SOLUTION

Name:	Final Exam
EE301 Signals and Systems	Wednesday, May 2, 2012

Cover Sheet

Write your name on this page and every page to be safe.

Test Duration: 120 minutes.

Coverage: Comprehensive

Open Book but Closed Notes. Four two-sided handwritten sheets.

Calculators NOT allowed.

This test contains **five** problems, each with multiple parts.

All plots should be done on the corresponding graphs provided.

You must show all work for each problem to receive full credit.

Good luck! It was great having you in class this semester! Have a great summer!

VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. For each plot, the abscissa goes from $-\pi$ to π with tic marks every $\pi/16$. There is a dashed vertical line at $\omega = -\frac{\pi}{2}$ and another dashed vertical line at $\omega = +\frac{\pi}{2}$, just so you can locate those 2 frequencies quickly. You only have to plot any DTFT over $-\pi < \omega < \pi$.*

Problem 1.

(a) Consider the continuous-time signal $x_1(t)$ below A discrete-time signal is created by sampling $x_1(t)$ according to $x_1[n] = x_1(nT_s)$ for $T_s = \frac{2\pi}{160}$. Plot the magnitude of the DTFT of $x_1[n]$, $|X_1(\omega)|$, over $-\pi < \omega < \pi$.

$$x_1(t) = T_s \frac{5}{\pi} \left\{ \frac{\sin(5t)}{\pi t} \right\}^2$$

$\frac{\pi}{5}$

(b) Consider the continuous-time signal $x_2(t)$ below A discrete-time signal is created by sampling $x_2(t)$ according to $x_2[n] = x_2(nT_s)$ for $T_s = \frac{2\pi}{160}$. Plot the magnitude of the DTFT of $x_2[n]$, $|X_2(\omega)|$, over $-\pi < \omega < \pi$.

$$x_2(t) = T_s \frac{1}{2} \left\{ \frac{\sin(10(t - \frac{\pi}{20}))}{\pi(t - \frac{\pi}{20})} + \frac{\sin(10(t + \frac{\pi}{20}))}{\pi(t + \frac{\pi}{20})} \right\}$$

(c) Given $x_1[n]$ and $x_2[n]$ defined above, the signal $x[n]$ is created as shown below. Determine the DTFT, $X(\omega)$, of $x[n]$ and plot the magnitude $|X(\omega)|$ over $-\pi < \omega < \pi$ showing as much detail as possible.

$$x[n] = 2x_1[n] \cos\left(\frac{3\pi}{8}n\right) + 2x_2[n] \cos\left(\frac{7\pi}{8}n\right)$$

(d) Determine and plot the magnitude of the DTFT $Z(\omega)$ of the signal $z[n]$ defined below, where $x[n] = 2x_1[n]\cos\left(\frac{3\pi}{8}n\right) + 2x_2[n]\cos\left(\frac{7\pi}{8}n\right)$ as defined in part (c). The trig identity $2\cos(\theta)\cos(\phi) = \cos(\theta + \phi) + \cos(\theta - \phi)$ should be useful.

$$z[n] = 2x[n] \cos\left(\frac{3\pi}{8}n\right)$$

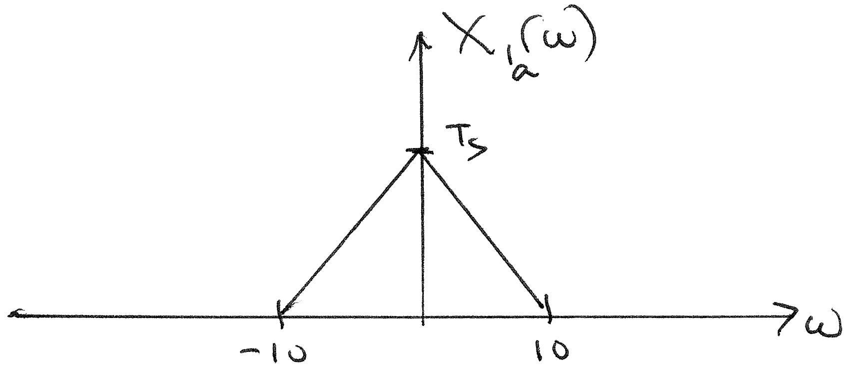
(e) The signal $w[n]$ is the output obtained with $z[n] = 2x[n]\cos\left(\frac{3\pi}{8}n\right)$ from part (d) as the input to the DT lowpass filter with impulse response $h[n]$ defined below. That is, $w[n] = z[n] * h[n]$, where $z[n] = 2x[n]\cos\left(\frac{3\pi}{8}n\right)$ and

$$h[n] = 8 \left\{ \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n} \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \right\}$$

(i) Plot the magnitude of the DTFT $H(\omega)$ of $h[n]$ over $-\pi < \omega < \pi$.

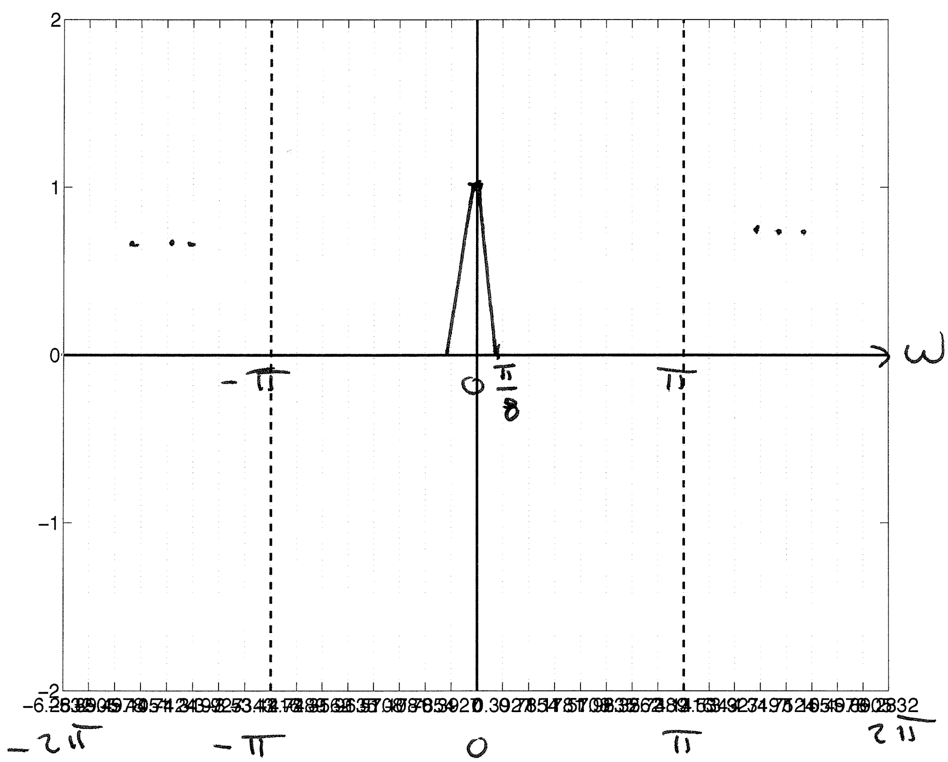
(ii) Plot the magnitude of the DTFT $W(\omega)$ of output signal $w[n]$ over $-\pi < \omega < \pi$.

(iii) Write a simple expression for $w[n]$. Is $w[n]$ equal to $x_1[n]$ except for a scalar amplitude gain?



$\omega_M = 10$ mapped to $10\,T_s = \frac{2\pi 10}{160} = \frac{\pi}{8}$

Plot your answer to Problem 1 (a) here. Show work above.

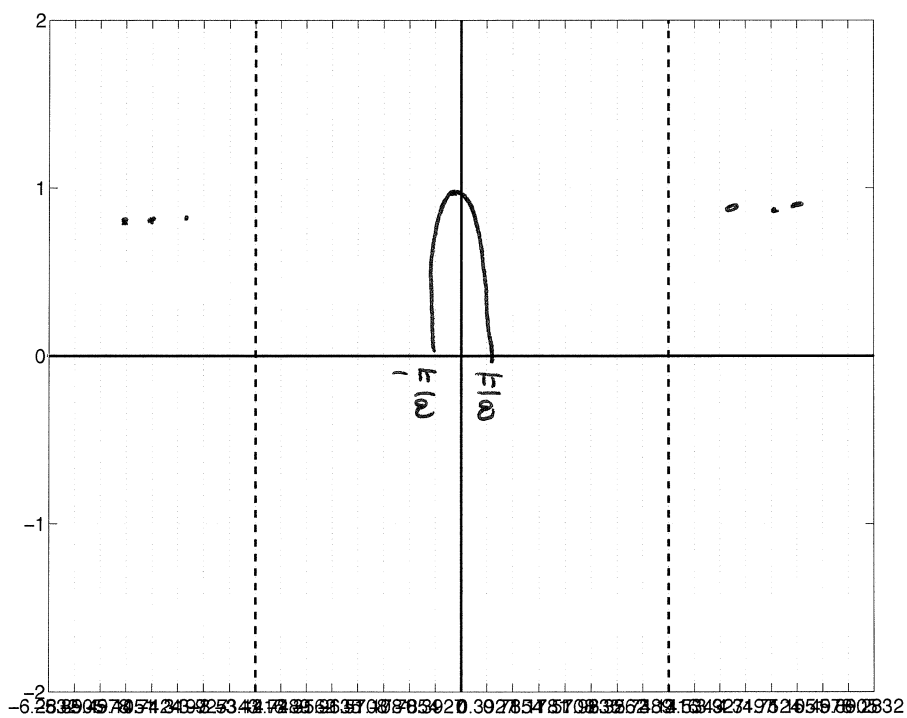


$$X_{2a}(\omega) = T_s \, \text{rect}\left(\frac{\omega}{20}\right) \frac{1}{2}\left\{ e^{-j\frac{\pi}{20}\omega} + e^{j\frac{\pi}{20}\omega}\right\}$$

$$= T_s \, \text{rect}\left(\frac{\omega}{20}\right) \cos\left(\frac{\pi}{20}\omega\right)$$

$\omega_M = 10$ mapped to $\pi/8$

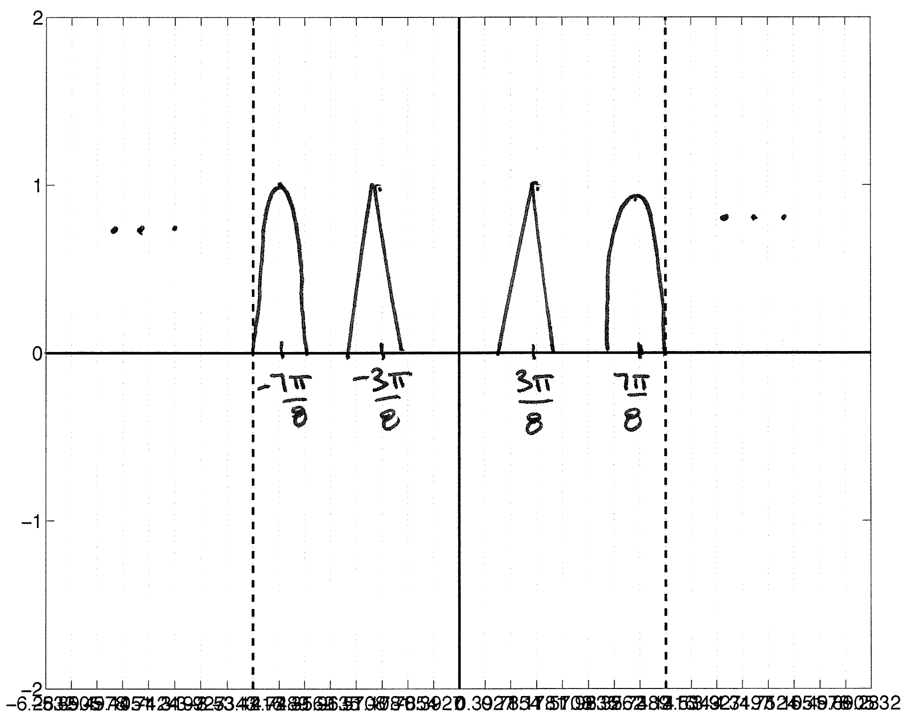
Plot your answer to Problem 1 (b) here. Show work above.



From the modulation prop. of the DTFT

$$X(\omega) = X_1\left(\omega - \frac{3\pi}{8}\right) + X_1\left(\omega + \frac{3\pi}{8}\right) + X_2\left(\omega - \frac{7\pi}{8}\right) + X_2\left(\omega + \frac{7\pi}{8}\right)$$

Plot your answer to Problem 1 (c) here. Show work above.



$$x[n] = \left\{ 2x_1[n]\cos\left(\frac{3\pi}{8}n\right) + 2x_2[n]\cos\left(\frac{7\pi}{8}n\right) \right\} 2\cos\left(\frac{3\pi}{8}n\right)$$

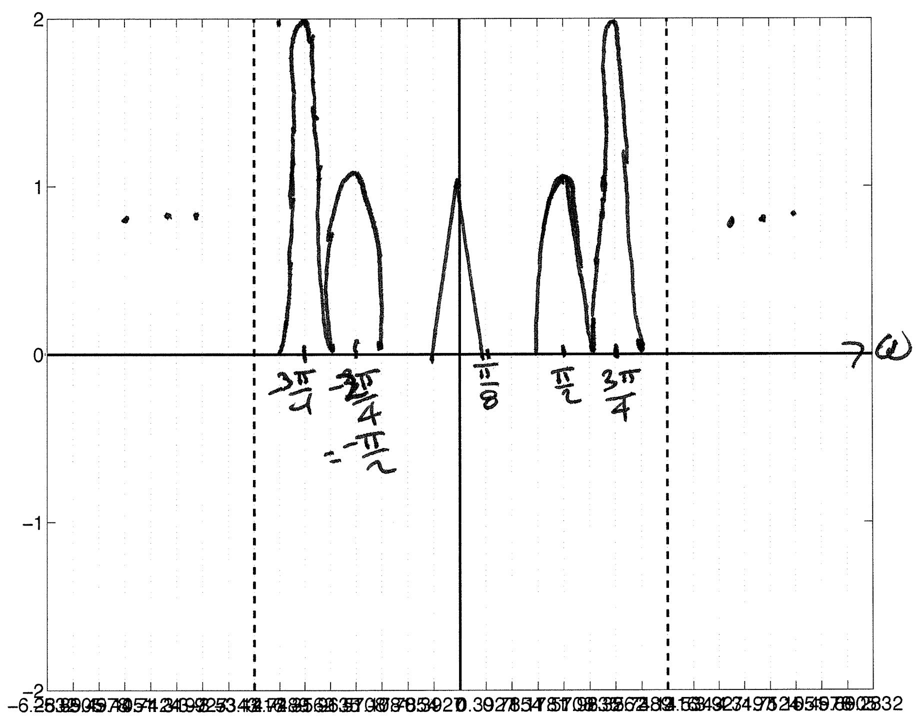
$$= 2\left\{ x_1[n] + x_1[n]\cos\left(\frac{6\pi}{8}n\right) + x_2[n]\cos\left(\frac{4\pi}{8}n\right) + x_2[n]\cdot\cos\left(\frac{10\pi}{8}n\right)\right.$$

$\frac{3\pi}{4}$ $\qquad$ $\frac{\pi}{2}$

$$\cos\left(\frac{10\pi}{8}n\right) = \cos\left(\frac{5\pi}{4}n\right) = \cos\left(\frac{-3\pi}{4}n\right) = \cos\left(\frac{3\pi}{4}n\right)$$

I am ignoring this factor of 2 in my plot below

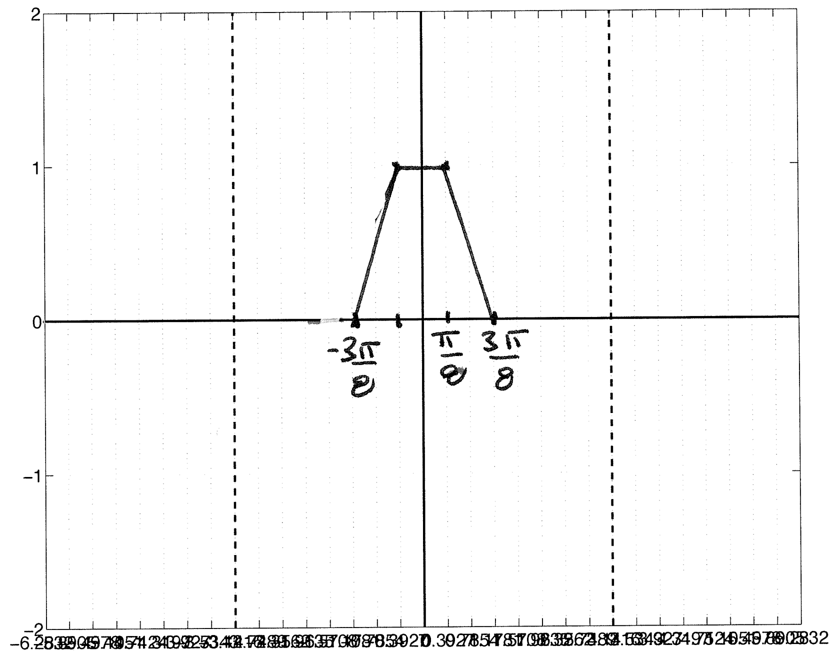
Plot your answer to Problem 1 (d) here. Show work above.



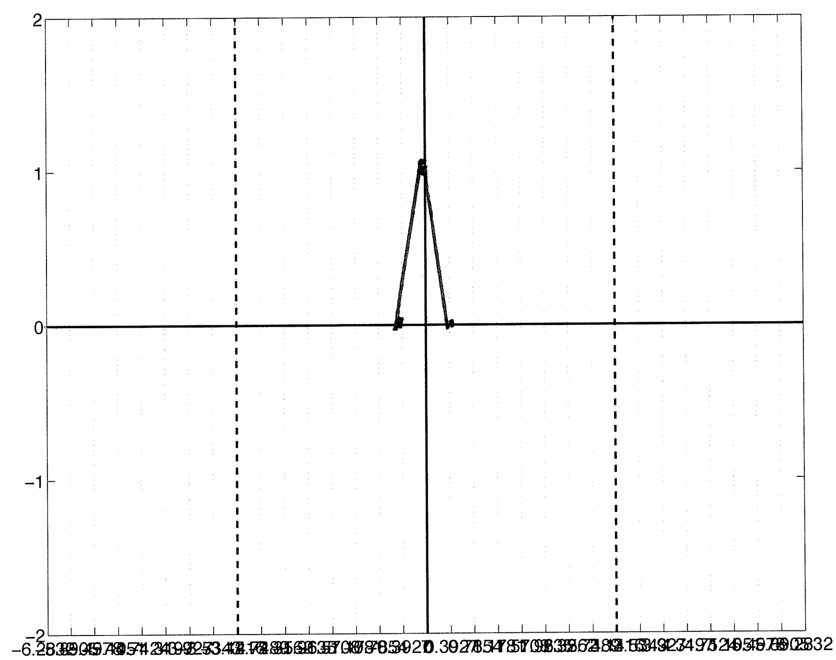
Plot your answers to Prob. 1 (e): $H(\omega)$ and $W(\omega)$.

$\frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$

$\frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$



again, ignoring add'l factor of 2



Write your simple expression for the output $w[n]$.

$x_1[n]$

(or $2x_1[n]$)

Yes!

Problem 2. The DT periodic signal $x[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\delta[n-4k]$ is input to a DT system characterized by the difference equation below

$$y[n] = \frac{2}{3}y[n-1] - \frac{2}{3}x[n] + x[n-1]$$

Plot the **magnitude** of the DTFT, $|Y(\omega)|$, of the output $y[n]$ on the graph below. *Hint:* $\sum_{k=-\infty}^{\infty}\delta[n-4k]$ is a DT periodic signal. See Table 5.2 in text.

$$Y(z) = \frac{2}{3}z^{-1}Y(z) - \frac{2}{3}X(z) + z^{-1}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{-\frac{2}{3} + z^{-1}}{1 - \frac{2}{3}z^{-1}}\,\frac{z}{z} = \frac{1 - \frac{2}{3}z}{z - \frac{2}{3}} = -\frac{2}{3}\,\frac{\left(z - \frac{3}{2}\right)}{\left(z - \frac{2}{3}\right)} = H(z)$$

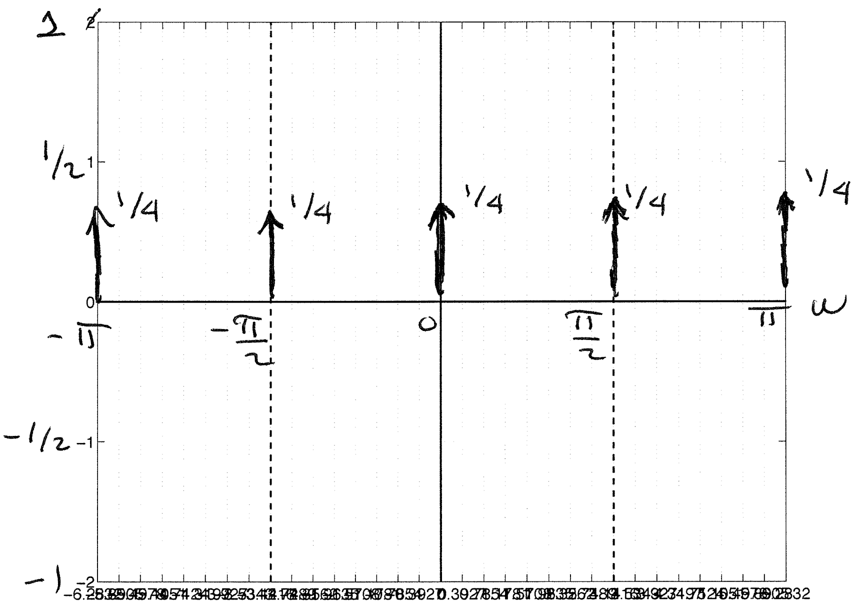
zero at $3/2$ and pole at $2/3$ $\Rightarrow$ all-pass filter

at $z=1$ $(\omega=0)$ $\Rightarrow H(z)\big|_{z=1} = H(\omega)\big|_{\omega=0} = -\frac{2}{3}\,\frac{\left(1-\frac{3}{2}\right)}{1-\frac{2}{3}} = 1$

$|H(\omega)| = 1$ for all ω

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\delta[n-4k] = \frac{1}{2\pi}\sum_{k=0}^{3}\frac{1}{4}e^{jk\frac{2\pi}{4}n} \xleftrightarrow{\text{DTFT}} \sum_{k=0}^{3}\frac{1}{4}\delta\left(\omega - k\frac{2\pi}{4}\right)$$

periodic with period 2π



Problem 3. The DT periodic signal $x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \delta[n-6k]$ is input to a DT system characterized by the difference equation below

$$y[n] = y[n-1] + x[n] - x[n-6]$$

Plot the **magnitude** of the DTFT, $|Y(\omega)|$, of the output $y[n]$ on the graph below. *Hint:* $\sum_{k=-\infty}^{\infty} \delta[n-6k]$ is a DT periodic signal. See Table 5.2 in text.

$$x[n] = \frac{1}{2\pi} \sum_{k=0}^{5} \frac{1}{6} e^{jk\frac{2\pi}{6}n}$$

Diff Eqn is form covered in class: impulse response

$$h[n] = u[n] - u[n-6] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\frac{(6-1)}{2}\omega} \frac{\sin\left(\frac{6}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} = H(\omega)$$

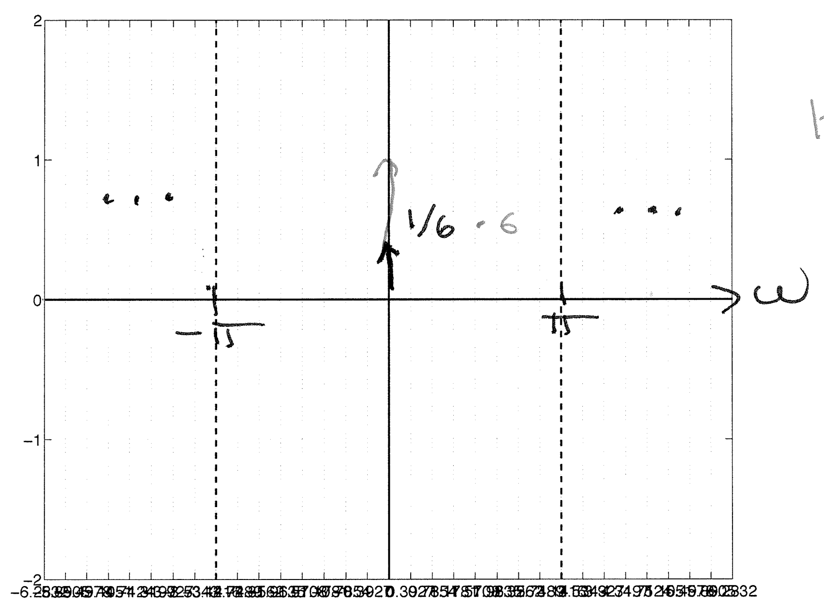
$$y[n] = \frac{1}{2\pi} \sum_{k=0}^{5} \frac{1}{6} H\left(k\frac{2\pi}{6}\right) e^{jk\frac{2\pi}{6}n}$$

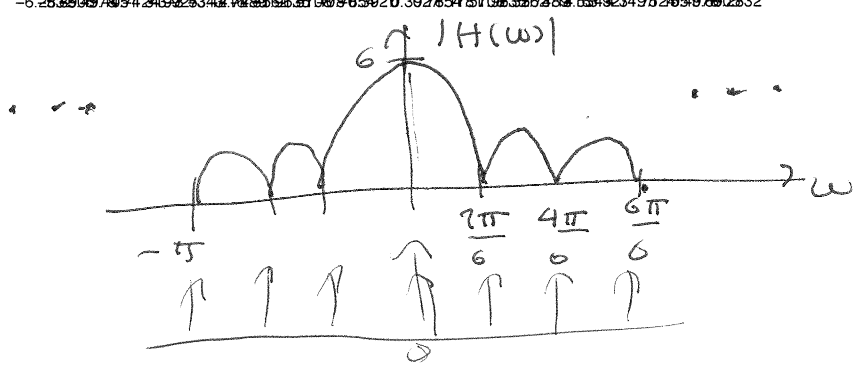
$$\frac{6}{2}\omega = l\pi$$

$H(\omega) = 0$ at $\omega_l = l\frac{2\pi}{6}$, $l \neq 0, \neq 6$, etc.

Thus, only $\omega = 0$ makes it thru the system

$$y[n] = \frac{1}{12\pi} \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{12\pi} 2\pi\,\delta(\omega) = \frac{1}{6}\delta(\omega)$$





Problem 4. Given $x(t) = \dfrac{\sin(10t)}{\pi t}$, $y(t)$ is defined as

$$y(t) = j2\pi t x^2(t)$$

$\omega_M = 20$ } since squaring doubles bandwidth

(i) What is the Nyquist sampling rate for $y(t)$? Write answer in this box: $\boxed{\omega_s = 40}$ $= 2\omega_M$

(ii) Define $\tilde{y}(t) = T_s y(t)$. Plot the DTFT of $\tilde{y}[n] = \tilde{y}(nT_s)$, where $T_s = \frac{2\pi}{40/3} = \frac{6\pi}{40}$. Plot $\tilde{Y}(\omega)$ over $-\pi < \omega < \pi$ on the graph provided below.

$$y(t) = j\,2\pi t \,\frac{\sin(10t)}{\pi t}\,\frac{\sin(10t)}{\pi t} = \frac{\sin(10t)}{\pi t}\, 2j \sin(10t)$$

$$Y(\omega) = \text{rect}\left(\frac{\omega - 10}{20}\right) - \text{rect}\left(\frac{\omega + 10}{20}\right)$$

$\omega_M = 20$ mapped to $\omega_M T_s = 20\,\dfrac{6\pi}{40} = 3\pi > \pi$ $\Rightarrow$ aliasing!

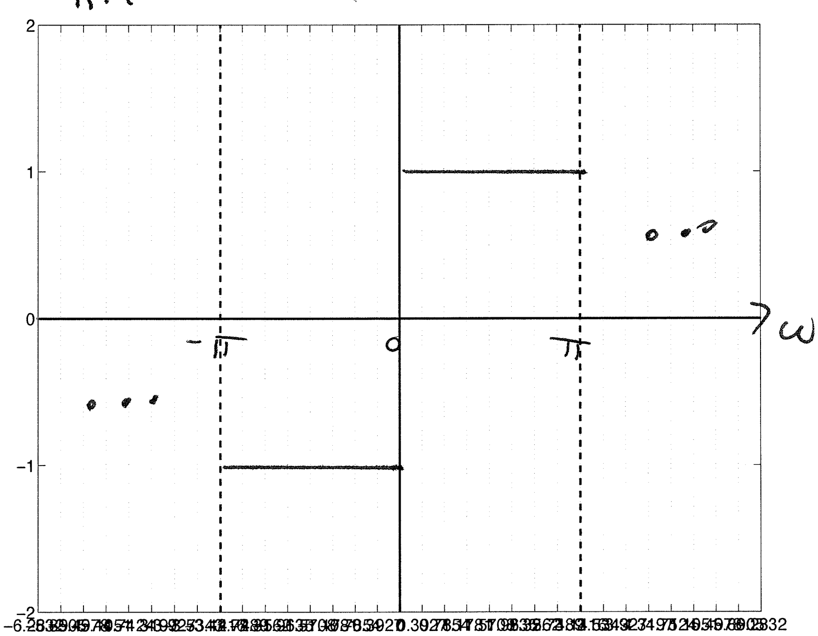
$\omega_s = \dfrac{40}{3} < 40 \Rightarrow$ aliasing

$$y[n] = y\left(n\frac{6\pi}{40}\right) = y\left(n\frac{3\pi}{20}\right) = \frac{\sin\left(10n\frac{3\pi}{20}\right)}{\pi n \frac{3\pi}{20}}\, 2j \sin\left(\frac{3\pi}{2}n\right)$$

$$\sin\left(\frac{3\pi}{2}n - \frac{4\pi n}{2}\right) = \sin\left(-\frac{\pi}{2}n\right) = -\sin\left(\frac{\pi}{2}n\right)$$

$$\tilde{y}[n] = T_s\, y[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} \cdot 2j\sin\left(\frac{\pi}{2}n\right)$$

$$= \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} e^{j\frac{\pi}{2}n} - \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n} e^{-j\frac{\pi}{2}n}$$



Problem 5 (a). You are given that the Fourier Transform of a Gaussian pulse $x(t) = e^{\frac{-t^2}{2}}$ is $X(\omega) = \sqrt{2\pi}\, e^{\frac{-\omega^2}{2}}$. That is,

$$x(t) = e^{\frac{-t^2}{2}} \longleftrightarrow X(\omega) = \sqrt{2\pi}\, e^{\frac{-\omega^2}{2}}$$

Determine the Fourier Transform of

$$y(t) = e^{\frac{-t^2}{2\sigma^2}} = e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2}$$

Write your expression for $Y(\omega)$ in the space directly below:

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$a = \frac{1}{\sigma}$$

$$\Rightarrow \frac{1}{a} = \sigma$$

$$Y(\omega) = \sigma\sqrt{2\pi}\, e^{-\frac{(\sigma\omega)^2}{2}}$$

$$= \sqrt{2\pi}\,\sigma\, e^{-\frac{\sigma^2\omega^2}{2}}$$

Problem 5 (b). You are given the Fourier Transform pair below

$$x(t) = \cos\left(\frac{\pi t}{2}\right) \text{rect}\left(\frac{t}{2}\right) \longleftrightarrow X(\omega) = \frac{4\pi\cos(\omega)}{\pi^2 - 4\omega^2}$$

Determine the Fourier Transform of

$$y(t) = \frac{4\pi\cos(t)}{\pi^2 - 4t^2}$$

Write your expression for $Y(\omega)$ in the space directly below.

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \qquad X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$$

$$Y(\omega) = 2\pi \cos\left(\frac{-\pi\omega}{2}\right) \text{rect}\left(-\frac{\omega}{2}\right)$$

$$= 2\pi \cos\left(\frac{\pi\omega}{2}\right) \text{rect}\left(\frac{\omega}{2}\right)$$