(t-nT_s)$$

only $n=0$ contributes

$$\left.\frac{\sin(\frac{\pi}{3}n)}{\pi n}\right|_{n=0} = \frac{1}{3}$$

$$= \frac{1}{T_s^2}\frac{1}{3} h(t)$$

$$= \frac{1}{T_s^2}\frac{1}{3} T_s \frac{\sin(15t)}{\pi t}$$

$$T_s = \frac{2\pi}{30} = \frac{\pi}{15}$$

$$\frac{1}{T_s} = \frac{15}{\pi}$$

$$= \frac{15}{\pi}\frac{1}{3}\frac{\sin(15t)}{\pi t}$$

$$= \frac{5}{\pi}\frac{\sin(15t)}{\pi t}$$ answer