SOLUTION

NAME: **NAME**
EE301 Signals and Systems **In-Class Exam**
Exam 3 **Wednesday, Apr. 17, 2013**

Cover Sheet

Test Duration: 60 minutes.
Coverage: Chaps. 5,7
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

VIP Note Regarding DTFT Plots: *The abscissa in each plot is the frequency axis. For each plot, the abscissa goes from $-\pi$ to π with tic marks every $\pi/16$. There is a dashed vertical line at $\omega = -\frac{\pi}{2}$ and another dashed vertical line at $\omega = +\frac{\pi}{2}$. You only have to plot any DTFT over $-\pi < \omega < \pi$.*

Problem 1. Short answer questions.

(a) The maximum frequency for the signal $x_a(t)$ is ω_{max}, such that the $X_a(\omega) = 0$ for $|\omega| > \omega_{max}$. What is the maximum frequency for the signal $y_a(t) = x_a^2(t)$ in terms of ω_{max}?

$$x_a^2(t) \xleftrightarrow{\mathcal{F}} \underbrace{\frac{1}{2\pi} X_a(\omega) * X_a(\omega)}$$

max frequency $= 2\,\omega_{max}$

$(\omega_{max} + \omega_{max})$

(b) If you sample at a rate $\omega_s = 40$ in terms of radians/sec, what analog frequency gets mapped unaliased to the Discrete-Time frequency $\omega_d = \frac{\pi}{2}$?

$$\omega_d = \omega_a T_s = \frac{\omega_a}{F_s} = \frac{2\pi\,\omega_a}{2\pi F_s} = 2\pi \frac{\omega_a}{\omega_s}$$

$$\omega_d = \frac{\pi}{2} = 2\pi \frac{\omega_a}{40}$$

$$\omega_a = \frac{40}{4} = 10 \text{ rads/sec}$$

(c) If you sample the sinewave $x_a(t) = \cos(8t)$ at a rate of $\omega_s = 14$ radians/sec, which is below the Nqyuist rate, what analog frequency (in radians/sec) does the under-sampled sinewave get aliased to?

$$\frac{\omega_s}{2} = \frac{14}{2} = 7$$

Since 8 is 1 above Half the Nyquist-Rate, it gets aliased to 1 below Half Nyquist Rate

$\Rightarrow$ 8 gets aliased to 6 rads/sec

(d) The decaying exponential signal $x_a(t) = e^{-ln(2)t}u(t)$ is sampled every $T_s = 1$ second to form $x[n] = x_a(nT_s)$, where, again, $T_s = 1$ second. The Fourier Transform of $x_a(t)$ is $X_a(\omega) = \frac{1}{ln(2)+j\omega}$ and is not strictly band limited so there will always be some amount of aliasing. We know that the DTFT of $x[n]$ is related to the CTFT $X_a(\omega)$ according to the expressions below, where $F_s = 1$ and $\omega_s = 2\pi$, since $T_s = 1$:

$$X(\omega) = X_s(F_s\omega) \quad \text{where:} \quad X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

Determine a closed-form expression for the DTFT $X(\omega)$. *Hint:* Trick question and $ln(2)$ equal to the natural logarithm of 2 was chosen to make the numbers work out nicely. Recall $e^{ln(x)} = x$.

$$x[n] = x_a(nT_s)\Big|_{T_s=1} = e^{-ln(2)n} u[n]$$

$$= \left(e^{-ln(2)}\right)^n u[n]$$

$$= \left(\frac{1}{e^{ln(2)}}\right)^n u[n]$$

$$= \left(\frac{1}{2}\right)^n u[n]$$

From Table 5.2 $\quad X(\omega) = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}}$

Problem 2 (a). Consider a CT signal $x_a(t)$ with bandwidth (maximum frequency) W in rads/sec. The sampling rate is chosen to be above the Nyquist rate at $\omega_s = 3.5W$, where $\omega_s = 2\pi/T_s$. $x_a(t)$ is reconstructed perfectly according to the formula below. Let $H(\omega)$ be the CTFT of $h(t)$. Determine the respective values of ω_1 and ω_2, both in terms of W, so that the CTFT $H(\omega)$ is flat up to the bandwidth W and then rolls off to zero at $\omega_s - W$.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h(t-nT_s) \quad \text{where} \quad h(t) = T_s\frac{\pi}{\omega_1}\frac{\sin(\omega_1 t)}{\pi t}\frac{\sin(\omega_2 t)}{\pi t} \quad \text{and} \quad \omega_s = \frac{7}{2}W$$

(1) $\omega_1 + \omega_2 = \omega_s - \omega_M = \frac{7}{2}W - W = \frac{5}{2}W$

(2) $-\omega_1 + \omega_2 = \omega_M = W$

(1) + (2) : $2\omega_2 = \frac{7}{2}W \Rightarrow \omega_2 = \frac{7}{4}W \quad \left(= \frac{\omega_s}{2}\right)$

(2) $\omega_1 = \omega_2 - W = \frac{7}{4}W - \frac{4}{4}W = \frac{3}{4}W$

$\omega_1 = \frac{3}{4}W \qquad \omega_2 = \frac{7}{4}W$

Problem 2 (b). Determine an expression for the DTFT $X(\omega)$ in terms of the integer-valued K for the finite-length geometric sequence $x[n]$ below. Be sure to indicate which DTFT properties and/or pairs you use to arrive at your answer.

$$x[n] = n\{u[n+K] - u[n-(K+1)]\}$$

From Table 5.2

$$u[n+K] - u[n-(K+1)] \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{(2K+1)}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

From Table 5.1

$$n\,x[n] \xleftrightarrow{\text{DTFT}} j\,\frac{dX(\omega)}{d\omega}$$

$$X(\omega) = j\,\frac{d}{d\omega}\left\{\frac{\sin\left(\frac{(2K+1)}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right\}$$

Use quotient rule

Problem 2 (c). Consider the signal $x_p(t)$ below, which is the Fourier Series expansion for a periodic waveform with fundamental frequency $\omega_0 = 4$ rads/sec. The values of the Fourier Series coefficients a_k are unspecified; just carry them along as constants through your analysis.

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk4t}$$

This signal is first low-passed filtered with an analog lowpass filter with impulse response

$$h_{LP}(t) = \frac{\pi}{4} \frac{\sin(4t)}{\pi t} \frac{\sin(20t)}{\pi t}$$

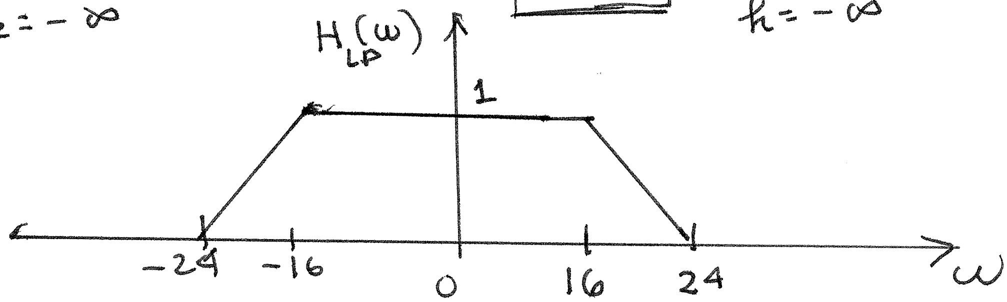
to form $x(t) = x_p(t) * h_{LP}(t)$, and then $x(t)$ is sampled at a rate of $\omega_s = 48$ to form $x[n]$, so that the time between samples is $T_s = \frac{2\pi}{48}$. The DT signal $x[n]$ thus obtained is then the input to a DT LTI system with impulse response

$$h[n] = 6 \, \frac{\sin\left(\frac{\pi}{6} n\right)}{\pi n} \frac{\sin\left(\frac{5\pi}{6} n\right)}{\pi n} \tag{1}$$

Show all work. Write your expression for the output $y[n] = x[n] * h[n]$ in the space below. Plot the DTFT of $h[n]$ to help determine the gain of each sinewave gets as it passes thru the system. Be very clear about the output amplitude and frequency of each Discrete-Time sinewave at the output in your final answer.

First, analog LPF:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk4t} \longrightarrow \boxed{h_{LP}(t)} \longrightarrow \sum_{k=-\infty}^{\infty} H(k4)\, a_k e^{j4kt}$$



Frequencies: $0, \pm 4, \pm 8, \pm 12, \pm 16$ pass thru system with unity gain

± 20 gets gain of $1/2$

$\pm k4$ for $k > 5$ are rejected

$$x(t) = \frac{1}{2} a_{-5} e^{-j20t} + \sum_{k=-4}^{4} a_k e^{jk4t} + \frac{1}{2} a_5 e^{j20t}$$

is sampled at a rate $\omega_s = 48$

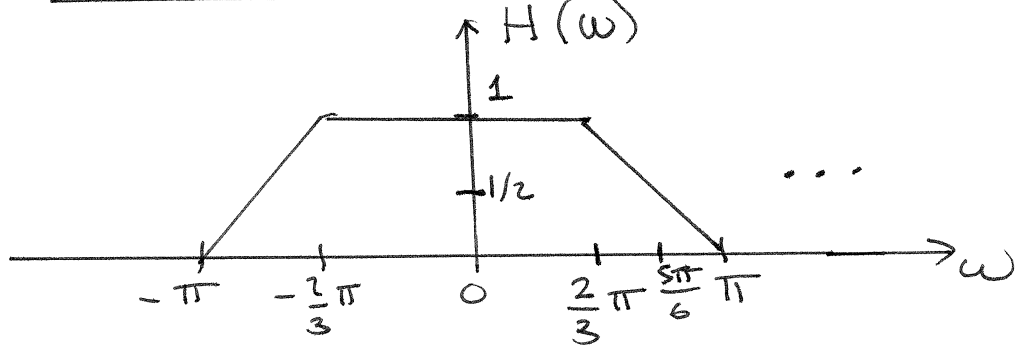
replace t by $nT_s = n \frac{2\pi}{48}$ to obtain $x[n]$

Problem 2 (c). You can continue your work for 2(c) here.

$$x[n] = \frac{1}{2} a_{-5} e^{-j\frac{40}{48}\pi n} + \sum_{k=-4}^{4} a_k e^{jk\,4\frac{2\pi}{48}n} + \frac{1}{2} a_5 e^{j\,20\frac{2\pi}{48}n}$$

$$= \frac{1}{2} a_{-5} e^{-j\frac{5}{6}\pi n} + \sum_{k=-4}^{4} a_k e^{jk\frac{\pi}{6}n} + \frac{1}{2} a_5 e^{j\frac{5}{6}\pi n}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



$$\frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6}$$

$$y[n] = \frac{1}{4} a_{-5} e^{-j\frac{5}{6}\pi n} + \sum_{k=-4}^{4} a_k e^{jk\frac{\pi}{6}n} + \frac{1}{4} a_5 e^{j\frac{5}{6}\pi n}$$

Workout Problem 3 Consider the continuous-time signal $x_a(t)$ below. Note that the multiplication by the scalar $-j$ is included to make the Fourier Transform $X_a(\omega)$ be purely real-valued, and the multiplication by the scalar T_s is intended to offset the amplitude-scaling by the sampling rate $F_s = \frac{1}{T_s}$ that inherently occurs in the process of sampling.

$$x_a(t) = -j\, T_s \frac{1}{5} \frac{d}{dt} \left\{ \frac{\sin(10t)}{\pi t} \right\}$$

For both parts below, indicate whether the sampling rate $\omega_s = \frac{2\pi}{T_s}$ is above or below the Nyquist rate and whether there is aliasing.

(a) A discrete-time signal is created by sampling $x_a(t)$ according to $x[n] = x_a(nT_s)$ for $T_s = \dfrac{3\pi}{40}$. Plot the DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$. Show your work on this page and the next page, and do your plot in the space provided on the next page.

(b) Repeat part (a) for $T_s = \dfrac{5\pi}{40}$. Plot the new DTFT of $x[n]$, $X(\omega)$, over $-\pi < \omega < \pi$. Show your work and do your plot in the space provided on the sheets attached.

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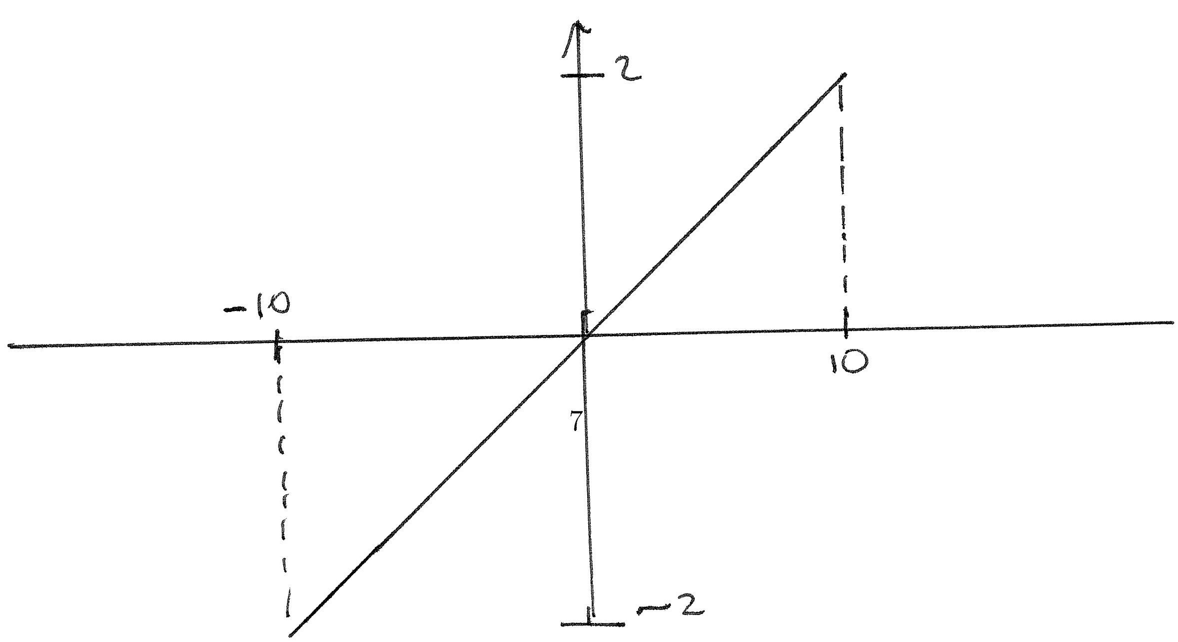
(a) $\omega_M = 10$ $\qquad \omega_s = \frac{2\pi}{T_s} = 2\pi \cdot \frac{40}{3\pi} = \frac{80}{3} > 2\omega_M = 20$

$\Rightarrow$ no aliasing!

$\omega_M = 10$ is mapped to $10 T_s = 10 \cdot \frac{3\pi}{40} = \frac{3\pi}{4}$



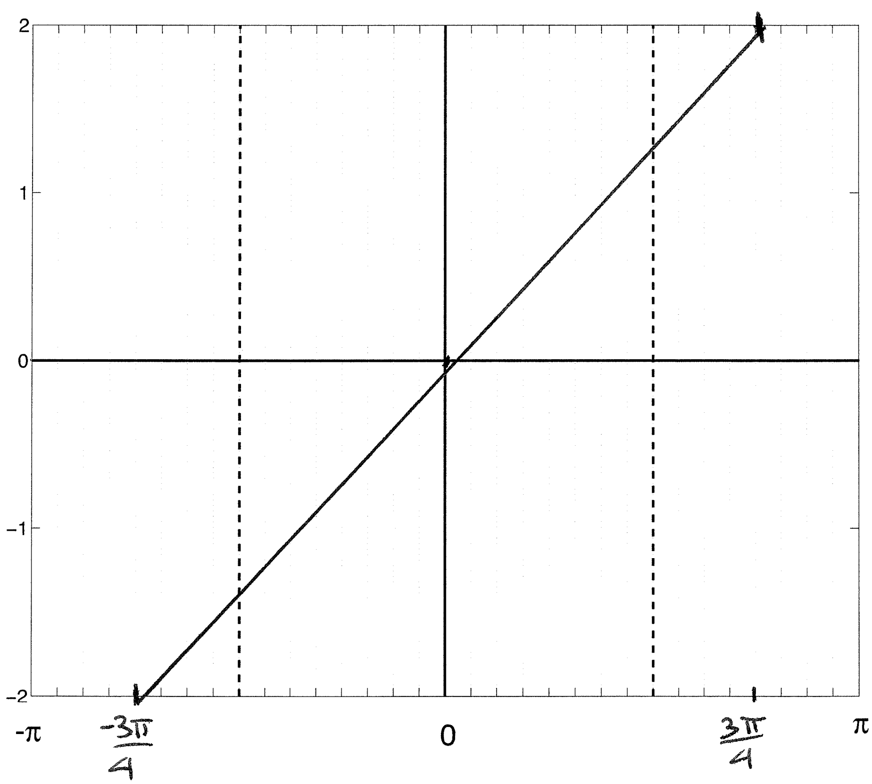
For plot on next page, sampling rate lowered so that $T_s = \frac{5\pi}{40} \Rightarrow \omega_s = \frac{2\pi}{T_s} = 2\pi \frac{40}{5\pi} = 16 < 2(10)$

$\omega_M = 10$ is mapped to $\omega_M T_s = 10 \cdot \frac{5\pi}{40} = \frac{5\pi}{4}$

Aliasing starts at $\omega_s - \omega_M = 16 - 10 = 6$ $\left(\text{height} = \frac{6}{5} = 1.2\right.$

is mapped to $6 T_s = 6 \frac{5\pi}{40} = \frac{30\pi}{40} = \frac{3}{4}\pi$

Plot your answer to Problem 3 (a) here. Show work above.



Plot your answer to Problem 3 (b) here. Show work above.