Solution Key

EE301 Signals and Systems — In-Class Exam
Exam 3 — Thursday, Apr. 21, 2011

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 5,7
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. Ignore the numbers at the bottom. For each plot, the abscissa goes from -2π to $+2\pi$ with tic marks every $\pi/8$. There is a dashed vertical line at $\omega = -\pi$ and another dashed vertical line at $\omega = +\pi$. You only have to plot over $-\pi < \omega < \pi$.*

Multiple Choice Question 1. Circle **EVERY** answer that is correct: Primary differences between the CTFT and the DTFT include

(a) The time-domain variable for the DTFT is a discrete (integer-valued) variable for the DTFT whereas it is continuous-valued for the CTFT

(b) The DTFT is always periodic with period 2π whereas the CTFT is generally not periodic except under special conditions

(c) The DTFT is a summation over time whereas the CTFT is an integration over time

(d) The inverse DTFT is a summation over frequency whereas the inverse CTFT is an integration over frequency The inverse DTFT is an integration

(e) The highest frequency for the DTFT is effectively π whereas the highest frequency for the CTFT is ∞

Multiple Choice Question 2. Circle **EVERY** answer that is correct: Properties of the CTFT that are very different for the DTFT are

(a) Time-Scaling Property

(b) Convolution Property

(c) Differentiation-in-Time Property

(d) Frequency-Shift Property } will accept either answer for this one

(e) Time-Shift Property

(f) Duality Property

Multiple Choice Question 3. Circle **EVERY** answer that is correct.

(a) When you pass an analog signal through an LTI filter, the Nyquist rate for the output signal is greater than the Nyquist rate for the input signal.

(b) When you form the product of two analog signals, the Nyquist rate for the product signal is the sum of the respective Nyquist rates for the two individual signals.

(c) The Nyquist rate for the square of a signal is twice the Nyquist rate of the original signal.

(d) The Nyquist rate for the derivative of a signal is greater than the Nyquist rate for the original signal.

(e) When you multiply a signal by a real-valued sinewave, the Nyquist rate of the resulting modulated signal is greater than the Nyquist rate for the original signal.

Problem 2 (a). Consider an analog signal $x_a(t)$ with a bandwidth (maximum frequency) of W in rads/sec. The sampling rate is twice the Nyquist rate: $\omega_s = 4W$ (where $\omega_s = 2\pi/T_s$.) Determine the range for acceptable values for the cut-off frequency, ω_c, of the Ideal Lowpass Filter that allows $x_a(t)$ to be reconstructed perfectly according to the formula below.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h(t - nT_s) \qquad \text{where:} \quad h(t) = T_s \frac{\sin(\omega_c t)}{\pi t}$$

$W < \omega_c < \omega_s - W$ $\qquad$ $\omega_s = 4W$

$W < \omega_c < 3W$

Problem 2 (b). Consider a CT signal $x_a(t)$ with bandwidth (maximum frequency) W in rads/sec. The sampling rate is chosen to be above the Nyquist rate at $\omega_s = 3W$, where $\omega_s = 2\pi/T_s$. $x_a(t)$ is reconstructed perfectly according to the formula below. Let $H(\omega)$ be the CTFT of $h(t)$. Determine the respective values of ω_1 and ω_2, both in terms of W, so that the CTFT $H(\omega)$ is flat up to the bandwidth W and then rolls off to zero at $\omega_s - W$.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h(t - nT_s) \quad \text{where} \quad h(t) = T_s \frac{\pi}{\omega_1} \frac{\sin(\omega_1 t)}{\pi t} \frac{\sin(\omega_2 t)}{\pi t} \quad \text{and} \quad \omega_s = 3W$$

$\omega_1 + \omega_2 = \omega_s - W = 3W - W = 2W$

$\omega_2 - \omega_1 = W$

$\omega_1 + \omega_2 = 2W$

$-\omega_1 + \omega_2 = W$

$2\omega_2 = 3W$ $\qquad$ $\omega_1 = 2W - \omega_2$

$\omega_2 = \frac{3}{2}W$ $\qquad$ $= 2W - \frac{3}{2}W = \frac{W}{2}$

$\omega_1 = \frac{W}{2} \qquad \omega_2 = \frac{3W}{2}$

Problem 2 (c). Consider the continuous-time signal $x(t)$ below A discrete-time signal is created by sampling $x(t)$ according to $x[n] = x(nT_s)$ for $T_s = \frac{2\pi}{80}$. Plot the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$ in the space provided below.

$$x(t) = j2T_s \left\{ \frac{\sin(10t)}{\pi t} \right\} \sin(10t)$$

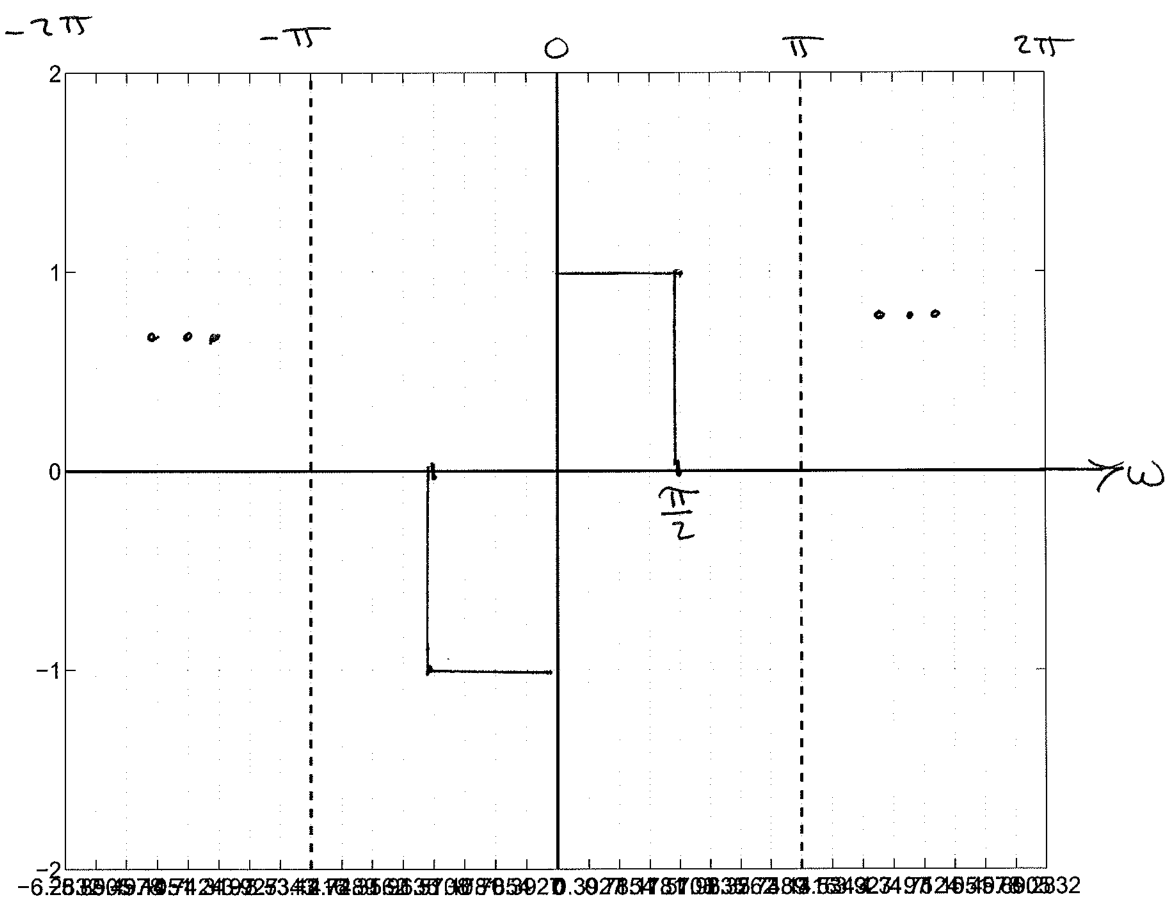
$$= T_s\, e^{j10t} \frac{\sin(10t)}{\pi t} - T_s e^{-10jt} \frac{\sin(10t)}{\pi t}$$

$X(\omega) =$

$$X(\omega) = T_s \ \mathrm{rect}\left(\frac{\omega - 10}{20}\right) - T_s \ \mathrm{rect}\left(\frac{\omega + 10}{20}\right)$$

ω_M = max freq = 20 Nyquist rate = 40 80 > 40 no aliasing

$\omega_M = 20$ mapped to $\omega_M T_s = 20 \times \frac{2\pi}{80} = \frac{\pi}{2}$



Problem 2 (d). Consider the continuous-time signal $x(t)$ below A discrete-time signal is created by sampling $x(t)$ according to $x[n] = x(nT_s)$ for $T_s = \frac{2\pi}{30}$. Plot the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$ in the space provided below.

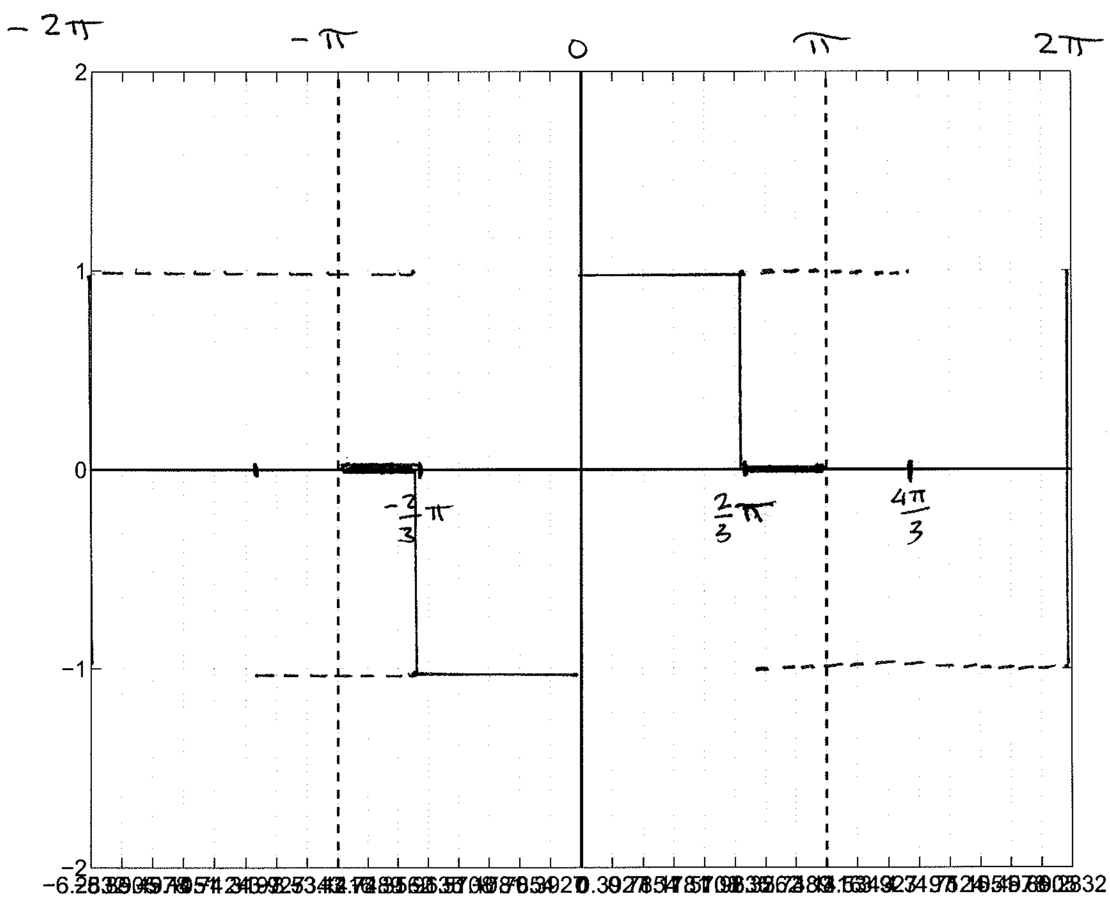
$$x(t) = j2T_s \left\{ \frac{\sin(10t)}{\pi t} \right\} \sin(10t)$$

$\omega_M = 20$ $\quad$ $\omega_s = 30$ $\quad$ $30 < 40 \Rightarrow$ Aliasing!

Aliasing starts at $\omega_s - \omega_M = 30 - 20 = 10$ and goes to $\frac{\omega_s}{2} = \frac{30}{2} = 15$ $\Big\}$ These are mapped to $10 \cdot \frac{2\pi}{30} = \frac{2}{3}\pi$

$15 \cdot \frac{2\pi}{30} = \pi$

and beyond to $\omega_M = 20$ $\xrightarrow{\text{mapped to}}$ $20 \cdot \frac{2\pi}{30} = \frac{4}{3}\pi$



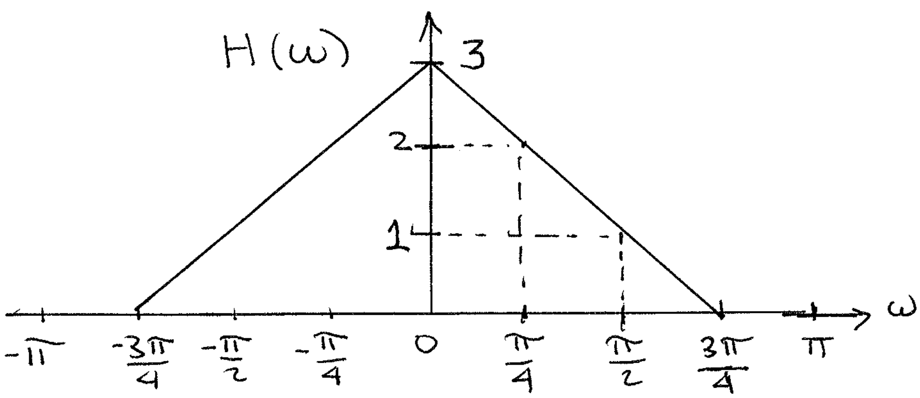
Problem 2 (e). Consider a DT LTI system with impulse response

$$h[n] = 8\left\{\frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n}\right\}^2$$

Determine the output $y[n]$ for the input $x[n]$ given by

$$x[n] = \sum_{k=0}^{4} \cos\left(k\frac{\pi}{4}n\right)$$

Show work and write your expression for $y[n]$ in the space directly below.



$2 \times \frac{3\pi}{8} = \frac{3\pi}{4}$

Height of triangle = $8 \times \frac{1}{\pi}\,\frac{3\pi}{8} = 3$

$x[n] = \cos(0 \cdot n) + \cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{3\pi}{4}n\right) + \cos(\pi n)$

$y[n] = H(0)\cos(0 \cdot n) + H\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}n\right) + H\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}n\right)$
$+ H\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}n\right) + H(\pi)\cos(\pi n)$

where: $H(0) = 3$ $\quad H\left(\frac{\pi}{4}\right) = 2$ $\quad H\left(\frac{\pi}{2}\right) = 1$ $\quad H\left(\frac{3\pi}{4}\right) = 0$
$H(\pi) = 0$

Thus:
$y[n] = 3 + 2\cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right)$

Workout Problem 3 Let $H_{LP}(\omega)$ be the Discrete-Time Fourier Transform (DTFT) of the impulse response $h_{LP}[n]$ defined below.

$$h_{LP}[n] = 8\,\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\,\frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \tag{1}$$

(a) Note: $h_{LP}[n]$ is real-valued and even-symmetric as a function of discrete-time. Thus, $H_{LP}(\omega)$ is both real-valued and symmetric as a function of frequency. Plot $H_{LP}(\omega)$ in the space provided over $-\pi < \omega < \pi$ showing as much detail as possible.

(b) $h[n]$ is defined in terms of $h_{LP}[n]$ as:

$$h[n] = 2\,h_{LP}[n]\,\sin\left(\frac{\pi}{2}n\right) \tag{2}$$

Note that $h[n]$ is odd-symmetric as a function of time. Thus, $H(\omega)$ is purely imaginary for all frequencies. Plot $H(\omega)$ in the space provided. Note that the vertical axis values have the multiplicative scalar $j = \sqrt{-1}$ factored into them.

(c) Consider the input signal $x[n]$ below.

$$x[n] = 16\left\{\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\right\}^2 \cos\left(\frac{\pi}{2}n\right)$$

Determine and plot the DTFT $X(\omega)$ of the signal $x[n]$ in the space provided.

(d) Denote the output $y[n]$ when the signal in part (c) directly above is the input to the LTI system with impulse response $h[n]$ in part (b) defined by eqn. (2). Determine and plot $Y(\omega)$ in the space provided over $-\pi < \omega < \pi$ showing as much detail as possible.

(e) Create a complex-valued signal as

$$z[n] = x[n] + jy[n]$$

Determine and plot the DTFT $Z(\omega)$ in the space provided over $-\pi < \omega < \pi$ showing as much detail as possible.

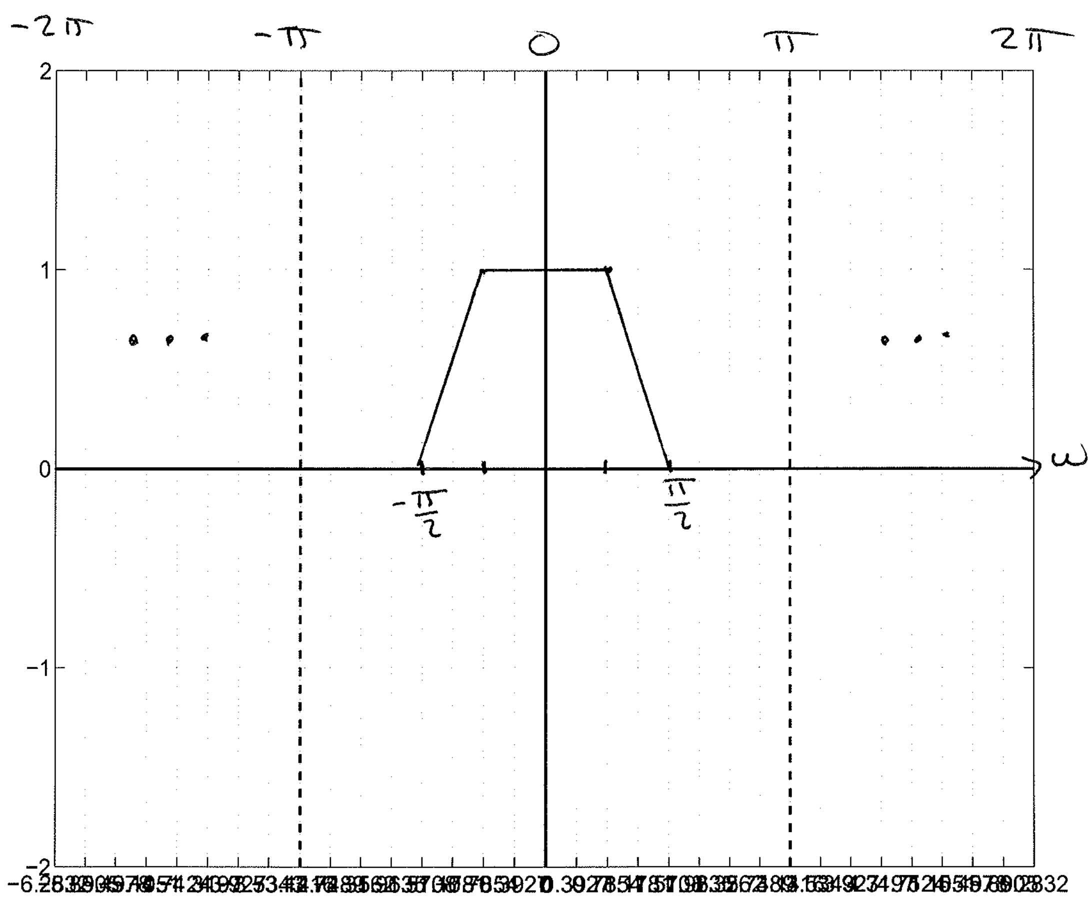
VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. Ignore the numbers at the bottom. For each plot, the abscissa goes from -2π to $+2\pi$ with tic marks every $\pi/8$. There is a dashed vertical line at $\omega = -\pi$ and another dashed vertical line at $\omega = +\pi$. You only have to plot over $-\pi < \omega < \pi$.*

Height $= 8 \times \dfrac{\frac{\pi}{8}}{\pi} = 1$

flat up to: $\dfrac{3\pi}{8} - \dfrac{\pi}{8} = \dfrac{\pi}{4}$

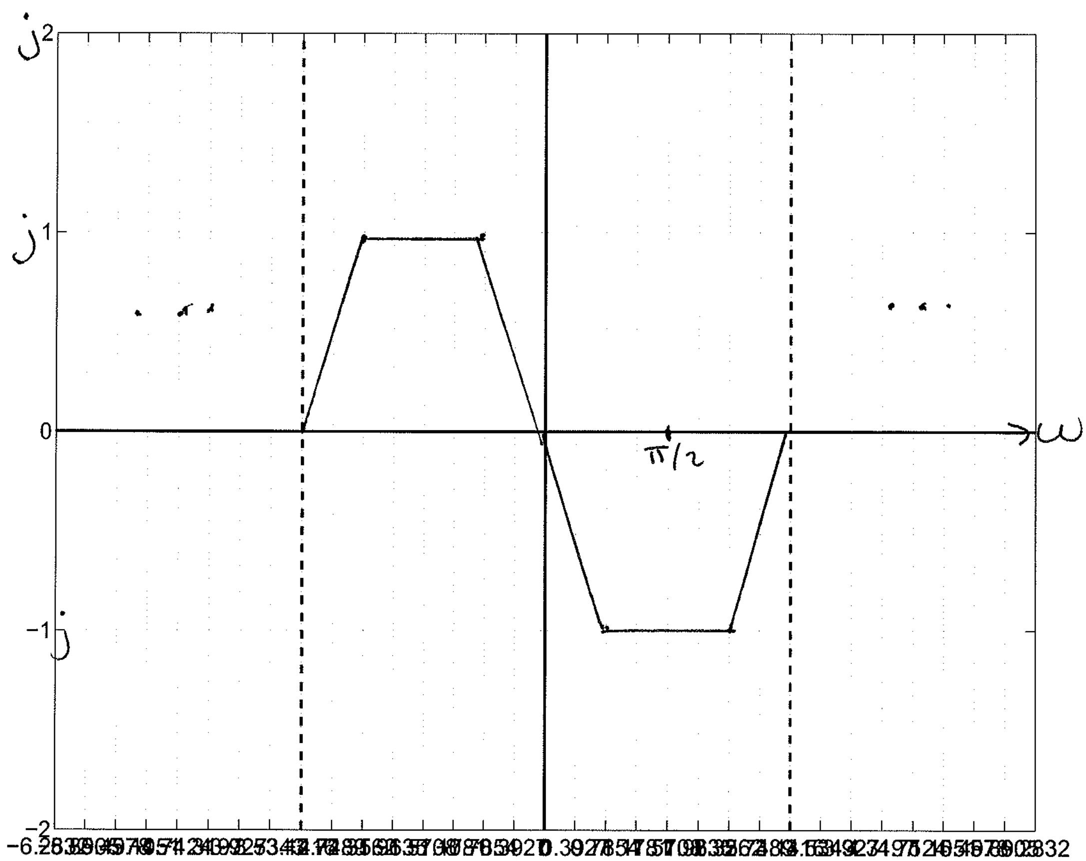
goes to zero at: $\dfrac{\pi}{8} + \dfrac{3\pi}{8} = \dfrac{\pi}{2}$

Plot your answer to Problem 3 (a) here. Show work above.



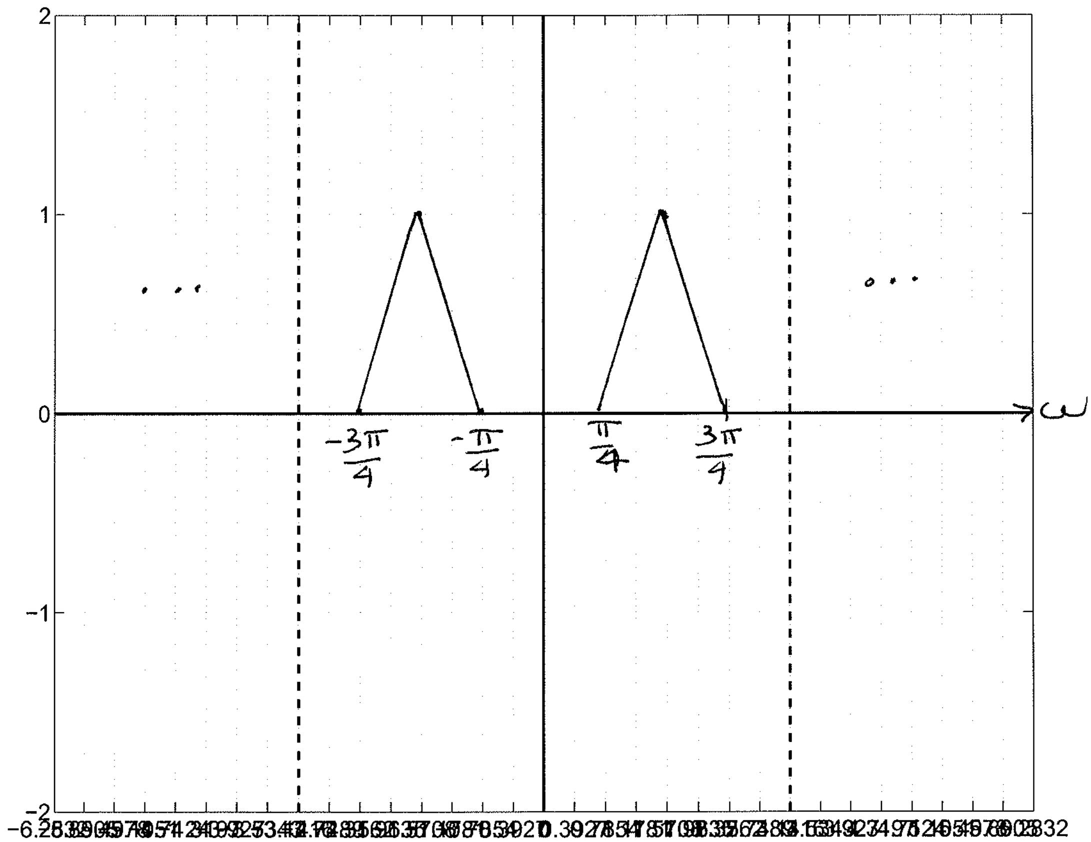
$$H(\omega) = -j\, H_{LP}\left(\omega - \frac{\pi}{2}\right) + j\, H_{LP}\left(\omega + \frac{\pi}{2}\right)$$

Plot your answer to Problem 3 (b) here. Show work above.

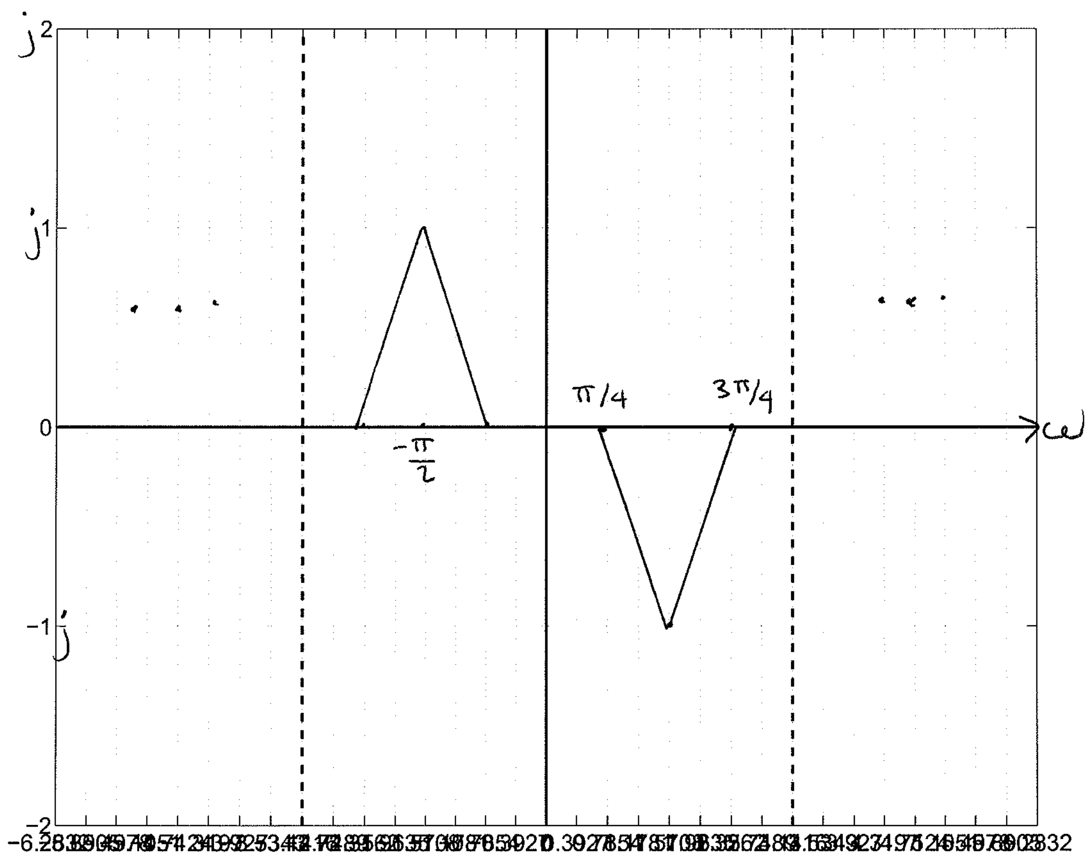


$$\text{Height} = \frac{1}{2} \times 16 \times \frac{1}{\pi}\, \frac{\pi}{8} = 1$$

Plot your answer to Problem 3 (c) here. Show work above.



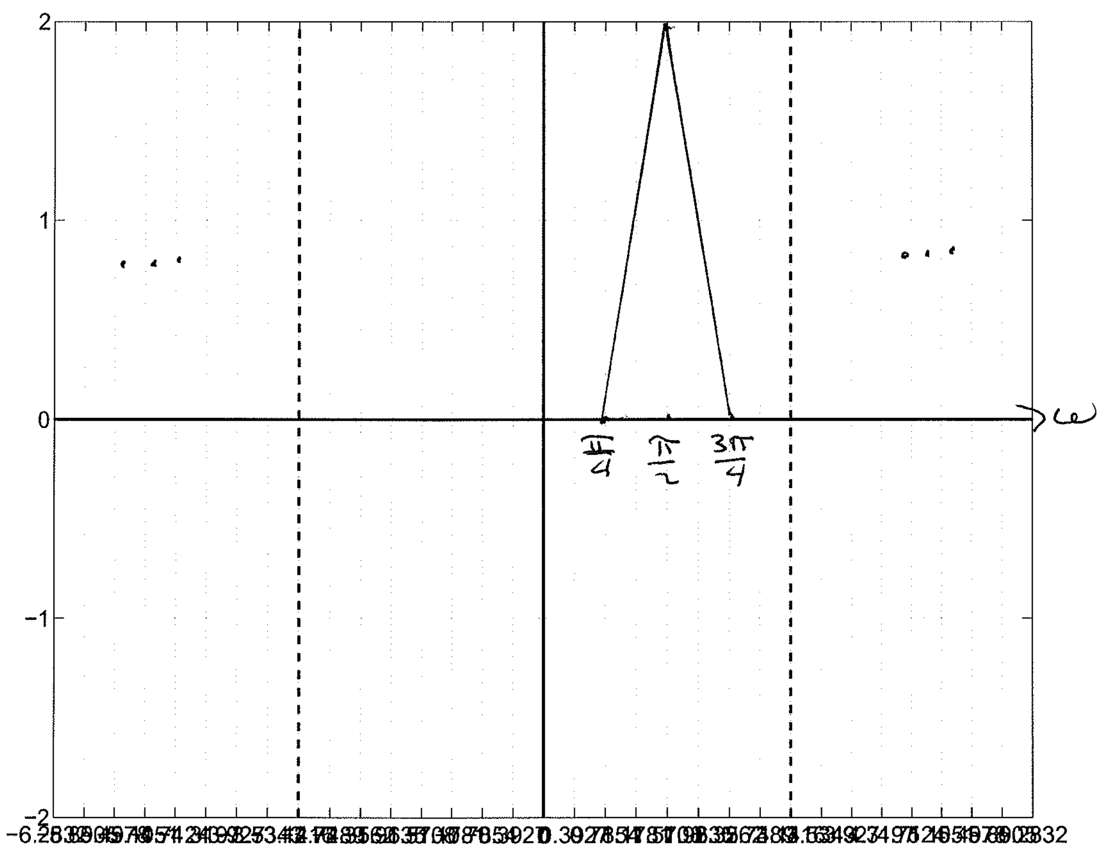
Plot your answer to Problem 3 (d) here. Show work above.



over $0 < \omega < \pi$, we have: $1 + j(-j) = 2$

over $-\pi < \omega < 0$, we have: $1 + j(j) = 0$

Plot your answer to Problem 3 (e) here. Show work above.



Note re: practical implications of Problem 3

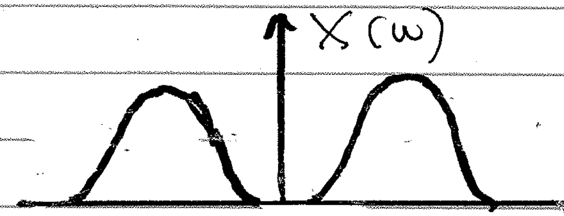
$$z[n] = x[n] + j\,y[n] \overset{DTFT}{\longleftrightarrow} Z(\omega)$$

$Z(\omega) = 0$

over $-\pi < \omega < 0$

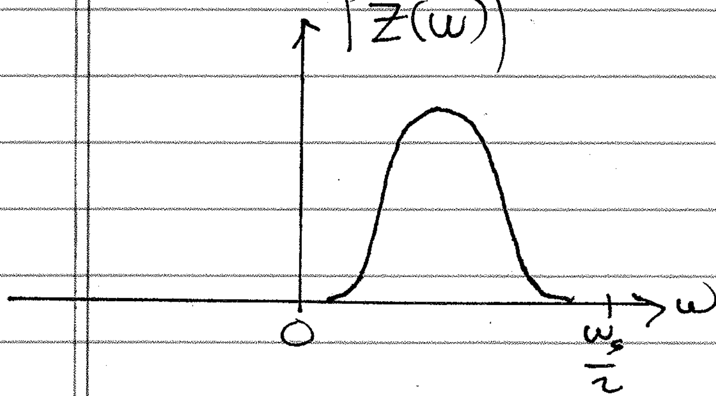
$x[n] \rightarrow \boxed{D/A} \rightarrow x(t)$

F_s



$y[n] \rightarrow \boxed{D/A} \rightarrow y(t)$

$$z(t) = x(t) + j\,y(t) \overset{\mathcal{F}}{\longleftrightarrow} Z(\omega)$$



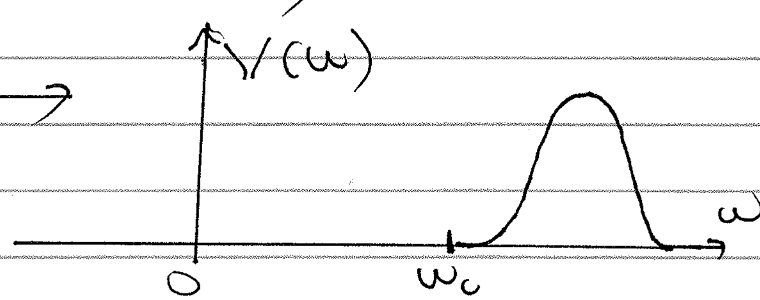
$Z(\omega) = 0$

for $\omega < 0$

$$\boxed{\begin{array}{l} \text{also: } Z(\omega) = 0 \\ \quad \text{for } \omega > \frac{\omega_s}{2} \\ \quad \omega_s = 2\pi F_s \end{array}}$$

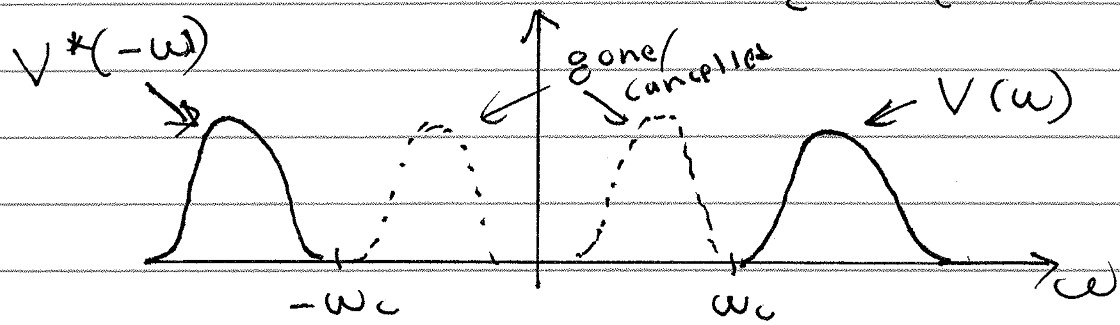
Consider: $w(t) = Re\{\underbrace{z(t)\,e^{j\omega_c t}}_{v(t)}\}$

$$v(t) = z(t)\,e^{j\omega_c t} \overset{\mathcal{F}}{\longleftrightarrow}$$



$$w(t) = \mathrm{Re}\{v(t)\} \xleftrightarrow{\mathcal{F}} W(\omega) = \frac{1}{2}V(\omega) + \frac{1}{2}V^*(-\omega)$$

$$= \frac{1}{2}v(t) + \frac{1}{2}v^*(t)$$



Examine $w(t)$:

$$w(t) = \mathrm{Re}\left\{ z(t)\, e^{j\omega_c t} \right\}$$

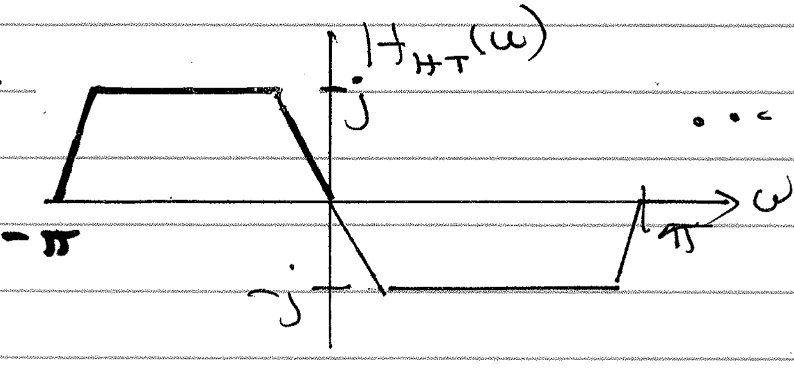
$$= \mathrm{Re}\left\{ \big(x(t) + j\,y(t)\big)\big(\cos(\omega_c t) + j\sin(\omega_c t)\big) \right\}$$

$$= x(t)\cos(\omega_c t) - y(t)\sin(\omega_c t)$$

Again, recall: $y[n] \rightarrow \boxed{D/A} \rightarrow y(t)$

where: $y[n] = x[n] * h_{HT}[n]$

$h_{HT}[n] \xleftrightarrow{DTFT}$



HT: Hilbert Transformer