Name: ______________________ **Exam 3**

ECE301 Signals and Systems Thursday, April 18, 2019

Cover Sheet

Write your name on this page and every page to be safe.

Test Duration: 80 minutes.

Coverage: Comprehensive

Open Book but Closed Notes. Three two-sided handwritten sheets.

Calculators NOT allowed.

This test contains **two** problems, each with multiple parts.

You have to draw your own plots.

You must show all work for each problem to receive full credit.

Problem 1 Consider an analog signal $x_a(t)$ below with maximum frequency $\omega_M = 40$ rads/sec: the Fourier Transform, $X_a(\omega)$, of $x_a(t)$ is exactly zero for $|\omega| > 40$ rads/sec.

$$x_a(t) = \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t} \right\} 2j \sin(20t)$$

This is the same signal that is sampled for each and every part of entire Problem 1.

VIP: For EACH part, you are required to plot the DTFT of $h(t)$ which is the lowpass filter impulse response that is used for interpolation and determine (and state) whether it is flat over the band $-\omega_M < \omega < \omega_M$ and whether it filters out all the spectral replicas outside of the band $-\omega_M < \omega < \omega_M$, where $\omega_M = 40$ rads/sec for all parts.

Problem 1 (a). The signal $x_a(t)$ is sampled at a rate $\omega_s = 100$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{100}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{ \frac{\pi}{5} \frac{\sin\left(\frac{\pi}{10}n\right)}{\pi n T_s} \frac{\sin\left(\frac{3\pi}{10}n\right)}{\pi n T_s} \right\} 2j \sin\left(\frac{2\pi}{5}n\right) \quad \text{where:} \quad T_s = \frac{2\pi}{100}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{100} \quad \text{and} \quad h(t) = T_s \frac{\pi}{10} \frac{\sin(10t)}{\pi t} \frac{\sin(50t)}{\pi t}$$

Problem 1 (b). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 100$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{100} \quad \text{and} \quad h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(40t)}{\pi t} + \frac{\sin(60t)}{\pi t} \right\}$$

Problem 1 (c). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 100$ rads/sec., but is reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \quad \text{where:} \quad T_s = \frac{2\pi}{100} \quad \text{and} \quad h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(45t)}{\pi t} + \frac{\sin(55t)}{\pi t} \right\}$$

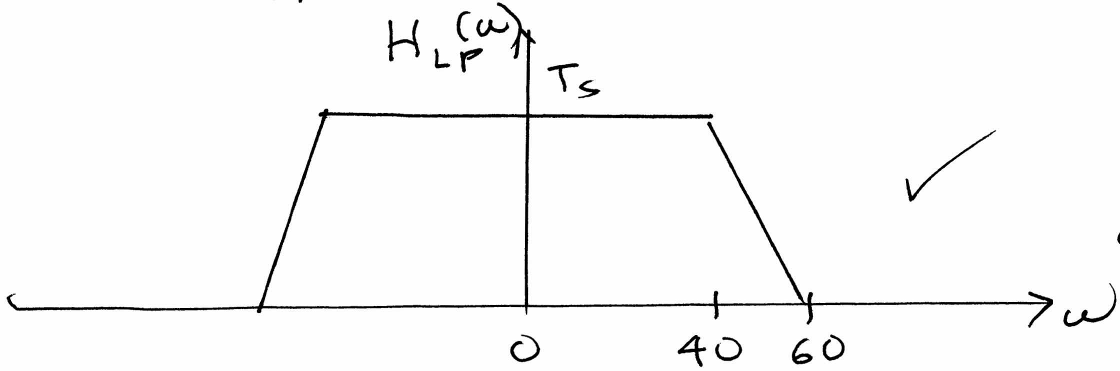
Show all your work for Prob. 1, parts (a)-(b)-(c) .

For parts (a) thru (c), sampling above Nyquist Rate

$\omega_s = 100 > 2\omega_M = 2(40) = 80$ ✓

- So, just need to check 2 conditions on lowpass filter
  - is $H_{LP}(\omega) = T_s$ over $-40 < \omega < 40$?
  - is $H_{LP}(\omega) = 0$ for $|\omega| > \omega_s - \omega_M = 100 - 40 = 60$?

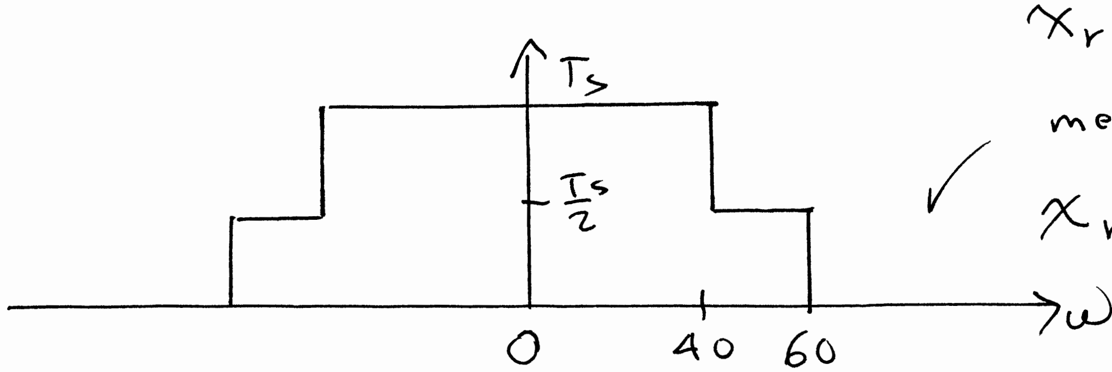
(a)



✓ meets both conditions

$x_r(t) = x_a(t)$

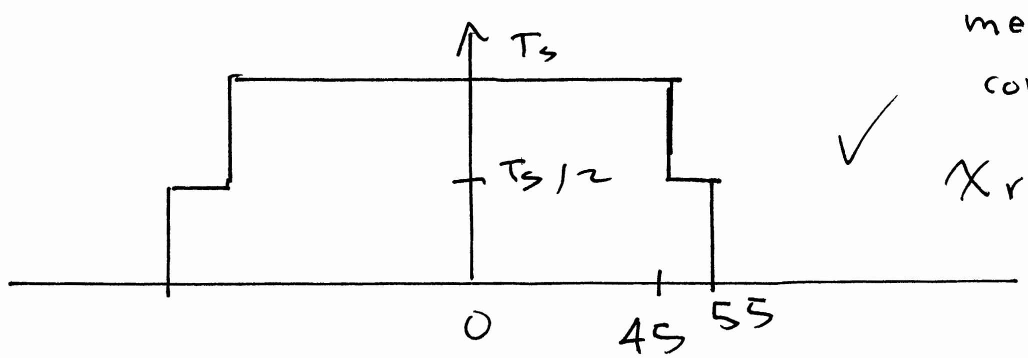
(b)



✓ meets both conditions

$x_r(t) = x_a(t)$

(c)



✓ meets both conditions

$x_r(t) = x_a(t)$

Problem 1 (d). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. That is, the Fourier Transform of the analog signal $x_a(t)$ is exactly zero for $|\omega| > 40$ rads/sec. This signal is sampled at a rate $\omega_s = 80$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{80}$ sec. This yields the discrete-time sequence

$$x[n] = x_a(nT_s) = \left\{ \frac{\pi}{5} \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n T_s} \frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n T_s} \right\} 2j \sin\left(\frac{\pi}{2}n\right) \qquad \text{where: } \; T_s = \frac{2\pi}{80}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. Show work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where: } \; T_s = \frac{2\pi}{80} \quad \text{and} \quad h(t) = T_s \frac{\sin(40t)}{\pi t}$$

Problem 1 (e). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 80$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where: } \; T_s = \frac{2\pi}{80} \quad \text{and} \quad h(t) = T_s \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(40t)}{\pi t} \right\}$$

Problem 1 (f). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 80$ rads/sec., but reconstructed with a different lowpass interpolating filter according to the formula below. Does this achieve perfect reconstruction, that is, does $x_r(t) = x_a(t)$? **For this part, you do not need to determine $x_r(t)$, just need to explain whether $x_r(t) = x_a(t)$ or not.**

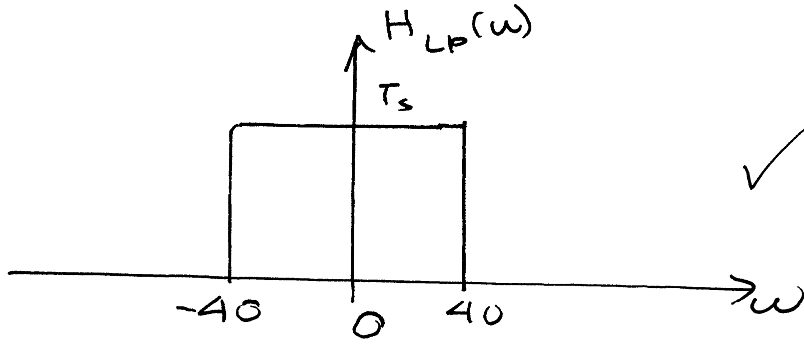
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h(t - nT_s) \qquad \text{where: } \; T_s = \frac{2\pi}{80} \quad \text{and} \quad h(t) = T_s \frac{1}{2} \left\{ \frac{\sin(40t)}{\pi t} + \frac{\sin(10t)}{\pi t} \right\}$$

Show all your work for Prob. 1, parts (d)-(e)-(f).

Show your work for Prob. 1, parts (d)-(e)-(f).

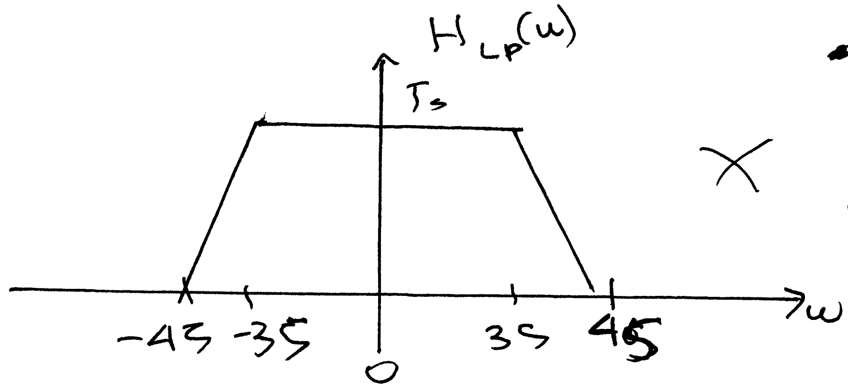
For parts (d), (e), (f), we sample right at Nyquist Rate ⇒ only ILPF in part (d) will work

(d)



✓ $x_r(t) = x_a(t)$

(e)

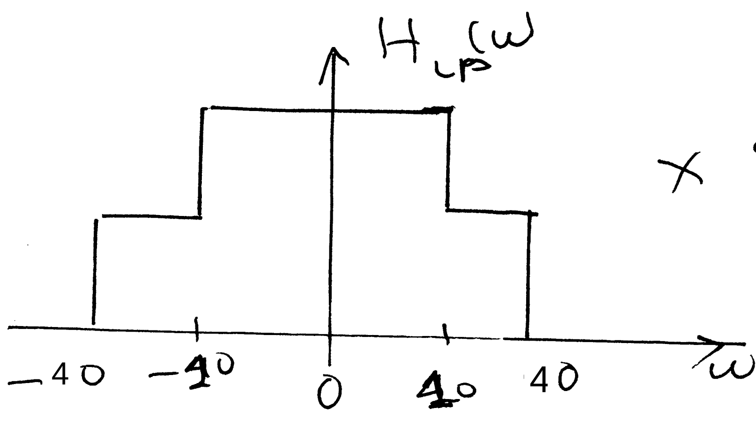


✗

- not flat over $-40 < \omega < 40$
- not 0 for $|\omega| > \omega_s - \omega_M = 40$

$x_r(t) \neq x_a(t)$

(f)



✗

- not flat over $-40 < \omega < 40$ ✗
- is 0 for $|\omega| > 40$ ✓

$x_r(t) \neq x_a(t)$

Problem 1 (g). Consider an analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at a rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ such the time between samples is $T_s = \frac{2\pi}{50}$ sec, yielding the following discrete-time sequence:

$$x[n] = x_a(nT_s) = \left\{ \frac{\pi}{5} \frac{\sin\left(\frac{\pi}{5}n\right)}{\pi n T_s} \frac{\sin\left(\frac{3\pi}{5}n\right)}{\pi n T_s} \right\} 2j \sin\left(\frac{4\pi}{5}n\right) \qquad \text{where: } T_s = \frac{2\pi}{50}$$

A reconstructed signal is formed from the samples above according to the formula below. Determine a closed-form expression for the reconstructed signal $x_r(t)$. Show all work.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h\left(t - nT_s\right) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s \frac{\sin(25t)}{\pi t}$$

Problem 1 (h). Consider the SAME analog signal $x_a(t)$ with maximum frequency (bandwidth) $\omega_M = 40$ rads/sec. This signal is sampled at the same rate $\omega_s = 50$ rads/sec., where $\omega_s = 2\pi/T_s$ and the time between samples is $T_s = \frac{2\pi}{50}$ sec, but at a different starting point. This yields the Discrete-Time $x[n]$ signal below:

$$x_\epsilon[n] = x_a(nT_s + 0.5T_s) = \left\{ \frac{\pi}{5} \right\} \left\{ \frac{\sin(\frac{\pi}{5}(n+0.5))}{\pi(n+0.5)T_s} \frac{\sin(\frac{3\pi}{5}(n+0.5))}{\pi(n+0.5)T_s} \right\} 2j \sin\left(\frac{4\pi}{5}(n+0.5)\right)$$

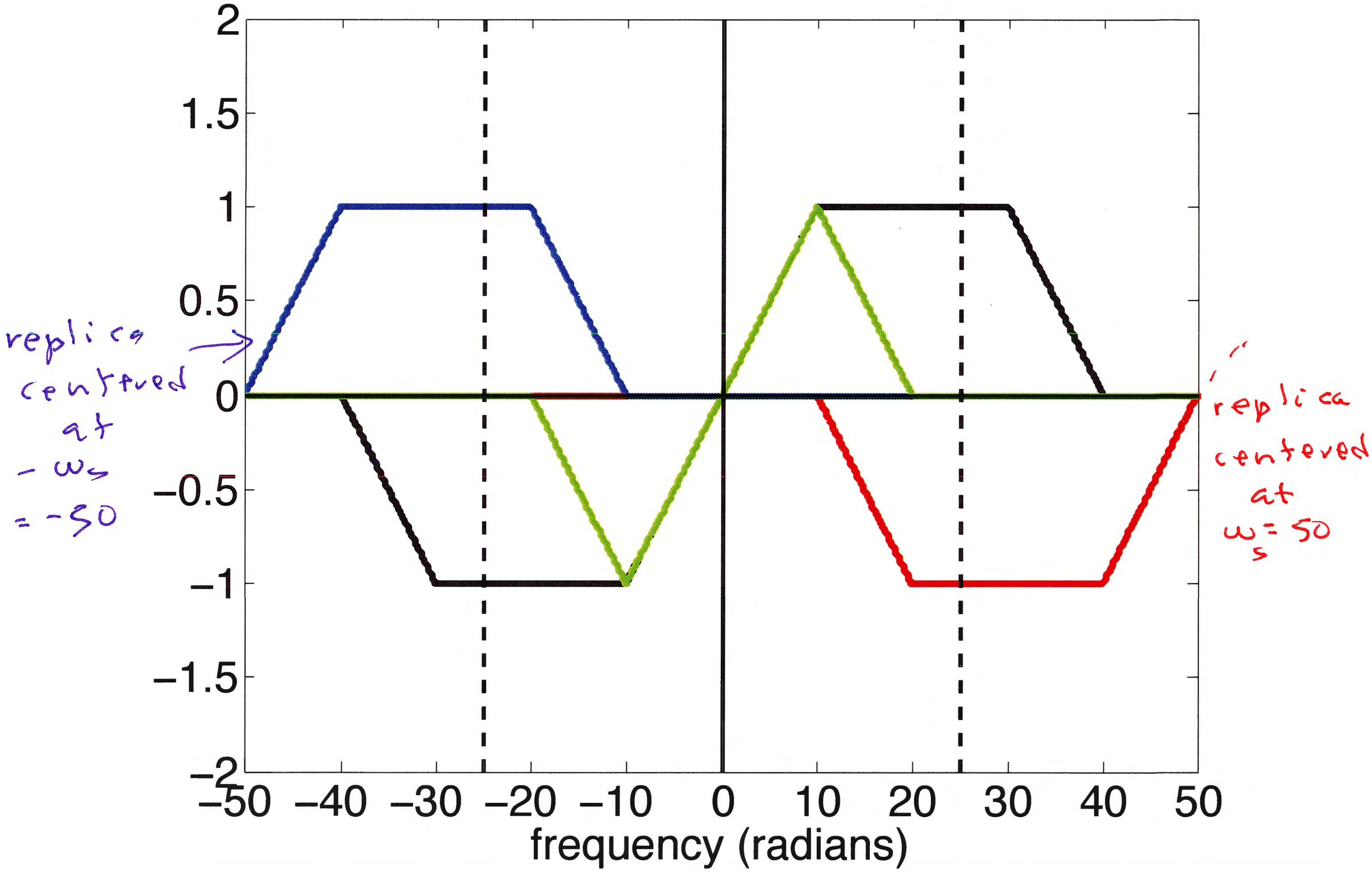
A reconstructed signal is formed from the samples above according to the formula below. Determine a simple, closed-form expression for the reconstructed signal $x_r(t)$. *Hint:* before you do a lot of work, look at the interpolating lowpass filter being used below.

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_\epsilon[n] h\left(t - (n+0.5)T_s\right) \qquad \text{where: } T_s = \frac{2\pi}{50} \quad \text{and} \quad h(t) = T_s \frac{\sin(25t)}{\pi t}$$

Show your work for Prob. 1, parts (g)-(h) .

Part (g)

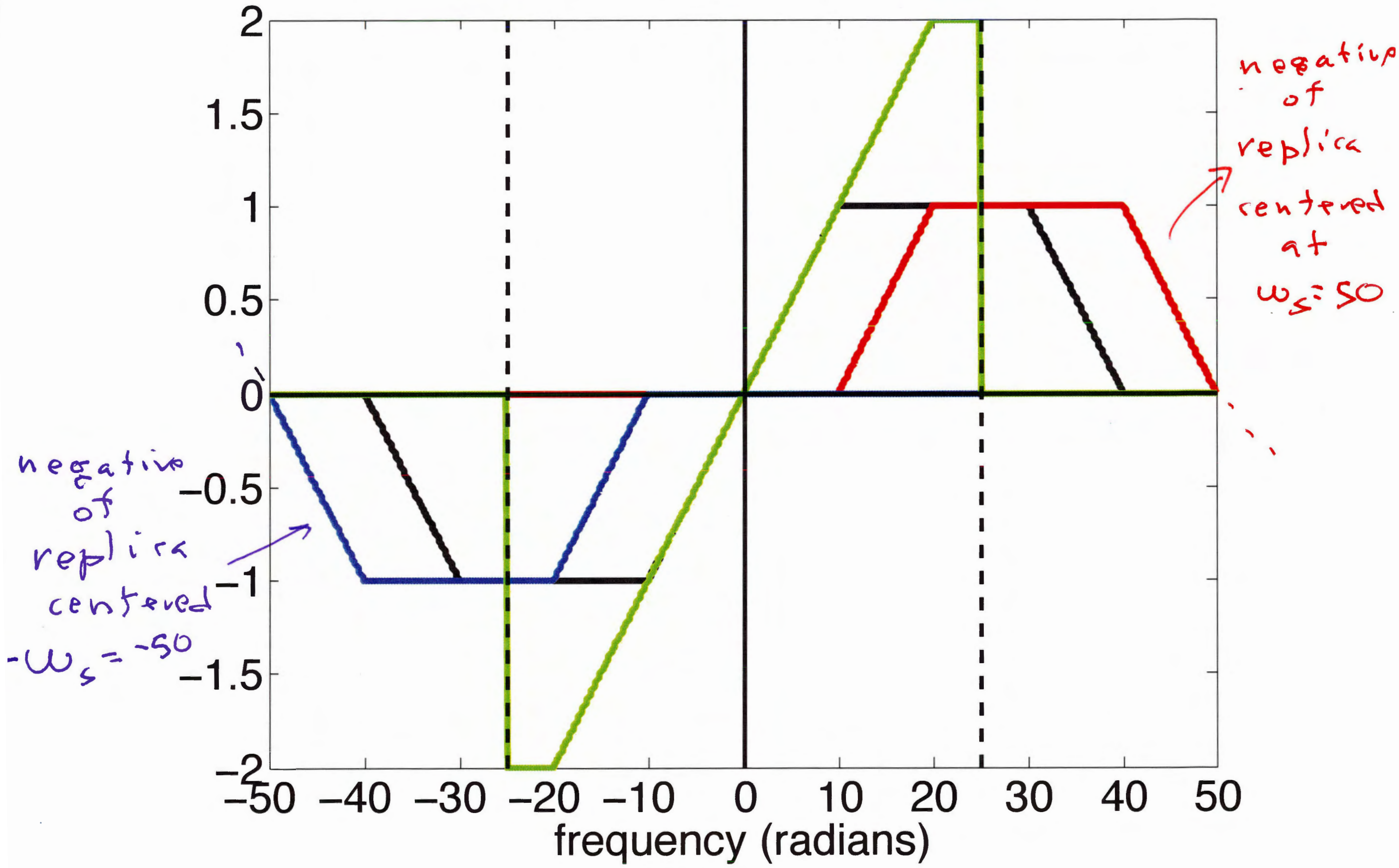
black = original Fourier Transform



$$x_r(t) = \frac{\pi}{5}\left\{\frac{\sin(5t)}{\pi t}\right\}^2 2j\sin(10t)$$

Part (h)

black = original Fourier Transform



$$x_r(t) = \frac{2}{20}\frac{1}{j}\frac{d}{dt}\left\{\frac{\sin(20t)}{\pi t}\right\}$$

$$+ 2\left\{\frac{\sin(2.5t)}{\pi t}\right\} 2j \sin(22.5t)$$

Problem 2. Consider the input signal $x_0(t)$ below.

$$x_0(t) = e^{-j32t}+e^{-j28t}+e^{-j24t}+e^{-j16t}+e^{-j12t}+e^{-j8t}+e^{-j4t}+1+e^{j4t}+e^{j8t}+e^{j12t}+e^{j16t}+e^{j24t}+e^{j28t}+e^{j32t}$$

This signal is first input to an analog filter with impulse response

$$h_{LP}(t) = 3\frac{\sin(5t)}{\pi t}2j\sin(5t) + 2\frac{\sin(5t)}{\pi t}2j\sin(15t) + \frac{\sin(5t)}{\pi t}2j\sin(25t)$$

to form $x(t) = x_0(t) * h_{LP}(t)$, and then $x(t)$ is sampled at a rate of $\omega_s = 64$ to form $x[n]$, so that the time between samples is $T_s = \frac{2\pi}{64}$. The DT signal $x[n]$ thus obtained is then input to a DT LTI system with impulse response

$$h[n] = 16\left\{\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right\}^2 2j\sin\left(\frac{\pi}{2}n\right) \qquad (1)$$

Show all work. Write your expression for the output $y[n] = x[n] * h[n]$ in the space below. Plot both the Fourier Transform of $h_{LP}(t)$ and the DTFT of $h[n]$ to help solve the problem, but since the input is a sum of sinewaves, it is not necessary to plot the Fourier Transform of $x_0(t)$ or the DTFT of the sampled signal $x[n]$.

$x(t) =$

$0 \cdot e^{-j32t} + 0 \cdot e^{j32t}$ $(k=8)$

$-1 \cdot e^{-j28t} + 1 \cdot e^{j28t}$ $(k=7)$

$-1\, e^{-j24t} + 1 \cdot e^{j24t}$ $(k=6)$

$-2\, e^{-j16t} + 2\, e^{j16t}$ $(k=4)$

$-2\, e^{-j12t} + 2\, e^{j12t}$ $(k=3)$

$-3\, e^{-j8t} + 3\, e^{j8t}$ $(k=2)$

$-3\, e^{-j4t} + 3\, e^{j4t}$ $(k=1)$

$0 \cdot 1$

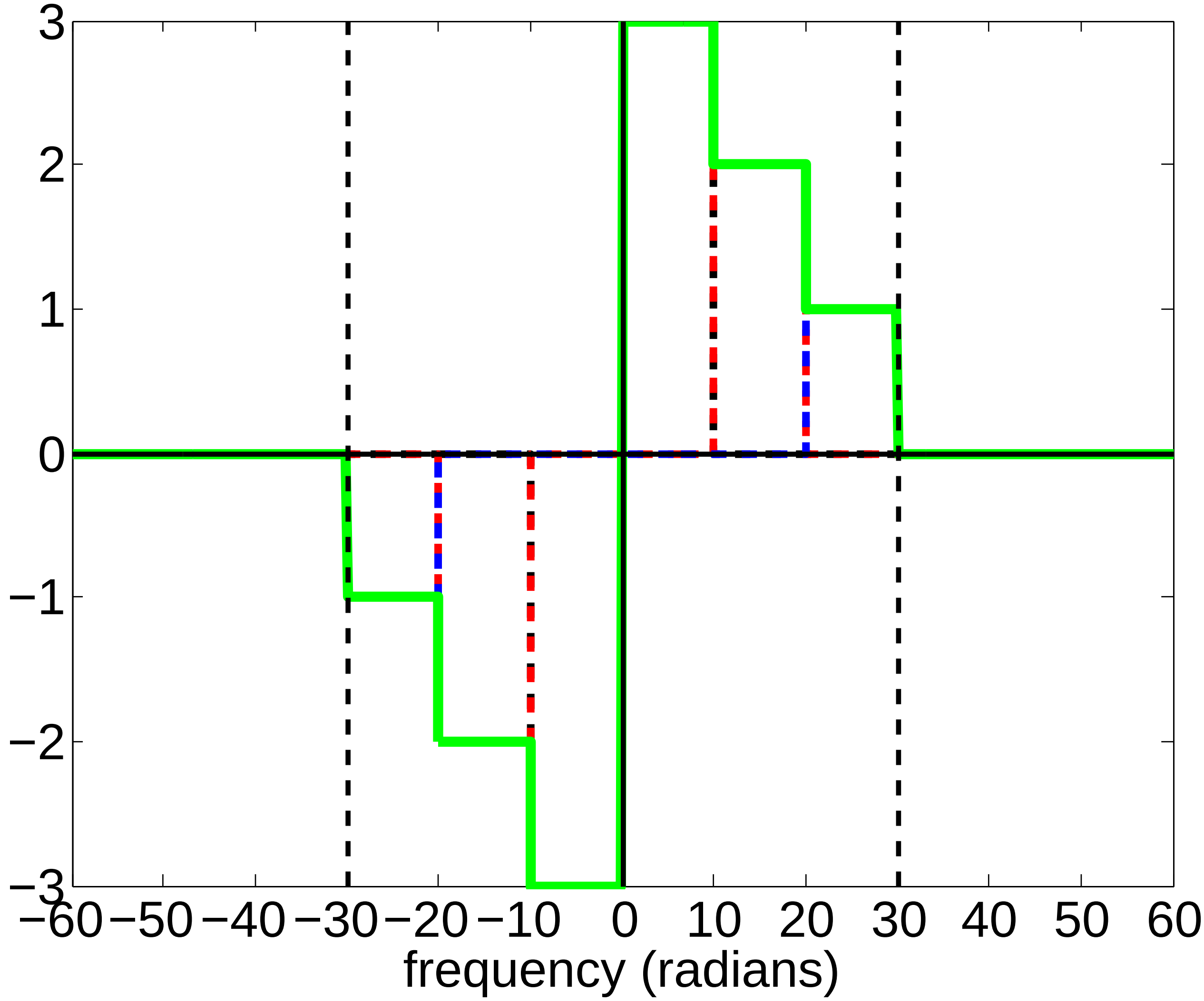
substitute $t = nT_s$

$\Rightarrow$

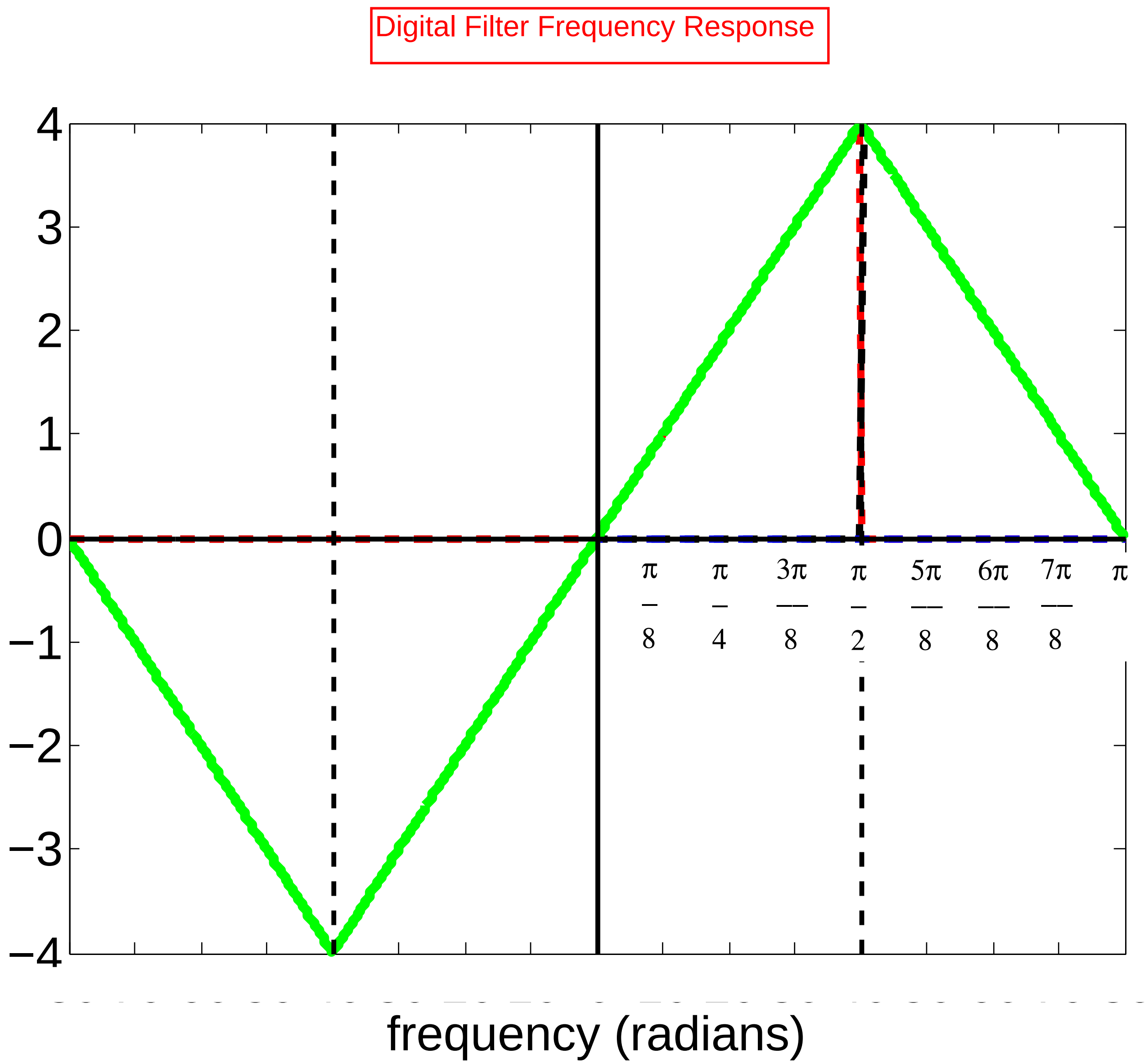
$\omega_d = \omega_a T_s$

$= k\,4\,\frac{2\pi}{64}$

$= k\frac{\pi}{8}$



3
2
1
0
−1
−2
−3
−60 −50 −40 −30 −20 −10 0 10 20 30 40 50 60
frequency (radians)


Digital Filter Frequency Response
4
3
2
1
0
−1
−2
−3
−4
π/8
π/4
3π/8
π/2
5π/8
6π/8
7π/8
π
frequency (radians)

Problem 2. You can continue your work for 2 here.

$$y[n] = (-1)(-1)e^{-j\frac{7\pi}{8}n} + (1)(1)e^{j\frac{7\pi}{8}n}$$

$$(-2)(-1)e^{-j\frac{6\pi}{8}n} + (2)(1)e^{j\frac{6\pi}{8}n}$$

$$(-4)(-2)e^{-j\frac{4\pi}{8}n} + (4)(2)e^{j\frac{4\pi}{8}n}$$

$$(-3)(-2)e^{-j\frac{3\pi}{8}n} + (3)(2)e^{j\frac{3\pi}{8}n}$$

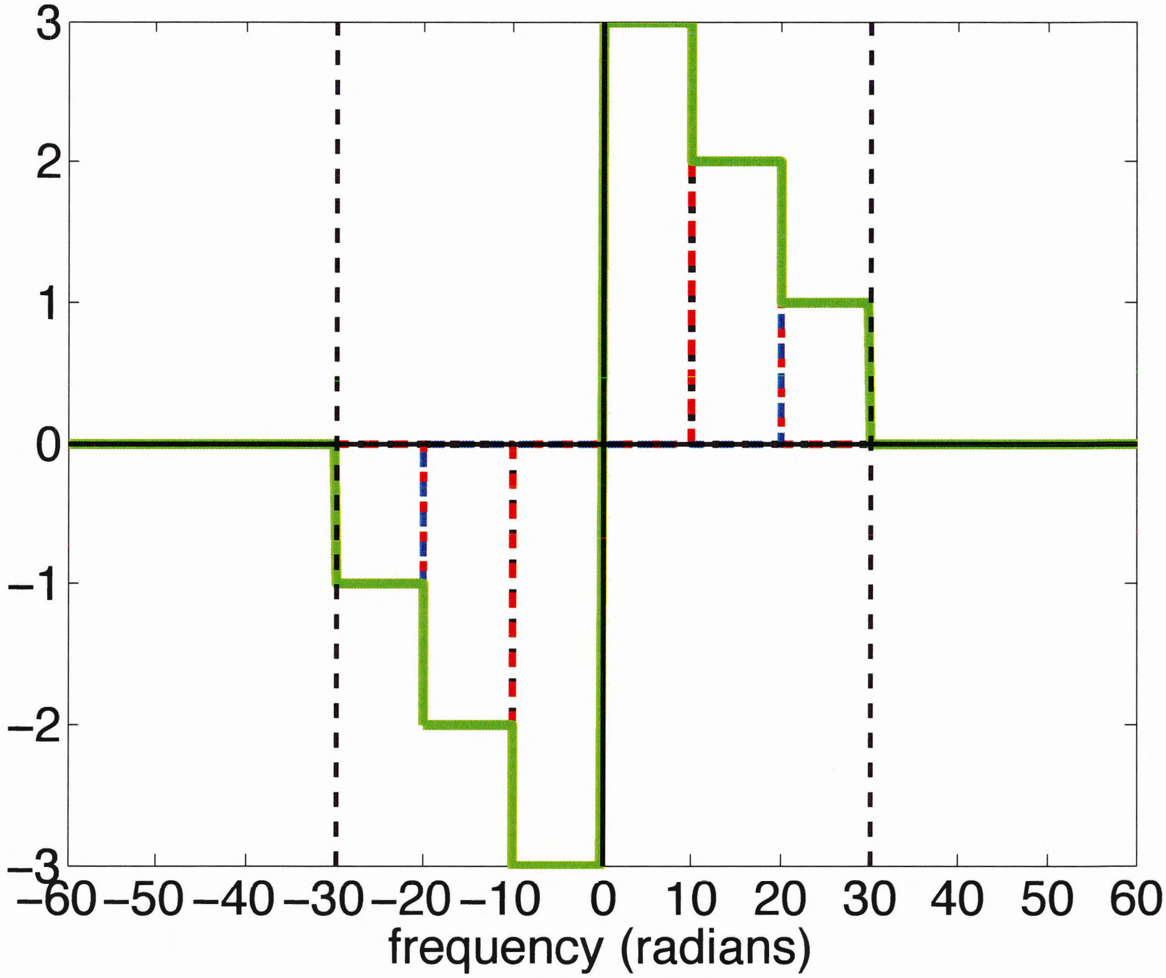
$$(-2)(-3)e^{-j\frac{2\pi}{8}n} + (2)(3) + e^{j\frac{2\pi}{8}n}$$

$$(-1)(-3)e^{-j\frac{\pi}{8}n} + (1)(3)e^{j\frac{\pi}{8}n}$$

$$= 2\cos\left(\frac{7\pi}{8}n\right) + 4\cos\left(\frac{6\pi}{8}n\right)$$

$$+ 16\cos\left(\frac{4\pi}{8}n\right) + 12\cos\left(\frac{3\pi}{8}n\right)$$

$$+ 12\cos\left(\frac{2\pi}{8}n\right) + 6\cos\left(\frac{\pi}{8}n\right)$$



3
2
1
0
−1
−2
−3
−60 −50 −40 −30 −20 −10 0 10 20 30 40 50 60
frequency (radians)