Convolution of Two Gaussian Pulses

$$\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma_3^2}} \qquad \sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

And, again, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}}\,dt = 1$

$$\boxed{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-\frac{\omega^2\sigma^2}{2}}}$$

- We know from Time-Invariance

$$\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(t-t_1)^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(t-t_2)^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{(t-(t_1+t_2))^2}{2\sigma_3^2}}$$

where: $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$ Variances add

two :)

Fourier Transform of Gaussian Pulse

Preliminary Result:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx = 1$$

regardless of value of m

- This is a well Known result from probability theory
- Holds if m is ~~even~~ purely imaginary
- For exam, helpful to use this result:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi}\,\sigma$$

- where independent variable x could be time t or frequency ω

σ = standard deviation

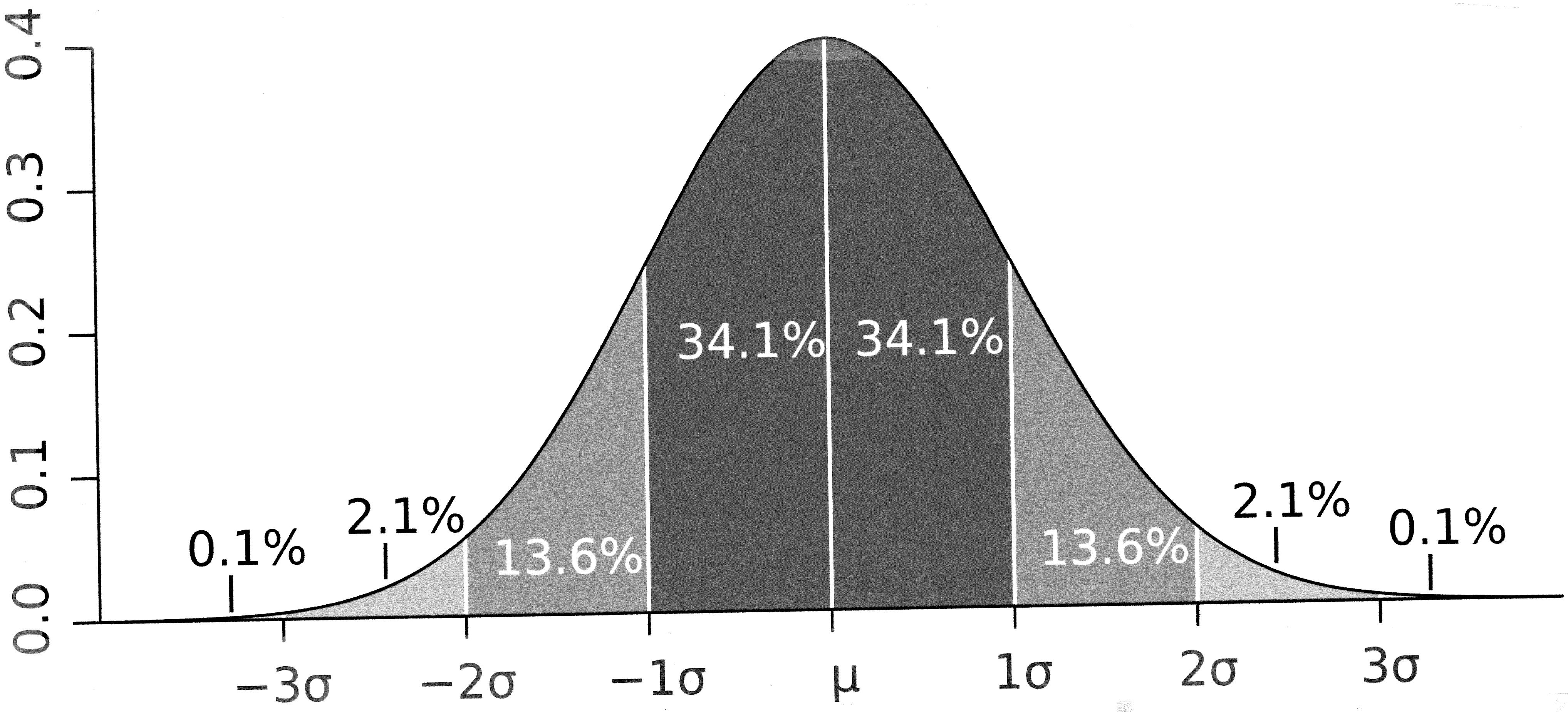
σ^2 = variance

Over 99% of the area is within

$$-3\sigma < x - m < 3\sigma$$

or: $$m - 3\sigma < x < m + 3\sigma$$

m = mean
denoted μ
in diagram below



• **Show:**

$$x(t) = e^{-\frac{t^2}{2}} \xleftrightarrow{\;\mathcal{F}\;} X(\omega) = e^{-\frac{\omega^2}{2}} \sqrt{2\pi}$$

$-\infty < t < \infty$ $\qquad\qquad$ $-\infty < \omega < \infty$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \, e^{-j\omega t} \, dt$$

"complete the square"

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(t^2 + 2j\omega t - \omega^2\right)} \, dt \; e^{-\frac{\omega^2}{2}}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t + j\omega)^2} \, dt}_{\sqrt{2\pi}} \; e^{-\frac{\omega^2}{2}}$$

$$= \sqrt{2\pi} \; e^{-\frac{\omega^2}{2}}$$

from previous page with $\sigma = 1$ and $m = j\omega$

- Recall basic Fourier Transform Property

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- Thus, we have the basic FT pair:

$$\underbrace{e^{-\frac{t^2}{2\sigma^2}}}_{e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2}} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2\sigma^2}{2}}$$

$$\Rightarrow a = \frac{1}{\sigma}$$

- Rearranging:

$$e^{-\frac{t^2}{2\sigma^2}} \overset{\mathcal{F}}{\longleftrightarrow} \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2}{2/\sigma^2}}$$

$-\infty < t < \infty$ $\qquad$ $-\infty < \omega < \infty$

$$\boxed{e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} \sqrt{2\pi}\,\sigma\, e^{-\frac{\omega^2}{2\frac{1}{\sigma^2}}}}$$

std dev $= \sigma$ std dev $= \frac{1}{\sigma}$

- What if you square a Gaussian or multiply two Gaussians?

$$e^{-\frac{t^2}{2\sigma_1^2}}\, e^{-\frac{t^2}{2\sigma_2^2}} = e^{-\frac{t^2}{2}\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)}$$

$$= e^{-\frac{t^2}{2\sigma^2}}$$

- <u>where:</u> $\sigma^2 = \dfrac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$

Product of 2 Gaussians (both zero mean) is another Gaussian with zero mean

Harmonic Mean (like adding 2 resistors in parallel)

Convolution of Two Gaussian Pulses

$$e^{-\frac{t^2}{2\sigma_1^2}} * e^{-\frac{t^2}{2\sigma_2^2}} = ??$$

$$\overset{\mathcal{F}}{\longleftrightarrow} \sqrt{2\pi}\,\sigma_1 \sqrt{2\pi}\,\sigma_2 \, e^{-\frac{\omega^2\sigma_1^2}{2}} e^{-\frac{\omega^2\sigma_2^2}{2}}$$

$$= 2\pi\sigma_1\sigma_2 \frac{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} e^{-\frac{\omega^2}{2}(\sigma_1^2+\sigma_2^2)}$$

Going back to time-domain:

$$\frac{\sqrt{2\pi}\,\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$$

} convolution of two Gaussians is a Gaussian with the sum of the variances

$$e^{-\frac{t^2}{2\sigma_1^2}} * e^{-\frac{t^2}{2\sigma_2^2}} = \frac{\sqrt{2\pi}}{\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}}\, e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$$

. And from time-invariance, we can deduce

$$e^{-\frac{(t-t_1)^2}{2\sigma_1^2}} * e^{-\frac{(t-t_2)^2}{2\sigma_2^2}} =$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}}\, e^{-\frac{(t-(t_1+t_2))^2}{2(\sigma_1^2+\sigma_2^2)}}$$

Energy of a Gaussian Signal:

- If: $x(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$, then:

Ⓐ

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\sigma\sqrt{\pi}}$$

- If: $x(t) = e^{-\frac{t^2}{2\sigma^2}}$, then:

Ⓑ

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \sigma\sqrt{\pi} \quad \left(= \left(\sigma\sqrt{2\pi}\right)^2 \frac{1}{2\sigma\sqrt{\pi}} \right)$$

Proof of Ⓑ: $\boxed{\underline{\text{Recall:}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx = \sqrt{2\pi}\,\sigma}$

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} e^{-\frac{t^2}{\sigma^2}}\,dt = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2(\sigma/\sqrt{2})^2}}\,dt = \sqrt{2\pi}\left(\frac{\sigma}{\sqrt{2}}\right) = \sigma\sqrt{\pi}$$

Another interesting result re: Gaussian Signals

FT Pair

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\frac{\omega^2}{2/\sigma^2}}$$

FT Property

$$t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(\omega)}{d\omega}$$

Thus:

$$\frac{1}{\sigma\sqrt{2\pi}}\, t e^{-\frac{t^2}{2\sigma^2}} \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left\{e^{-\frac{\omega^2}{2/\sigma^2}}\right\} = j\left(-\frac{\sigma^2}{2} 2\omega\right) e^{-\omega^2/2\sigma^2}$$

$$= -j\sigma^2\omega e^{-\frac{\omega^2}{2/\sigma^2}}$$

$\boxed{\text{except for scalars: same functional form in both domains}}$

Consider $\sigma = 1$:

$$t e^{-\frac{t^2}{2}} \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{-j\sqrt{2\pi}}_{\text{scalar}}\,\omega e^{-\frac{\omega^2}{2}}$$