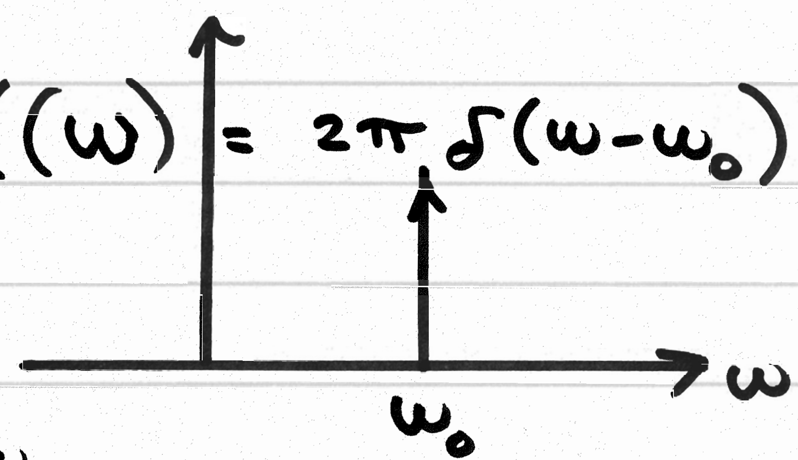
Fourier Transform of a Finite-Length Sinewave

Recall:

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$-\infty < t < \infty$$



- only frequency content is ω_0, so all energy concentrated at $\omega = \omega_0$

- Consider finite-duration sinewave "turned-on" for T secs (WLOG centered at $t=0$)

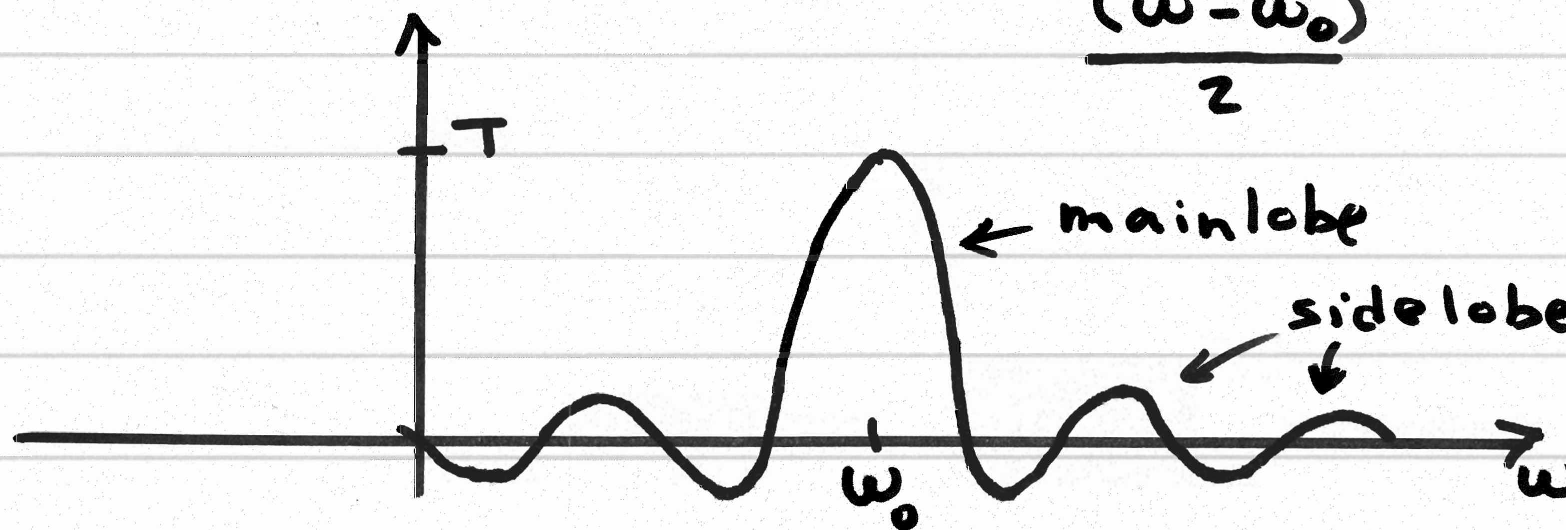
$$y(t) = e^{j\omega_0 t}\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} Y(\omega) = ?$$

Recall: **FT pair:** $\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \dfrac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$

FT property: $x(t)\,e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$

- Thus, the FT of a finite-duration sinewave is

$$e^{j\omega_0 t}\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{\sin\left(T\,\frac{(\omega-\omega_0)}{2}\right)}{\frac{(\omega-\omega_0)}{2}}$$



- zero crossings at $\omega = \omega_0 + m\frac{2\pi}{T}$ $m \to$ integer, $m \neq 0$, $-\infty < m < \infty$

- a finite-duration sinewave has energy over a continuum of frequencies, although something like 88% of the energy is in $\omega_0 - \frac{2\pi}{T} < \omega < \omega_0 + \frac{2\pi}{T}$

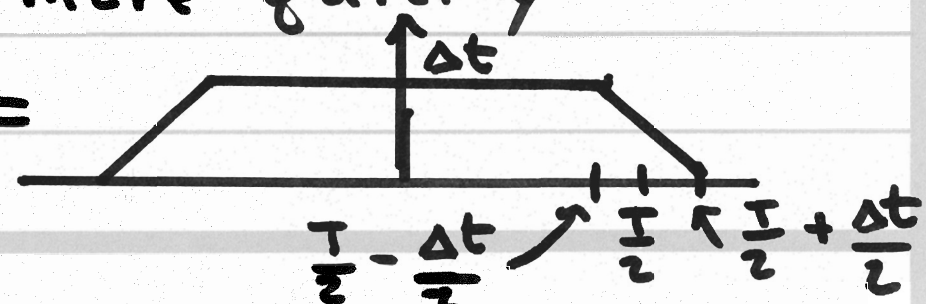
• **Note:** as $T \to \infty$:

$$\lim_{T\to\infty} e^{j\omega_0 t}\,\text{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \lim_{T\to\infty} \frac{\sin\left(T\frac{(\omega-\omega_0)}{2}\right)}{\frac{(\omega-\omega_0)}{2}}$$

$$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega-\omega_0)$$

Typically, in practice, we taper at both end - i.e., have the sinewave turn on "gradually" rather than instantaneously (and turn off "gradually") so that there is more energy in the mainlobe and the sidelobes decay more quickly

• **Recall:** $\text{rect}\left(\frac{t}{T}\right) * \text{rect}\left(\frac{t}{\Delta t}\right) =$



Thus:

$$e^{j\omega_0 t}\left(\text{rect}\left(\frac{t}{T}\right) * \frac{1}{\Delta t}\,\text{rect}\left(\frac{t}{\Delta t}\right)\right) \xleftrightarrow{\mathcal{F}} Z(\omega-\omega_0)$$

<u>where:</u>

$$Z(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\omega/2} \cdot \frac{1}{\Delta t}\,\frac{\sin\left(\Delta t\frac{\omega}{2}\right)}{\omega/2}$$

- notice: ω^2 in denominator

$\Rightarrow$ sidelobes decay more quickly

- Fourier Transform of a Periodic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta\left(\omega - k\frac{2\pi}{T}\right)$$

- Alternative method for computing Fourier Series Coefficients (related to Prob. 4.27 on Hmwk7)

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} \left\{x(t)\,\mathrm{rect}\left(\frac{t}{T}\right)\right\} e^{-j\omega t}\, dt \Bigg|_{\omega = k\frac{2\pi}{T}}$$

$$= \frac{1}{T}\mathcal{F}\left\{\begin{matrix}\text{one period}\\ \text{of } x(t)\end{matrix}\right\}\Bigg|_{\omega = k\frac{2\pi}{T}}$$

- As a check, we previously determined FS coefficients for periodic train of rectangular pulses

$$a_k = \frac{\sin\left(k\pi \frac{\tau}{T}\right)}{k\pi} \qquad \text{for } x(t) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t-nT}{\tau}\right)$$

- One period is $\text{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \dfrac{\sin\left(\tau \frac{\omega}{2}\right)}{\omega/2}$

$$a_k = \frac{1}{T} \left. \frac{\sin\left(\tau \frac{\omega}{2}\right)}{\omega/2} \right|_{\omega = k\frac{2\pi}{T}} = \frac{1}{T} \frac{\sin\left(\tau \frac{1}{2} k \frac{2\pi}{T}\right)}{\frac{1}{2} k \frac{2\pi}{T}}$$

$$= \frac{\sin\left(k\pi \frac{\tau}{T}\right)}{k\pi}$$

checks!

- Suppose signal is only periodic for N periods

- $y(t) = x(t)\, \mathrm{rect}\left(\frac{t}{NT}\right)$

where:
$x(t) = x(t+T)$
for all t

- Using Multiplication Property of FT:

$$Y(\omega) = \frac{1}{2\pi} X(\omega) * \frac{\sin\left(NT\frac{\omega}{2}\right)}{\omega/2}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k\, 2\pi\, \delta\left(\omega - k\frac{2\pi}{T}\right) * \frac{\sin\left(NT\frac{\omega}{2}\right)}{\omega/2}$$

$$= \sum_{k=-\infty}^{\infty} a_k \frac{\sin\left(\frac{NT}{2}\left(\omega - k\frac{2\pi}{T}\right)\right)}{\frac{1}{2}\left(\omega - k\frac{2\pi}{T}\right)}$$

- See vowel example in Matlab