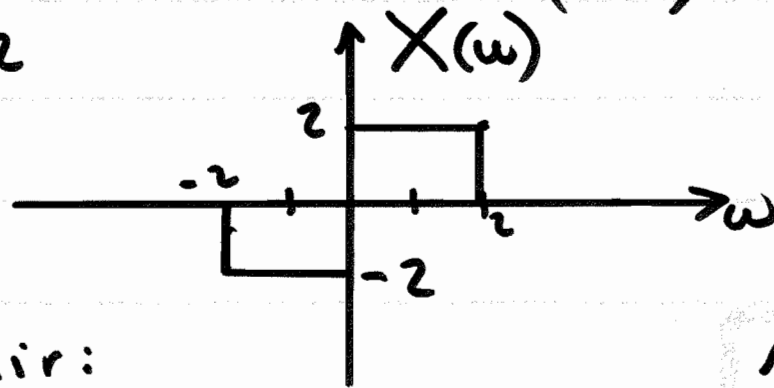


# Example Problems on Fourier Transform (FT)

## Using FT Pairs and FT Properties

Prob. 4.4 (Hmwk) Part (b)

$$X(\omega) = \begin{cases} 2, & 0 \leq \omega \leq 2 \\ -2, & -2 \leq \omega < 0 \\ 0, & |\omega| > 2 \end{cases} = -2 \operatorname{rect}\left(\frac{\omega+1}{2}\right) + 2 \operatorname{rect}\left(\frac{\omega-1}{2}\right)$$

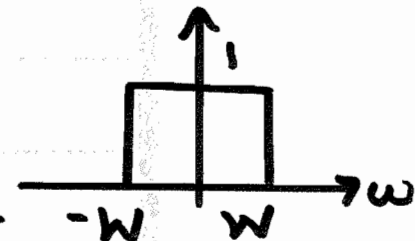


Recall basic FT pair:

$$\frac{\sin(\omega t)}{\pi t}$$

$\longleftrightarrow$

$$\operatorname{rect}\left(\frac{\omega}{2W}\right)$$



Consider  $W=1$

Also, recall FT property:  $e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$   
 consider  $\omega_0 = 1, \omega_0 = -1$

• also, using linearity property of FT, we have: ②

$$x(t) = -2 \frac{\sin(t)}{\pi t} e^{-jt} + 2 \frac{\sin(t)}{\pi t} e^{jt}$$


$$= \frac{\sin(t)}{\pi t} 4j \left\{ \frac{e^{jt} - e^{-jt}}{2j} \right\}$$

$$= 4j \frac{\sin(t)}{\pi t} \sin(t) = 4j \frac{\sin^2(t)}{\pi t}$$

Prob. 4.21 Hmwk 6: a, b, c, e, g

Do part (c) first:

$$x(t) = \begin{cases} 1 + \cos(\pi t), & |t| < 1 \\ 0, & |t| > 1 \end{cases} = (1 + \cos(\pi t)) \text{rect}\left(\frac{t}{2}\right)$$

• Basic FT pair:   $\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{\sin\left(\omega \frac{T}{2}\right)}{\frac{\omega}{2}}$  (3)

• Basic FT property:  $e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$

leads to:

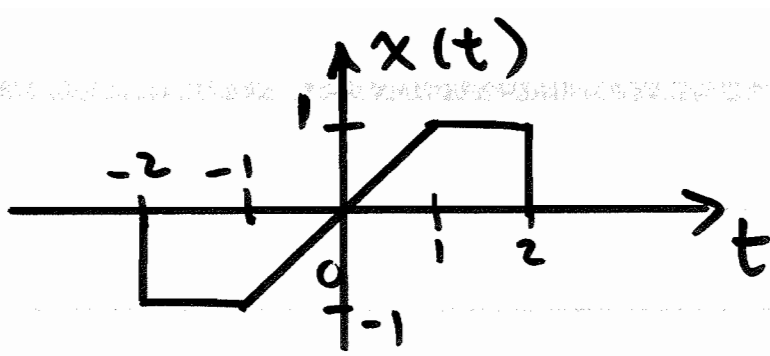
$$x(t) \cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$$

since  $\cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$  and linearity prop. of FT

Consider  $T=2$  and  $\omega_0 = \pi$  and  $\omega_0 = 0$ :

$$\begin{aligned} X(\omega) &= \frac{\sin(\omega)}{\omega/2} + \frac{1}{2} \frac{\sin(\omega + \pi)}{\frac{(\omega + \pi)}{2}} + \frac{1}{2} \frac{\sin(\omega - \pi)}{\frac{(\omega - \pi)}{2}} \\ &= \sin(\omega) \left\{ \frac{2}{\omega} - \frac{1}{\omega + \pi} - \frac{1}{\omega - \pi} \right\} \\ &= \sin(\omega) \left\{ \frac{2}{\omega} - \frac{2\omega}{\omega^2 - \pi^2} \right\} = 2 \sin(\omega) \left\{ \frac{1}{\omega} - \frac{\omega}{\omega^2 - \pi^2} \right\} \end{aligned}$$

Prob. 4.21 (g)



(4)

$$x(t) = -\text{rect}\left(\frac{t + \frac{3}{2}}{1}\right) + t \text{rect}\left(\frac{t}{2}\right) + \text{rect}\left(\frac{t - \frac{3}{2}}{1}\right)$$

$$\text{FT pair: } \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{\sin\left(\frac{T\omega}{2}\right)}{\omega/2}$$

$$\text{FT property: } x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$$

$$t x(t) \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} X(\omega)$$

$$\begin{aligned} X(\omega) &= -\frac{\sin\left(\frac{3\omega}{2}\right)}{\omega/2} e^{j\frac{3\omega}{2}} + j \frac{d}{d\omega} \left\{ \frac{\sin(\omega)}{\omega/2} \right\} + \frac{\sin\left(\frac{3\omega}{2}\right)}{\omega/2} e^{j\frac{3\omega}{2}} \\ &= -4j \frac{\sin\left(\frac{3\omega}{2}\right)}{\omega} \sin\left(\frac{3\omega}{2}\right) + j2 \left\{ \frac{\cos(\omega)\omega - \sin(\omega)}{\omega^2} \right\} \end{aligned}$$

• Other variation on modulation property: (5)

$$x(t) \sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j} X(\omega - \omega_0) - \frac{1}{2j} X(\omega + \omega_0)$$

Use for Probs. 4.21 (b) and (e)

$$4.21 (b) \quad e^{-3|t|} \sin(2t) \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

FT pair from Ex. 4.2 in text on pg. 292:

$$e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{\omega^2 + a^2}$$

THUS:

$$\begin{aligned} X(\omega) &= \frac{1}{2j} \frac{2(3)}{(\omega+2)^2 + 3^2} - \frac{1}{2j} \frac{2(3)}{(\omega-2)^2 + 3^2} \\ &= -3j \left\{ \frac{1}{(\omega+2)^2 + 9} - \frac{1}{(\omega-2)^2 + 9} \right\} \end{aligned}$$

Prob. 4.38 We previously proved modulation property as requested in part (a):

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega - \omega_0) = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt$$

$$= \int_{-\infty}^{\infty} (x(t) e^{j\omega_0 t}) e^{-j\omega t} dt$$

Since FT is unique, this proves:

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$$

Part (b): prove same property using multiplication property:  $x(t) y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$

$$\text{So, let: } y(t) = e^{j\omega_0 t} \xleftrightarrow{\widehat{F}} 2\pi \delta(\omega - \omega_0) \quad (7)$$

Thus:

$$\begin{aligned} x(t) e^{j\omega_0 t} &\xleftrightarrow{\widehat{F}} \frac{1}{2\pi} X(\omega) * 2\pi \delta(\omega - \omega_0) \\ &= X(\omega) * \delta(\omega - \omega_0) \\ &= X(\omega - \omega_0) \end{aligned}$$

Alternative proof of modulation property.

• On an exam, simply use modulation property

$$x(t) \cos(\omega_0 t) \xleftrightarrow{\widehat{F}} \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$$

• no need to use multiplication property  
if multiplying by a sinewave

Prob. 4.39 See proof of Duality Property (8)  
posted at course web site under "Chapter 4"

$$\text{If: } x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

$$\text{Then } X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$$

$$\text{For example: } \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{\sin\left(T\frac{\omega}{2}\right)}{\omega/2}$$

$$\text{Thus: } \frac{\sin\left(T\frac{t}{2}\right)}{t/2} \xleftrightarrow{\mathcal{F}} 2\pi \text{rect}\left(\frac{-\omega}{T}\right)$$

We previously showed with  $T=2W$ :

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{\mathcal{F}} \text{rect}\left(\frac{\omega}{2W}\right)$$



• For part (b), an alternative derivation of the FT of a sine wave  $e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0)$  using Duality Property and basic FT pair

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$

Time-Shift Property dictates:

$$\delta(t+B) \xleftrightarrow{\mathcal{F}} e^{jB\omega} (1) = e^{jB\omega}$$

• Applying Duality Property:

$$e^{jBt} \xleftrightarrow{\mathcal{F}} 2\pi\delta(-\omega + B)$$

$$\delta(-t) = \delta(t)$$

• note: Dirac Delta fn. is a symmetric fn.:  $\delta(-\omega) = \delta(\omega)$

• Thus:

$$e^{jBt} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - B)$$

• which we had proved previously ( $\omega_0 = B$ )

• For Hmwk. 7 (later), Prob. 4.12 uses Duality to find the FT of  $x(t) = \frac{2a}{t^2+a^2}$   $a = \text{const.}$

$$X(\omega) = \int_{-\infty}^{\infty} \left( \frac{2a}{t^2+a^2} \right) e^{-j\omega t} dt = ? \quad \text{hard integral to do!}$$

• instead, employ duality:

Since:  $e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{\omega^2+a^2}$

Then:  $\frac{2a}{t^2+a^2} \xleftrightarrow{\mathcal{F}} 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$

Divide by 2 on both sides

$$\frac{a}{t^2+a^2} \longleftrightarrow \pi e^{-a|\omega|}$$

Prob. 4.23 exercises on even and odd part properties of FT which are proved in a separate handout posted at course web site

• For  $x(t)$  real-valued, if  $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$  then: even part of  $x(t)$ .

$$x_e(t) = \frac{x(t) + x(-t)}{2} \xleftrightarrow{\mathcal{F}} \text{Re}\{X(\omega)\} = \text{real part of } X(\omega)$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} \xleftrightarrow{\mathcal{F}} j\text{Im}\{X(\omega)\} = \text{imaginary part of } X(\omega)$$

Note: for  $x(t)$  real-valued:  $X(-\omega) = X^*(\omega)$   
 $\Rightarrow |X(-\omega)| = |X(\omega)| \Rightarrow$  magnitude is even fn. of  $\omega$   
 $\Rightarrow \angle X(-\omega) = -\angle X(\omega) \Rightarrow$  phase is odd fn. of  $\omega$

$X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$   
polar form

Prob. 4.23 (cont.)  $x_0(t) = \begin{cases} e^{-t}, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

$$x_0(t) = e^{-t} \{u(t) - u(t-1)\}$$

$$= e^{-t} u(t) - e^{-1} e^{-(t-1)} u(t-1)$$

Since:  $e^{-at} u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$   $x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$

We have:  $X_0(\omega) = \frac{1}{1+j\omega} (1 - e^{-1} e^{-j\omega})$

$$= \frac{1 - e^{-1} (\cos\omega - j \sin\omega)}{1 + j\omega} \cdot \frac{(1 - j\omega)}{(1 - j\omega)}$$

$$= \underbrace{\frac{2 - 2e^{-1} \cos(\omega) - 2e^{-1} \omega \sin(\omega)}{1 + \omega^2}}_{\text{Re}\{X_0(\omega)\}} + j \underbrace{\left\{ \frac{-2\omega + 2e^{-1} \sin(\omega) + 2e^{-1} \omega \cos(\omega)}{1 + \omega^2} \right\}}_{\text{Im}\{X_0(\omega)\}}$$

## Example. Prob. 4.21 (d)

$$x(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - kT) \xleftrightarrow{\mathcal{F}} X(\omega) = ?$$

Basic FT Pair:  $\delta(t) \xleftrightarrow{\mathcal{F}} 1$

Basic FT Property:  $x(t - t_0) \xleftrightarrow{\mathcal{F}} X(\omega) e^{-j\omega t_0}$

Thus:  $\delta(t - kT) \xleftrightarrow{\mathcal{F}} 1 \cdot e^{-j\omega kT}$   
 $= (e^{-j\omega T})^k$

Since FT satisfies linearity property

$$X(\omega) = \sum_{k=0}^{\infty} \alpha^k e^{-j\omega kT} = \sum_{k=0}^{\infty} (\alpha e^{-j\omega T})^k$$

$$= \frac{1}{1 - \alpha e^{-j\omega T}} \quad \text{since } |\alpha| < 1$$

## Example. Prob. 4.21 (i)

$$x(t) = \begin{cases} 1-t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} = (1-t^2) \text{rect}\left(\frac{t-\frac{1}{2}}{1}\right)$$

$$= \text{rect}\left(\frac{t-\frac{1}{2}}{1}\right) - t^2 \text{rect}\left(\frac{t-\frac{1}{2}}{1}\right)$$

Basic FT pair:  $\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \frac{\sin\left(\frac{\omega}{2}T\right)}{\omega/2}$  ( $T=1$  here)

Two FT properties:

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(\omega) e^{-j\omega t_0} \quad \text{and} \quad t x(t) \xleftrightarrow{\mathcal{F}} j \frac{dX(\omega)}{d\omega}$$

Generalization:  $t^n x(t) \xleftrightarrow{\mathcal{F}} (j)^n \frac{d^n X(\omega)}{d\omega^n}$

Thus:

$$X(\omega) = \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} e^{-j\omega/2} - (-1) \frac{d^2}{d\omega^2} \left\{ \frac{\sin\left(\frac{\omega}{2}\right) e^{-j\omega/2}}{\omega/2} \right\}$$

Look at term to be differentiated:

$$\sin\left(\frac{\omega}{2}\right) = \frac{1}{2j} e^{j\frac{\omega}{2}} - \frac{1}{2j} e^{-j\frac{\omega}{2}}$$

Thus, we have:

$$\begin{aligned} & + j \frac{d^2}{d\omega^2} \left\{ \frac{1 - e^{j\omega}}{\omega} \right\} = j \frac{d^2}{d\omega^2} (\omega^{-1}) + j \frac{d^2}{d\omega^2} \left\{ \frac{e^{j\omega}}{\omega} \right\} \\ & = -j \omega^{-3} + j \frac{d}{d\omega} \left\{ \frac{-j \omega e^{j\omega} - e^{j\omega}}{\omega^2} \right\} \\ & = -j \omega^{-3} + \frac{d}{d\omega} \left\{ \frac{e^{j\omega}}{\omega^2} \right\} - j \frac{d}{d\omega} \left\{ \frac{e^{j\omega}}{\omega^2} \right\} \\ & = -j \omega^{-3} + \left\{ \frac{\omega e^{j\omega} - e^{j\omega}}{\omega} \right\} - j \left\{ \frac{\omega^2 e^{j\omega} - 2\omega e^{j\omega}}{\omega^3} \right\} \end{aligned}$$

Still some sign errors and j's missing in last line - you can fix :)