NAME: SOLUTION
EE301 Signals and Systems
Exam 3

NAME
In-Class Exam
Tuesday, Apr. 22, 2014

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 5,7
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. For each plot, the abscissa goes from -2π to 2π with tic marks every $\pi/8$. There is a dashed vertical line at $\omega = -\pi$ and another dashed vertical line at $\omega = +\pi$. You only have to plot any DTFT over $-\pi < \omega < \pi$.*

Problem 1. Short answer questions.

(a) Determine the Nyquist Rate for the signal $x_a(t) = \left\{\frac{\sin(2.5t)}{\pi t}\frac{\sin(7.5t)}{\pi t}\right\}\sin(10t)$?

Max frequency $= \omega_M = 2.5 + 7.5 + 10 = 20$

Nyquist Rate $= 2\omega_M = 40$ rads/sec

(b) If you sample at a rate ω_s in terms of radians/sec, what discrete-time frequency is the analog frequency $\frac{\omega_s}{4}$ mapped to?

ω_s mapped to 2π

$\frac{\omega_s}{2}$ mapped to π

$\frac{\omega_s}{4}$ mapped to $\frac{\pi}{2}$

$\omega_d = \omega_a T_s$

$= \omega_a \frac{2\pi}{\omega_s}$

Thus:

$\omega_d = \frac{\omega_s}{4} \frac{2\pi}{\omega_s}$

$= \frac{\pi}{2}$

(c) The DT signal $x[n]$ is obtained by sampling the sinewave $x_{a_1}(t) = \cos(15t)$ at a rate of $\omega_s = 35$ radians/sec. Specify the frequency ω_{a_2} of another sinewave $x_{a_2}(t) = \cos(\omega_{a_2} t)$ at a higher frequency that will yield the exact same DT signal $x[n]$ when sampled at the same rate, $\omega_s = 35$ radians/sec.

$$x_{a_\ell}(t)\Big|_{t=\frac{2\pi}{35}} = \cos((15 + \ell 35)t)\Big|_{t=\frac{2\pi}{35}}$$

ℓ, integer

$$= \cos\left(2\pi \frac{15}{35} n + \ell 35 \frac{2\pi}{35} n\right)$$

$$= \cos\left(2\pi \frac{3}{7} n + \ell n 2\pi\right)$$

$$= \cos\left(2\pi \frac{3}{7} n\right)$$

Thus, one answer:

$15 + 35 = 50$

Problem 2 (a). Consider the discrete-time signal below.

$$x[n] = (0.5)^n \, e^{j\frac{\pi}{2}n} \, \{u[n] - u[n-5]\}$$

Determine a closed-form expression for the DTFT, $X(\omega)$, of $x[n]$. Be sure to indicate which DTFT properties and/or pairs you use to arrive at your answer.

Several ways to solve:

$$x[n] = \left(.5\, e^{j\frac{\pi}{2}}\right)^n \left\{u[n] - u[n-5]\right\}$$

define: $\alpha = .5\, e^{j\frac{\pi}{2}}$

$$x[n] = \alpha^n u[n] - \alpha^5 \alpha^{n-5} u[n-5]$$

$$X(\omega) = \frac{1}{1-\alpha e^{-j\omega}} - \frac{\alpha^5 e^{-j5\omega}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1 - \alpha^5 e^{-j\omega 5}}{1-\alpha e^{-j\omega}} = \frac{1 - \frac{j}{32} e^{-j5\omega}}{1 - \frac{1}{2} e^{j\frac{\pi}{2}} e^{-j\omega}}$$

$$\boxed{\alpha^5 = \left(\tfrac{1}{2}\right)^5 e^{j\frac{\pi}{2}5} = \frac{j}{32}}$$

OR: directly: $X(\omega) = \sum_{n=0}^{4} \alpha^n e^{-j\omega n} = \sum_{n=0}^{4} \left(\alpha e^{-j\omega}\right)^n$

$$= \frac{1 - \alpha^5 e^{-j5\omega}}{1 - \alpha e^{-j\omega}} \qquad \alpha = \frac{1}{2} e^{j\frac{\pi}{2}} = \frac{j}{2}$$

Problem 2 (b). Show how your answer to 2(a) changes for the discrete-time signal below.

$$x[n] = (0.5)^n \, e^{j\frac{\pi}{2}n} \, \{u[n-2] - u[n-7]\}$$

Determine a closed-form expression for the DTFT, $X(\omega)$, of $x[n]$.

$$x[n] = \alpha^n \{u[n-2] - u[n-7]\} \qquad \alpha = \frac{j}{2} = \frac{1}{2} e^{j\frac{\pi}{2}}$$

$$= \alpha^2 \alpha^{n-2} \{u[n-2] - u[n-7]\}$$

$$= \alpha^2 \underbrace{x_{2a}[n-2]}_{\text{from Prob. 2(a) denote } x_{2a}[n]} \qquad \alpha^2 = \left(\frac{j}{2}\right)^2 = -\frac{1}{4}$$

Thus:

$$X(\omega) = \alpha^2 e^{-j2\omega} \underbrace{X_{2a}(\omega)}_{\text{answer to 2(a)}}$$

$$= -\frac{1}{4} e^{-j2\omega} \frac{1 - \frac{j}{32} e^{-j5\omega}}{1 - \frac{j}{2} e^{-j\omega}}$$

Problem 2 (c). The damped sinusoidal signal $x_a(t) = e^{-4ln(2)t}e^{j2\pi t}\{u(t) - u(t-1)\}$ is sampled every $T_s = 0.25$ seconds to form $x[n] = x_a(nT_s)$, where, again, T_s is a quarter of a second. Determine a closed-form expression for the DTFT $X(\omega)$ of the $x[n]$ thus obtained. *Hint:* $ln(2)$ equal to the natural logarithm of 2 was chosen to make the numbers work out nicely; same with the factor of 4 in the exponent. Recall $e^{ln(x)} = x$.

$$x[n] = x_a(nT_s) \quad \text{where: } T_s = 0.25 \text{ secs} \quad \text{and} \quad x_a(t) = e^{-4ln(2)t}e^{j2\pi t}\{u(t) - u(t-1)\}$$

$$x[n] = e^{-4\ln(2)\frac{n}{4}} e^{j2\pi\frac{n}{4}} \underbrace{\left\{u\left(\frac{n}{4}\right) - u\left(\frac{n}{4}-1\right)\right\}}_{\text{explained below}}$$

$$= \left(e^{-\ln(2)}\right)^n e^{j\frac{\pi}{2}n} \{u[n] - u[n-5]\}$$

$$= \left(\frac{1}{2}e^{j\frac{\pi}{2}}\right)^n \{u[n] - u[n-5]\}$$

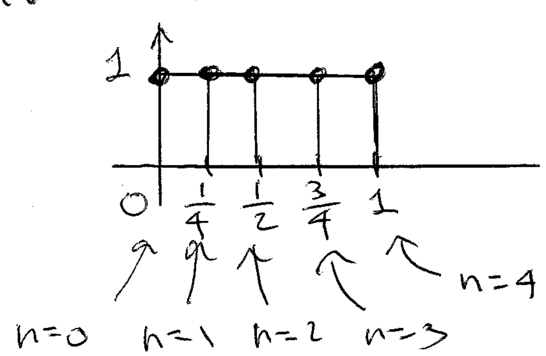
Same as $x[n]$ in 2(a) $\Rightarrow$ Thus:

$$X(\omega) = \frac{1 - \frac{j}{32}e^{-j5\omega}}{1 - \frac{j}{2}e^{-j\omega}}$$

$y(t) = x_1(t)\, x_2(t)$

$y(nT_s) = x_1(nT_s)\, x_2(nT_s)$ $\qquad y[n] = x_1[n]\, x_2[n]$

consider $x_2(t) = u(t) - u(t-1)$ sampled every $T_s = .25$ secs



$\Rightarrow u[n] - u[n-5]$

Problem 3. Consider the input signal $x_p(t)$ below.

$$x_p(t) = \cos(2.5t) + \cos(17.5t) + \left\{ \frac{\sin(5t)}{\pi t} 2\cos(10t) \right\}$$

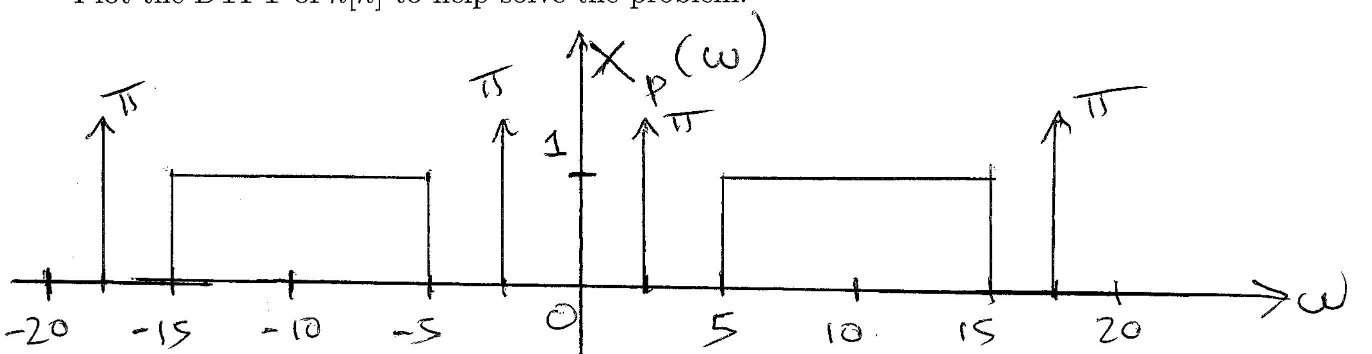
This signal is first low-passed filtered with an analog lowpass filter with impulse response

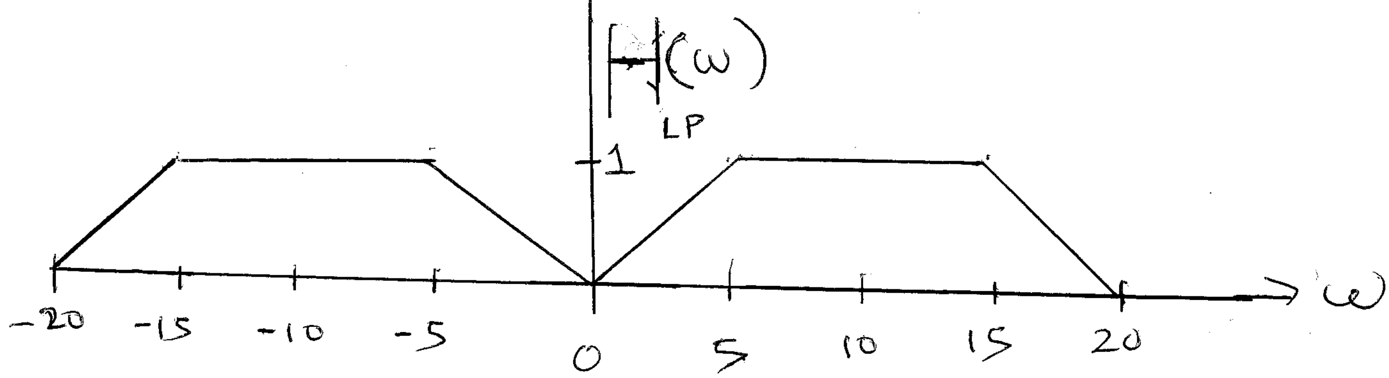
$$h_{LP}(t) = \frac{\pi}{2.5} \frac{\sin(2.5t)}{\pi t} \frac{\sin(7.5t)}{\pi t} 2\cos(10t)$$

to form $x(t) = x_p(t) * h_{LP}(t)$, and then $x(t)$ is sampled at a rate of $\omega_s = 40$ to form $x[n]$, so that the time between samples is $T_s = \frac{2\pi}{40}$. The DT signal $x[n]$ thus obtained is then the input to a DT LTI system with impulse response

$$h[n] = 16 \left\{ \frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n} \frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n} \right\} \cos\left(\frac{\pi}{2}n\right) \tag{1}$$

Show all work. Write your expression for the output $y[n] = x[n] * h[n]$ in the space below. Plot the DTFT of $h[n]$ to help solve the problem.





$$X(\omega) = X_p(\omega) H_{LP}(\omega)$$

Thus:

$$x(t) = \frac{1}{2}\cos(2.5t) + \frac{1}{2}\cos(17.5t) + \frac{\sin(5t)}{\pi t} 2\cos(10t)$$

Problem 3. You can continue your work for 3 here.

$$T_s = \frac{2\pi}{40} = \frac{\pi}{20}$$

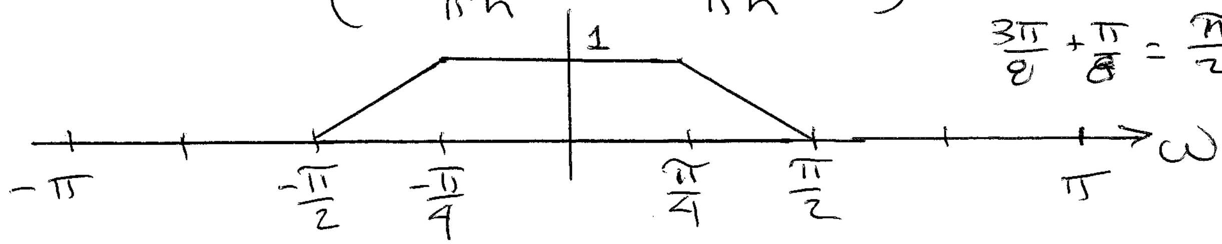
$$x[n] = \frac{1}{2}\cos\left(\frac{5}{2}\frac{\pi}{20}n\right) + \frac{1}{2}\cos\left(\frac{35}{2}\frac{\pi}{20}n\right) + \frac{\sin\left(5\frac{\pi}{20}n\right)}{\pi\frac{\pi}{20}n}\,2\cos\left(10\frac{\pi}{20}n\right)$$

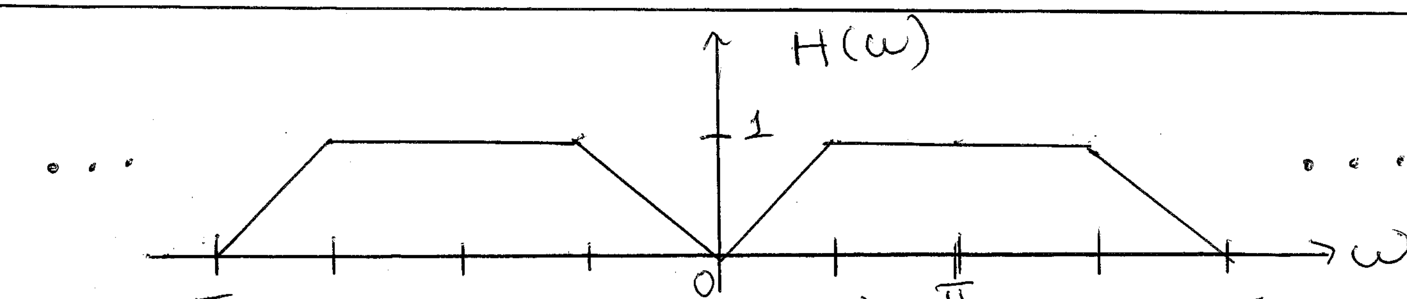
$$x[n] = \frac{1}{2}\cos\left(\frac{\pi}{8}n\right) + \frac{1}{2}\cos\left(\frac{7\pi}{8}n\right) + \frac{20}{\pi}\,\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\,2\cos\left(\frac{\pi}{2}n\right)$$

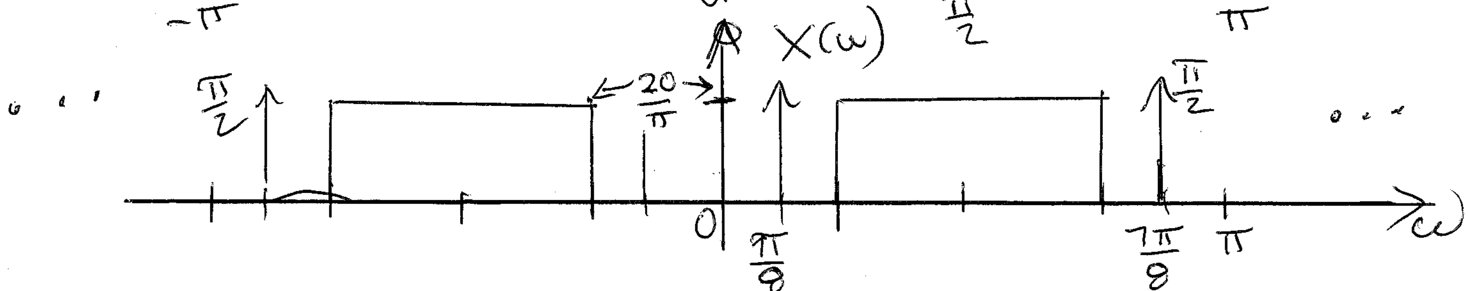
First, note: $8\left\{\frac{\sin\left(\frac{\pi}{8}n\right)}{\pi n}\,\frac{\sin\left(\frac{3\pi}{8}n\right)}{\pi n}\right\}$

$$\frac{3\pi}{8} - \frac{\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\frac{3\pi}{8} + \frac{\pi}{8} = \frac{\pi}{2}$$







$$y[n] = \frac{1}{4}\cos\left(\frac{\pi}{8}n\right) + \frac{1}{4}\cos\left(\frac{7\pi}{8}n\right) + \frac{20}{\pi}\,\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\,2\cos\left(\frac{\pi}{2}n\right)$$

Workout Problem 4. Consider the continuous-time signal $x_a(t)$ below. Note that the multiplication by the scalar j is included to make the Fourier Transform $X_a(\omega)$ be purely real-valued, and the multiplication by the scalar T_s is intended to offset the amplitude-scaling by the sampling rate $F_s = \frac{1}{T_s}$ that inherently occurs in the process of sampling.

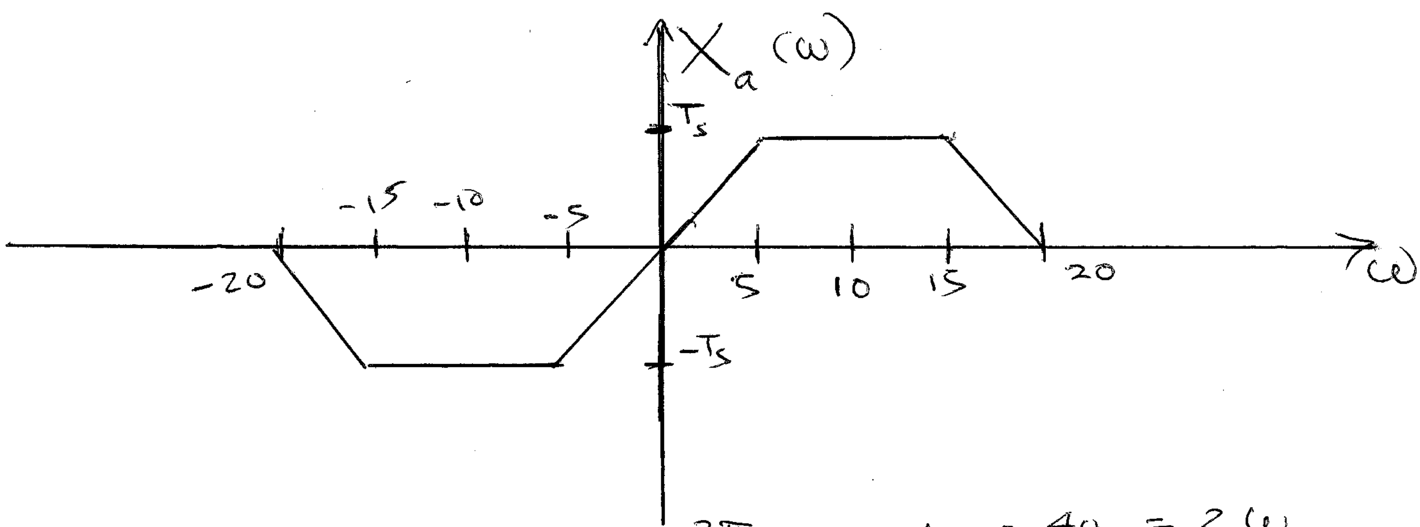
$$x_a(t) = T_s \left\{ \frac{\pi}{2.5} \frac{\sin(2.5t)}{\pi t} \frac{\sin(7.5t)}{\pi t} \right\} 2j \, \sin(10t)$$

For both parts below, indicate whether the sampling rate $\omega_s = \frac{2\pi}{T_s}$ is above or below the Nyquist rate and whether there is aliasing.

(a) A discrete-time signal is created by sampling $x_a(t)$ according to $x[n] = x_a(nT_s)$ for $T_s = \dfrac{2\pi}{40}$. Plot the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$. Show your work on this page and the next page, and do your plot in the space provided on the next page.

(b) Repeat part (a) for $T_s = \dfrac{2\pi}{25}$. Plot the new DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$. Show your work and do your plot in the space provided on the sheets attached.

VIP Note Regarding DTFT Plots: You only have to plot any DTFT over $-\pi < \omega < \pi$. You can label the abscissa any way that you want, but make it clear where π is.



$\omega_M = 20$ For $T_s = \frac{2\pi}{40}$ $\Rightarrow$ $\omega_s = 40 = 2\omega_M$

Nyquist Rate

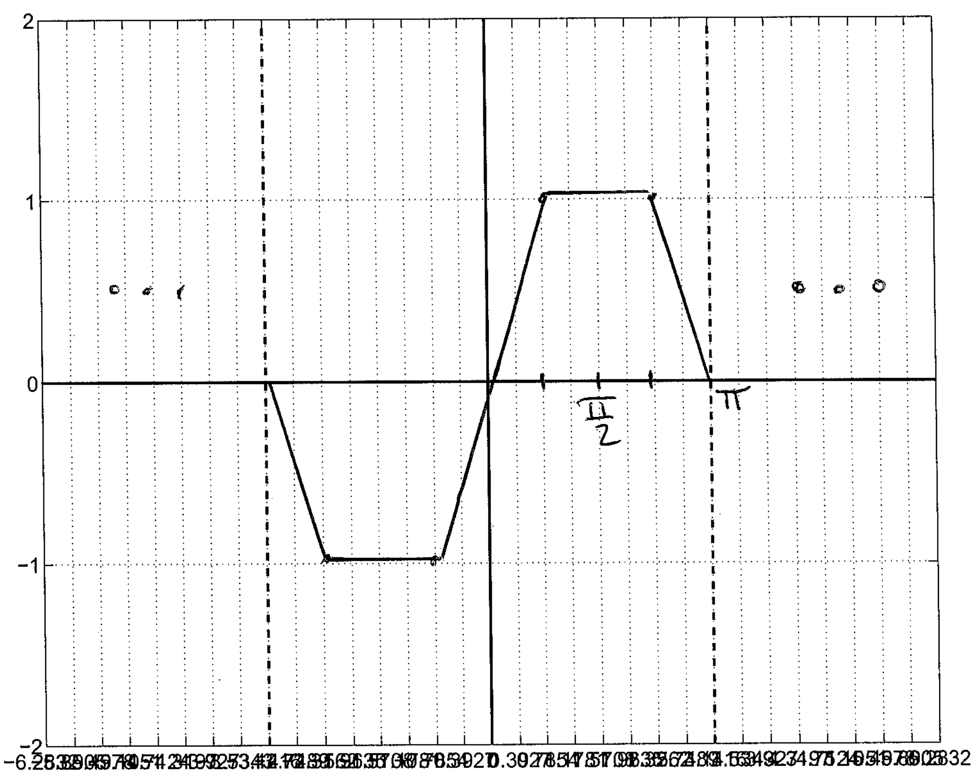
Thus, no aliasing and

$X(\omega) = F_s X_a(F_s \omega)$ for $-\pi < \omega < \pi$

consider m = integer } $5m$ mapped to $5mT_s = m\,5\,\frac{2\pi}{40}$

$= m\frac{\pi}{4}$

Plot your answer to Problem ~~3~~ 4 (a) here. Show work above.



4 (b) $T_s = \frac{2\pi}{25} \Rightarrow \omega_s = 25 < 40 \Rightarrow$ aliasing!

aliasing starts at $\omega_s - \omega_M = 25 - 20 = 5$

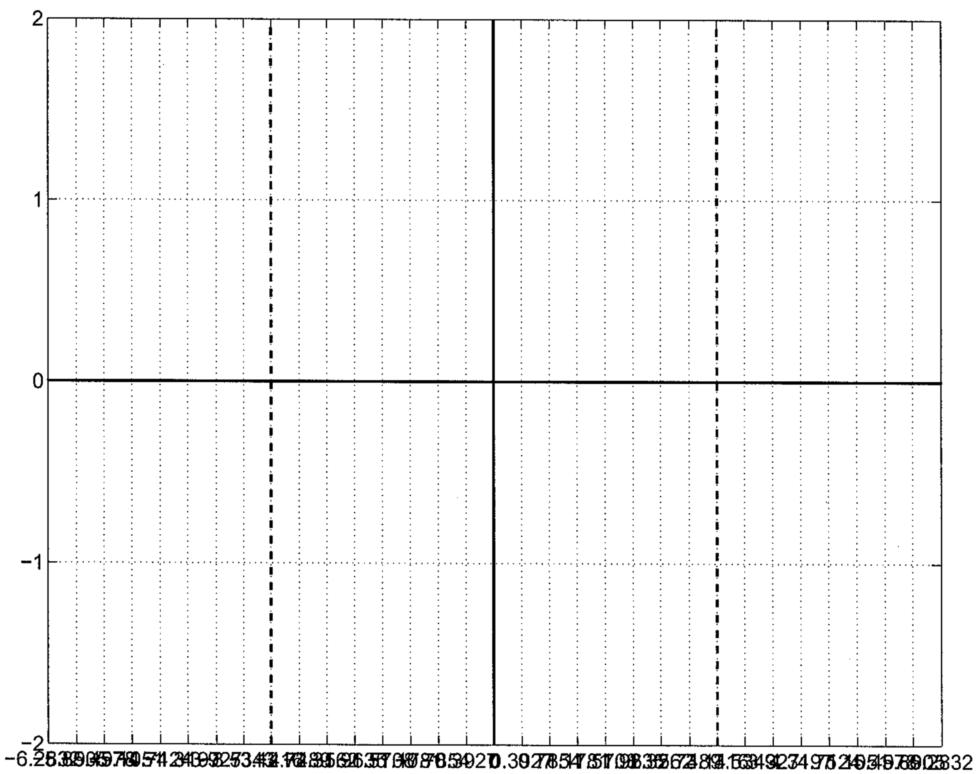
which maps to $5\,T_s = 5\,\frac{2\pi}{25} = \frac{2\pi}{5}$

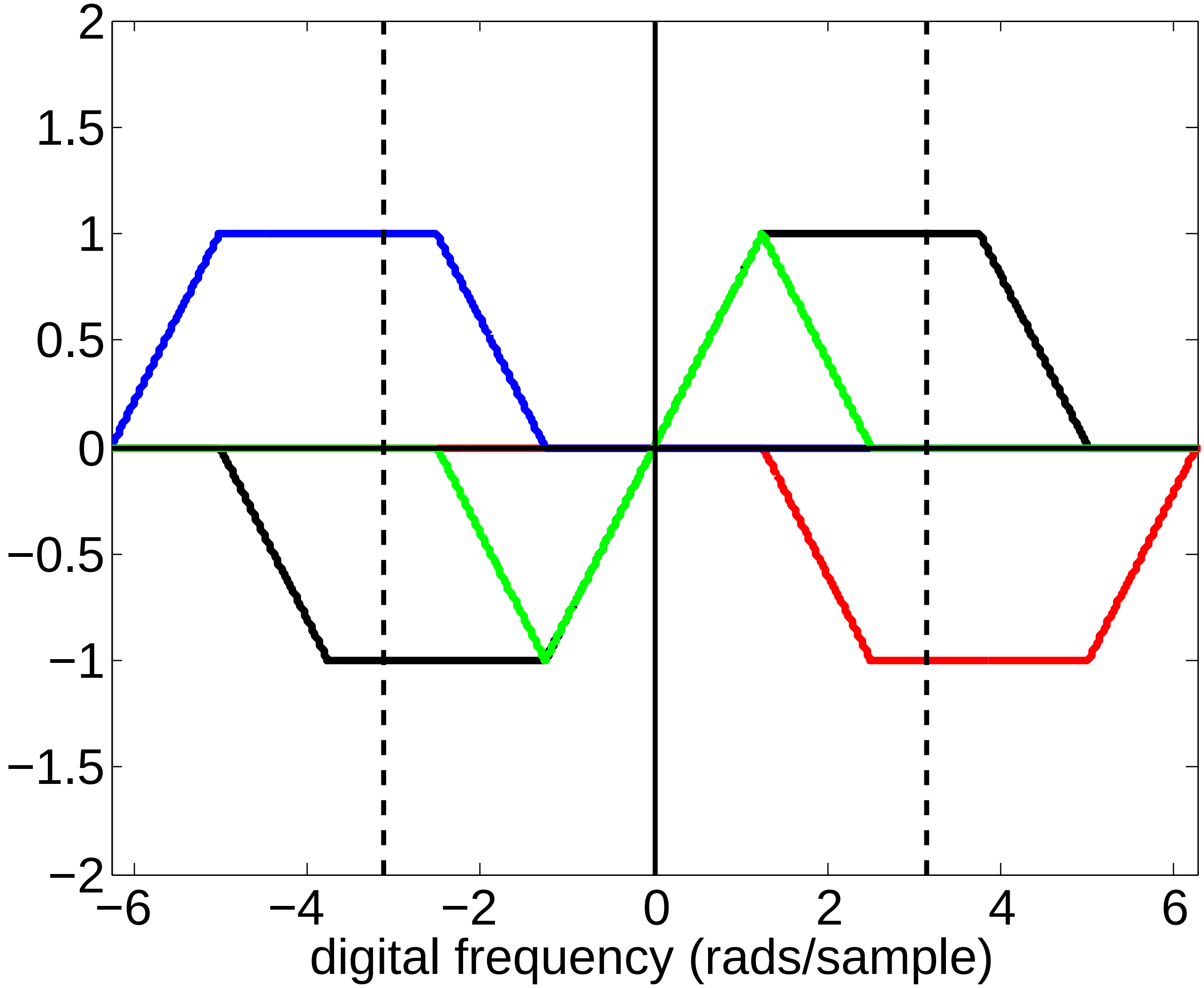
$$X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a\left(F_s\,(\omega - k2\pi)\right)$$

$k=0$ term $F_s X_a(F_s\omega)$ extends from $-20T_s < \omega < 20T_s$

$\Rightarrow -\frac{8\pi}{5} < \omega < +\frac{8\pi}{5}$

Plot your answer to Problem ~~3~~ 4 (b) here. Show work above.




2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−6
−4
−2
0
2
4
6
digital frequency (rads/sample)