ECE 301 Exam 3 Solutions Spring 2009

1) a: $x_1(t) = T_s \frac{d}{dt}\left[\frac{\sin(10t)}{\pi t}\right]$ $\quad x_1[n] = x_1(nT_s) \quad T_s = \frac{2\pi}{160}$

$$\frac{d}{dt}x(t) \xleftrightarrow{CTFT} j\omega X(\omega) \qquad \frac{\sin(10t)}{\pi t} \xleftrightarrow{CTFT} \begin{cases} 1 & |\omega| < 10 \\ 0 & |\omega| > 10 \end{cases}$$

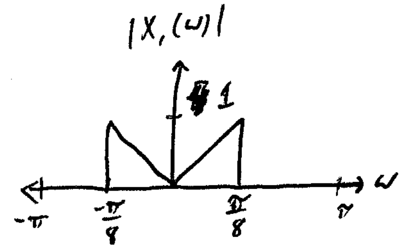
$$\left[X_1(\omega)\right]_{CT} = \begin{cases} T_s j\omega & |\omega| < 10 \\ 0 & |\omega| > 10 \end{cases}$$

$$\omega_s = \frac{2\pi}{T_s} = 160 \ \frac{rad}{s} \quad \text{sampling frequency}$$

In performing quantization, the spectrum gets re-mapped such that

$$\left[X_1\left(\tfrac{\omega_s}{2}\right)\right]_{CT} \longrightarrow \left[X_1(\pi)\right]_{DT}$$

$$\left[X_1(\omega)\right]_{DT} = \begin{cases} j\omega & |\omega| < 10\frac{\pi}{80} \\ 0 & ow \end{cases} = \begin{cases} j\omega & |\omega| < \frac{\pi}{8} \\ 0 & |\omega| > \frac{\pi}{8} \end{cases}$$



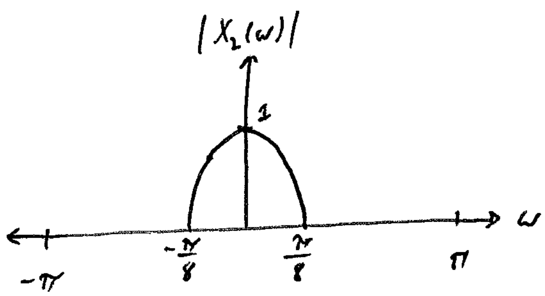
b: $x_2(t) = T_s \frac{1}{2}\left[\frac{\sin(10(t-\frac{\pi}{20}))}{\pi(t-\frac{\pi}{20})} + \frac{\sin(10(t+\frac{\pi}{20}))}{\pi(t+\frac{\pi}{20})}\right]$

$$= \frac{T_s}{2}\left[x_0(t-\tfrac{\pi}{20}) + x_0(t+\tfrac{\pi}{20})\right] \qquad x_0(t) = \frac{\sin(10t)}{(\pi t)} \qquad X_0(\omega) = \begin{cases} 1 & |\omega| < 10 \\ 0 & |\omega| > 10 \end{cases}$$

$$\left[X_2(\omega)\right]_{CT} = \frac{T_2}{2}\left[\left(e^{-j\frac{\pi}{20}\omega} + e^{j\frac{\pi}{20}\omega}\right) X_0(\omega)\right]$$

$$= \begin{cases} \frac{T_2}{2} 2\cos\left(\frac{\pi}{20}\omega\right) & |\omega| < 10 \\ 0 & |\omega| > 10 \end{cases} = \begin{cases} T_2 \cos\left(\frac{\pi}{20}\omega\right) & |\omega| < 10 \\ 0 & |\omega| > 10 \end{cases}$$

$$\left[X_2(\omega)\right]_{DT} = \begin{cases} \cos\left(\frac{\pi}{20}\frac{80}{\pi}\omega\right) & |\omega| < 10\frac{\pi}{80} \\ 0 & ow \end{cases} = \begin{cases} \cos(4\omega) & |\omega| < \frac{\pi}{8} \\ 0 & |\omega| > \frac{\pi}{8} \end{cases}$$



i: $x[n] = 2x_1[n]\cos\left(\frac{3\pi}{8}n\right) + 2x_2[n]\cos\left(\frac{7\pi}{8}n\right)$

$$\cos(\omega_0 n) \xleftrightarrow{DTFT} \pi \sum_{\ell} \left[\delta(\omega - \omega_0 - 2\pi\ell) + \delta(\omega + \omega_0 - 2\pi\ell)\right]$$

$$x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{2\pi} X(\theta)\,Y(\omega-\theta)\,d\theta$$

Let $y_1[n] = 2x_1[n]\cos\left(\frac{3\pi}{8}n\right)$, $y_2[n] = 2x_2 \cos\left(\frac{7\pi}{8}n\right)$

$$Y_1(\omega) = 2\frac{1}{2\pi}\int_{-\pi}^{\pi} \pi \sum_{\ell}\left[\delta\left(\theta - \frac{3\pi}{8} - 2\pi\ell\right) + \delta\left(\theta + \frac{3\pi}{8} - 2\pi\ell\right)\right] X_1(\omega-\theta)\,d\theta$$

only 2 δ's in $\theta \in (-\pi, \pi)$, $\ell = 0$

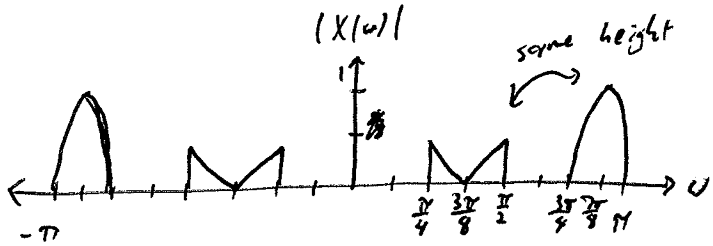
$$= X_1\left(\omega - \frac{3\pi}{8}\right) + X_1\left(\omega + \frac{3\pi}{8}\right)$$

$$Y_2(\omega) = 2\frac{1}{2\pi}\int_{-\pi}^{\pi} \pi \sum_{\ell}\left[\delta\left(\theta - \frac{7\pi}{8} - 2\pi\ell\right) + \delta\left(\theta + \frac{7\pi}{8} - 2\pi\ell\right)\right] X_2(\omega-\theta)\,d\theta$$

only 2 δ's in $\theta \in (-\pi, \pi)$, $\ell = 0$

$$= X_2\left(\omega - \frac{7\pi}{8}\right) + X_2\left(\omega + \frac{7\pi}{8}\right)$$

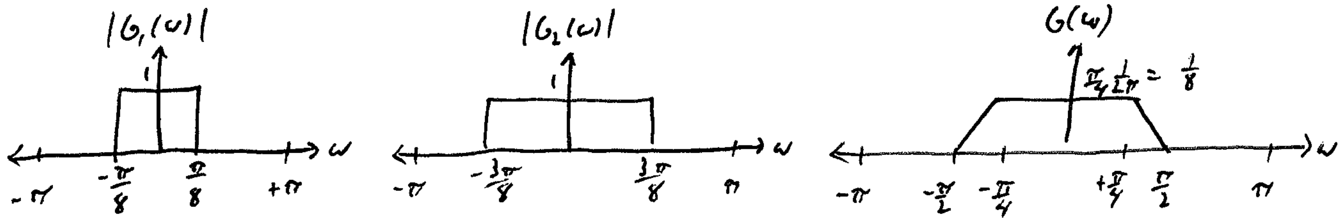
So, $X(\omega) = Y_1(\omega) + Y_2(\omega) = X_1\left(\omega - \frac{3\pi}{8}\right) + X_1\left(\omega + \frac{3\pi}{8}\right) + X_2\left(\omega - \frac{7\pi}{8}\right) + X_2\left(\omega + \frac{7\pi}{8}\right)$



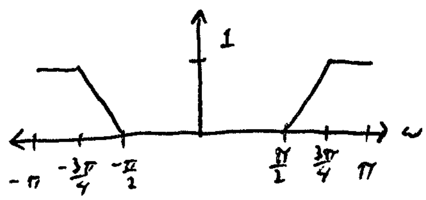
d: $h_1[n] = 8(-1)^n \left[\underbrace{\frac{\sin(\frac{\pi}{8}n)}{\pi n}}_{g_1[n]} \; \underbrace{\frac{\sin(\frac{3\pi}{8}n)}{\pi n}}_{g_2[n]} \right]$

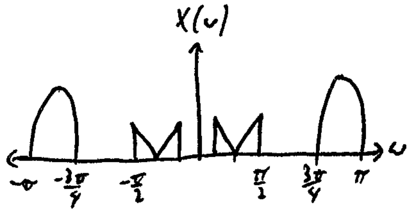
note : $(-1)^n = (e^{j\pi})^n$

$g[n] = g_1[n]\, g_2[n]$

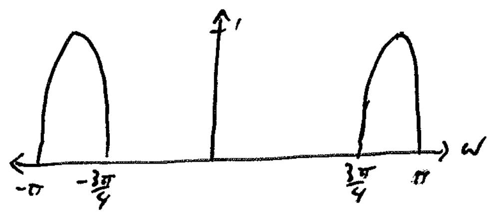


$H_1(\omega) = 8\, G(\omega - \pi)$





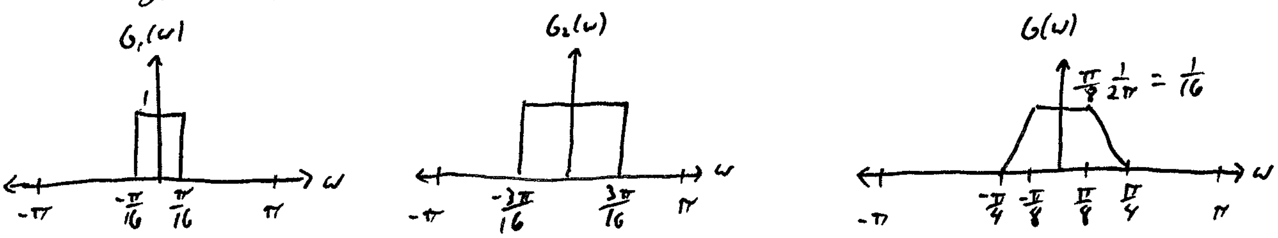
$Y_1(\omega) = H_1(\omega)\, X(\omega)$



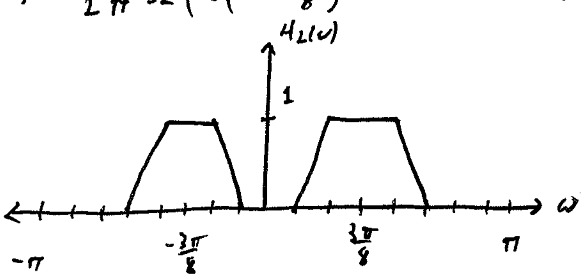
$y_1[n] = 2\, x_2[n] \cos(\frac{7\pi}{8} n)$

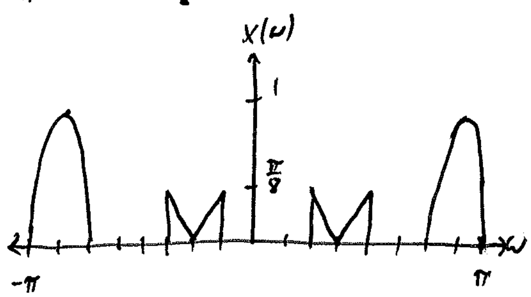
e: $h_2[n] = 32 \left[\underbrace{\frac{\sin(\frac{\pi}{16}n)}{\pi n}}_{g_1[n]} \; \underbrace{\frac{\sin(\frac{3\pi}{16}n)}{\pi n}}_{g_2[n]} \right] \cos(\frac{3\pi}{8} n)$

$g[n] = g_1[n]\, g_2[n]$

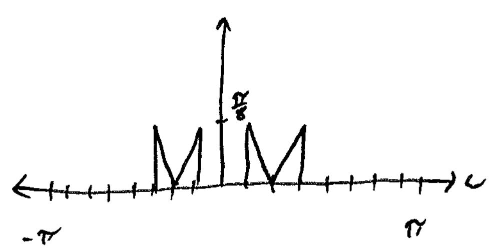


$H_2(\omega) = \frac{1}{2\pi}\, 32 \left(G(\omega - \frac{3\pi}{8}) + G(\omega + \frac{3\pi}{8}) \right)$



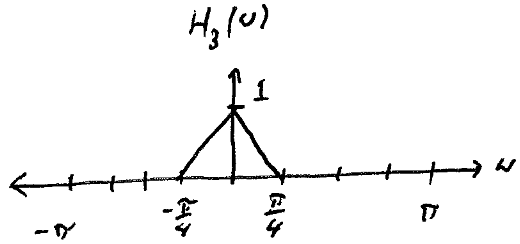


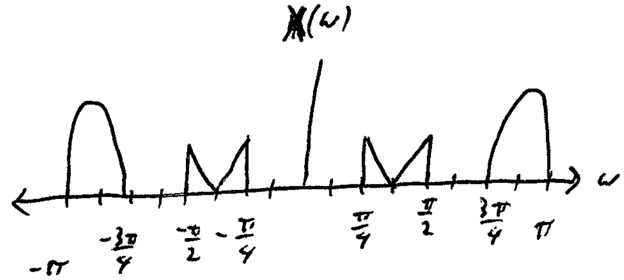
$Y_2(\omega) = H_2(\omega)\, X(\omega)$



$y_2[n] = 2\, x_1[n] \cos(\frac{3\pi}{8} n)$

f: $h_3[n] = 8\left[\frac{\sin(\frac{\pi}{8}n)}{\pi n}\right]^2$

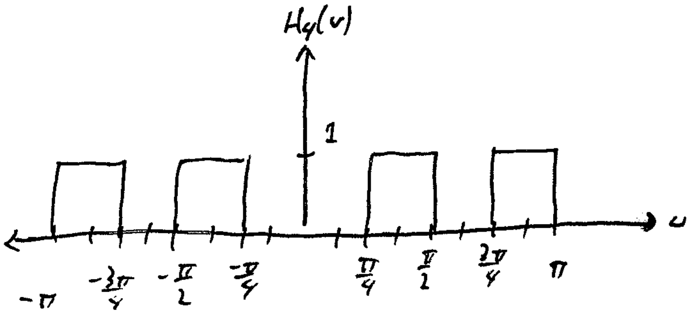


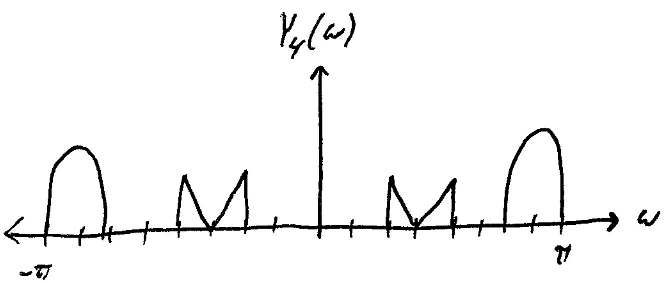


$Y_3(\omega) = X(\omega)\, H_3(\omega)$

$Y_3(\omega) = 0 \qquad y_3[n] = 0$

g: $h_4[n] = 2\,\frac{\sin(\frac{\pi}{8}n)}{\pi n}\cos\left(\frac{3\pi}{8}n\right) + 2\,\frac{\sin(\frac{\pi}{8}n)}{\pi n}\cos\left(\frac{7\pi}{8}n\right)$





$Y_4(\omega) = X(\omega)$

$y_4[n] = x[n]$

$= 2\,x_1[n]\cos\left(\frac{3\pi}{8}n\right) + 2\,x_2[n]\cos\left(\frac{7\pi}{8}n\right)$

h: $z[n] = 2x[n]\cos\left(\frac{3\pi}{8}n\right)$

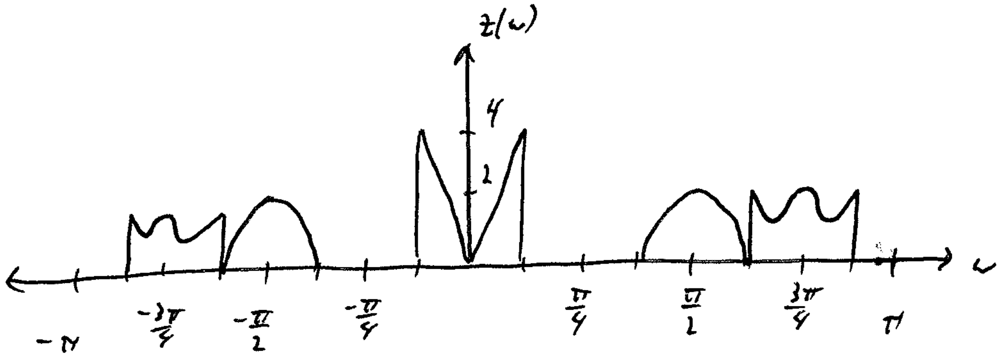
$$= 2\left[2x_1[n]\cos\left(\frac{3\pi}{8}n\right) + 2x_2[n]\cos\left(\frac{7\pi}{8}n\right)\right]\cos\left(\frac{3\pi}{8}n\right)$$

$$= 4x_1[n]\cos\left(\frac{3\pi}{8}n\right)\cos\left(\frac{3\pi}{8}n\right) + 4x_2[n]\cos\left(\frac{7\pi}{8}n\right)\cos\left(\frac{3\pi}{8}n\right)$$

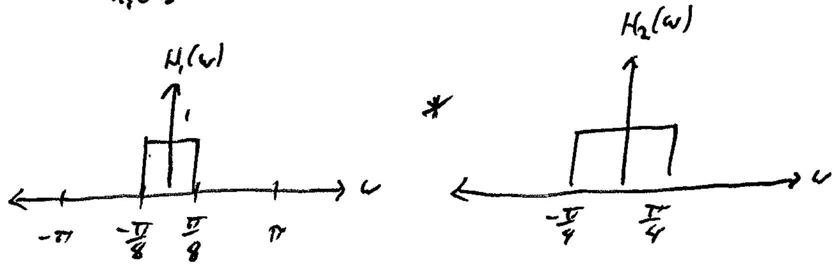
$$= 4x_1[n]\left[\cos\left(\frac{6\pi}{8}n\right) + \cos(0n)\right] + 4x_2[n]\left[\cos\left(\frac{10\pi}{8}n\right) + \cos\left(\frac{4\pi}{8}n\right)\right]$$

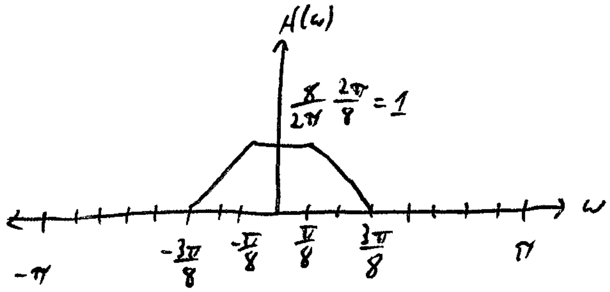
$$= 4x_1[n]\left[1 + \cos\left(\frac{3\pi}{4}n\right)\right] + 4x_2[n]\left[\cos\left(\frac{3\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n\right)\right]$$

$$Z(\omega) = 4X_1(\omega) + 2X_1\left(\omega - \frac{3\pi}{4}\right) + 2X_1\left(\omega + \frac{3\pi}{4}\right)$$
$$+ 2X_2\left(\omega - \frac{3\pi}{4}\right) + 2X_2\left(\omega + \frac{3\pi}{4}\right) + 2X_2\left(\omega - \frac{\pi}{2}\right) + 2X_2\left(\omega + \frac{\pi}{2}\right)$$

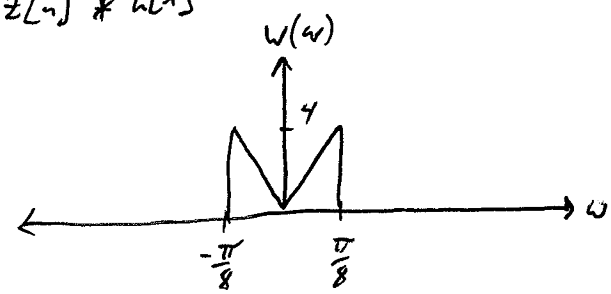


i: $h[n] = 8\left[\underbrace{\frac{\sin(\frac{\pi}{8}n)}{\pi n}}_{h_1[n]} \; \underbrace{\frac{\sin(\frac{\pi}{4}n)}{\pi n}}_{h_2[n]}\right]$





$w[n] = z[n] * h[n]$



$W(\omega) = 4X_1(\omega)$

$\Rightarrow w[n] = 4x_1[n]$

$\neq 2x_1[n]$

$$E_w = \sum_{n=-\infty}^{\infty} |w[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |W(\omega)|^2 d\omega \quad \text{by Parseval's Relation}$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \left|\frac{4}{\frac{\pi}{8}}\omega\right|^2 d\omega = \frac{1}{2\pi}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \left(\frac{32}{\pi}\omega\right)^2 d\omega$$

$$= \frac{1}{2\pi}\frac{(32)^2}{\pi^2}\left(\frac{\pi}{4}\right) = \frac{(32)(32)}{8}\frac{1}{\pi^2} = \frac{(4)(32)}{\pi^2} = \frac{128}{\pi^2}$$