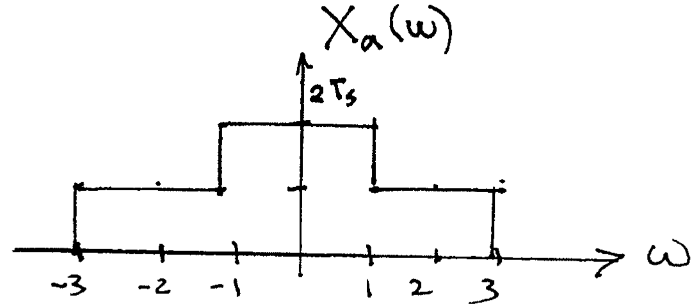
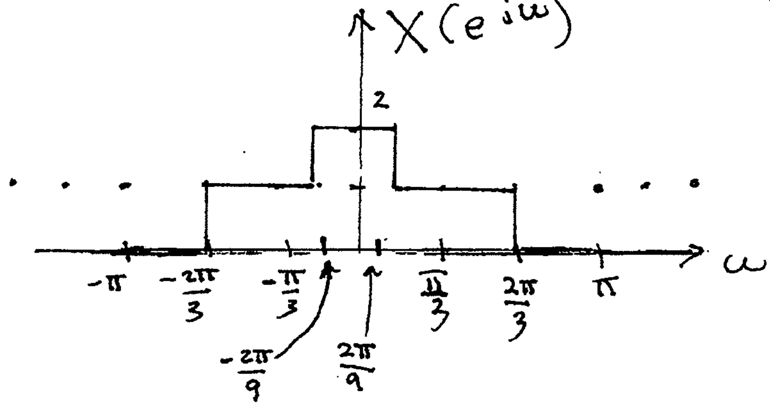
Solution to Exam 3

Problem 1 (a)



$\omega_M = 3 \qquad \omega_s = \frac{2\pi}{T_s} = 9 > 3(2) = 6 \Rightarrow$ no aliasing

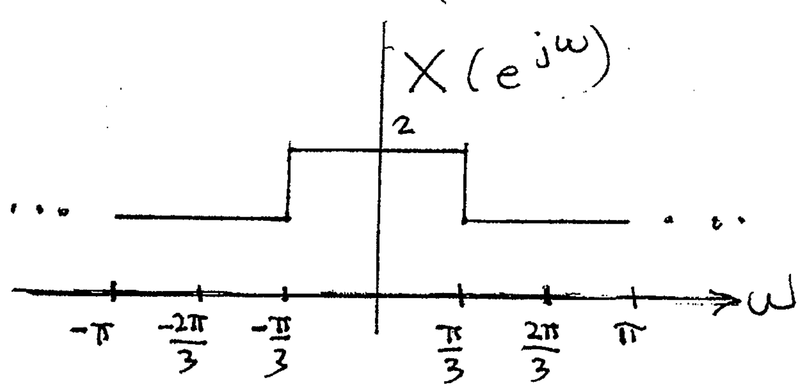
$\omega_M = 3$ is mapped to $3T_s = 3 \cdot \frac{2\pi}{9} = \frac{2\pi}{3}$



period $= 2\pi$

(b) same $x_a(t)$, now $\omega_s = \frac{2\pi}{T_s} = 6 = 2(3) \Rightarrow$ right at Nyquist rate

$\omega_M = 3$ mapped to $3\left(\frac{2\pi}{6}\right) = \pi$

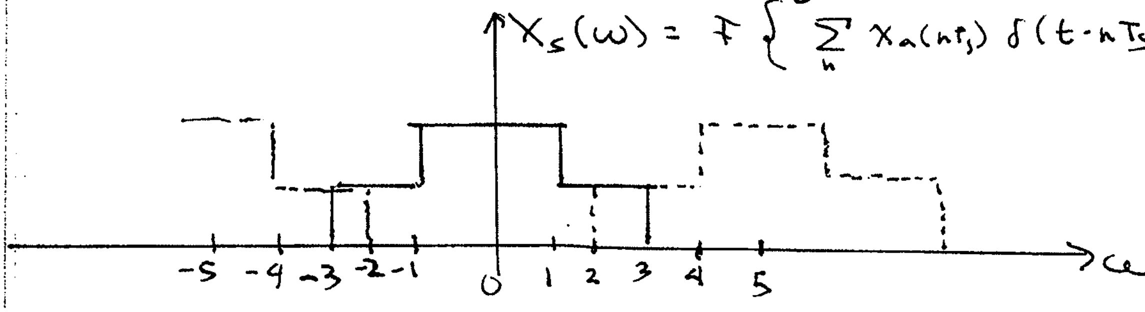


Exam 3 Prob. 1 Sol'n.

1(c) same $x_a(t)$, now $T_s = \frac{2\pi}{5}$

$\omega_s = 5 < 2(3) \Rightarrow$ aliasing!!

$X_s(\omega) = \mathcal{F}\left\{ \sum_n x_a(nT_s)\,\delta(t - nT_s) \right\}$

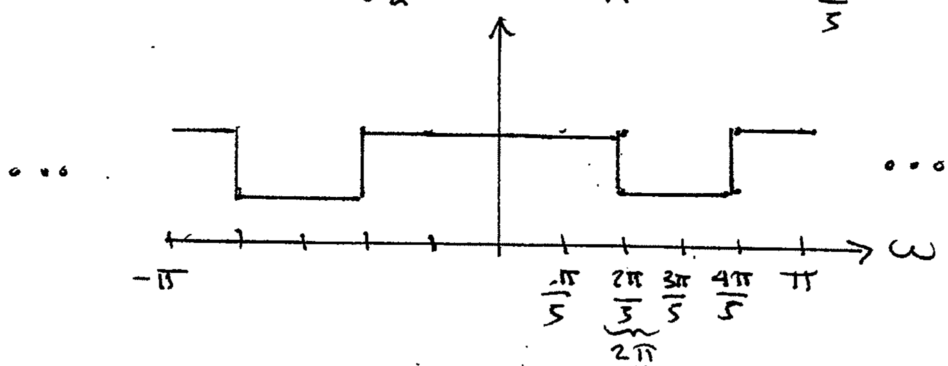


Every analog freq. mapped to digital frequency by multiplying by $T_s = \frac{2\pi}{5}$

For example $\omega_a = 2.5 = \frac{5}{2}$ mapped to $\frac{5}{2} \cdot \frac{2\pi}{5} = 5$

$\omega_a = 1$ mapped to $\frac{2\pi}{5}$

$\omega_a = 2$ mapped to $\frac{4\pi}{5}$



1(d) Same $x_a(t)$, now $T_s = \frac{2\pi}{4} = \frac{\pi}{2} \Rightarrow$ aliasing

$$x[n] = T_s \frac{\sin\left(n\frac{\pi}{2}\right)}{n\frac{\pi}{2}} + T_s \frac{\sin\left(\frac{3\pi}{2}n\right)}{n\frac{\pi}{2}}$$

$$\sin\left(\frac{3\pi}{2}n - \frac{4\pi}{2}n\right) = \sin\left(-\frac{\pi}{2}n\right) = -\sin\left(\frac{\pi}{2}n\right)$$

proportional to

so $x[n] = 0$ for all n except at $n=0$ $\Rightarrow x[n] \propto \delta[n]$

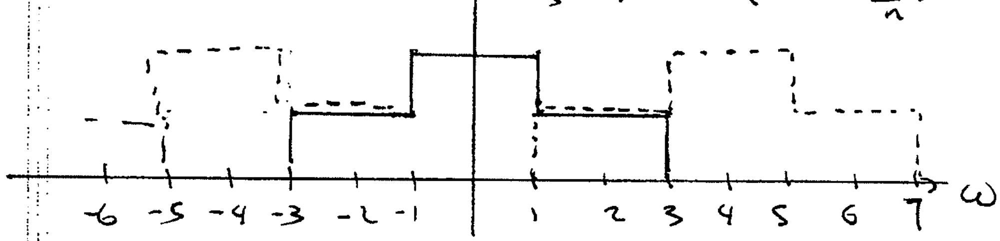
Prob. 1 (d) (cont.)

at $n=0$, we have:

$$\lim_{n\to 0} x[n] = T_s \frac{\frac{n\pi}{2}}{\frac{n\pi}{2}} + T_s \frac{\frac{3n\pi}{2}}{\frac{n\pi}{2}} = T_s(1+3) = \frac{2\pi}{4}(4) = 2\pi$$

$\omega_s = 4$

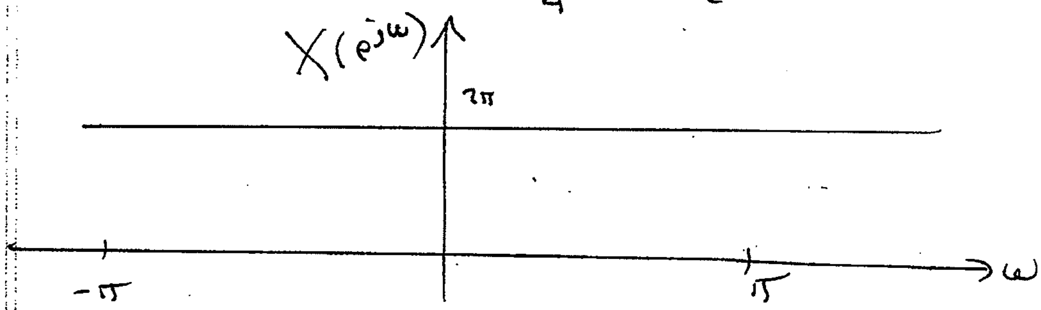
$X_s(\omega) = \mathcal{F}\{x(t) \sum_n \delta(t - nT_s)\}$



4 mapped to $\frac{2\pi}{4} 4 = 2\pi$

2 mapped to $\frac{2\pi}{4} \cdot 2 = \pi$

1 mapped to $\frac{2\pi}{4} \cdot 1 = \frac{\pi}{2}$



(e) $x_a(t) = 2T_s \left\{ \frac{\sin(t)}{\pi t} + \frac{\sin(3t)}{\pi t} \right\} \cos(3t)$

$\omega_M = 3 + 3 = 6$

$\omega_s = \frac{2\pi}{T_s} = 12 = 2(6)$ Nyquist rate

$\Rightarrow$ no aliasing

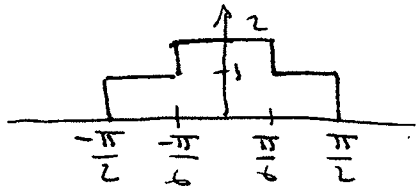
Prob. 1 (e) Sol'n.

• with no aliasing, we can simply substitute $t = nT_s = n\frac{\pi}{6}$

$$x[n] = 2\cancel{T_s}\left\{\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi \cancel{T_s} n} + \frac{\sin\left(\frac{3(\pi)}{6}n\right)}{\pi \cancel{T_s} n}\right\}\cos\left(\frac{3(\pi)}{6}n\right)$$

$$= \left\{\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n} + \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\right\}\left\{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right\}$$

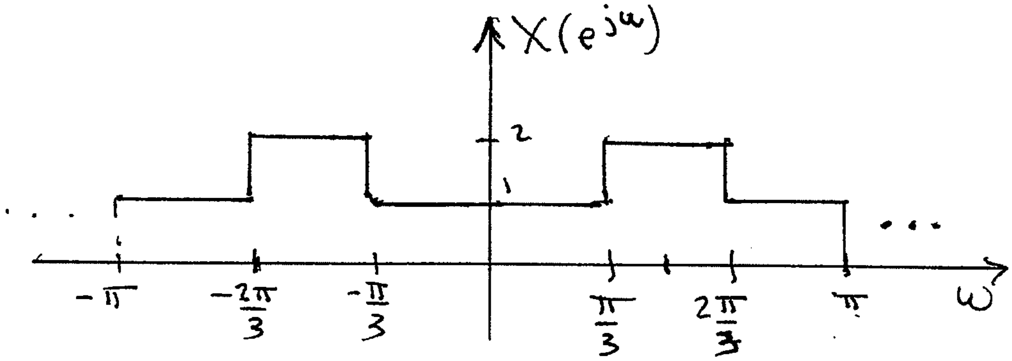
$X(e^{j\omega}) \Rightarrow$



shifted to right by $\frac{\pi}{2}$

$+$

shifted to left by $\frac{\pi}{2}$



$\frac{\pi}{2} - \frac{\pi}{6}$

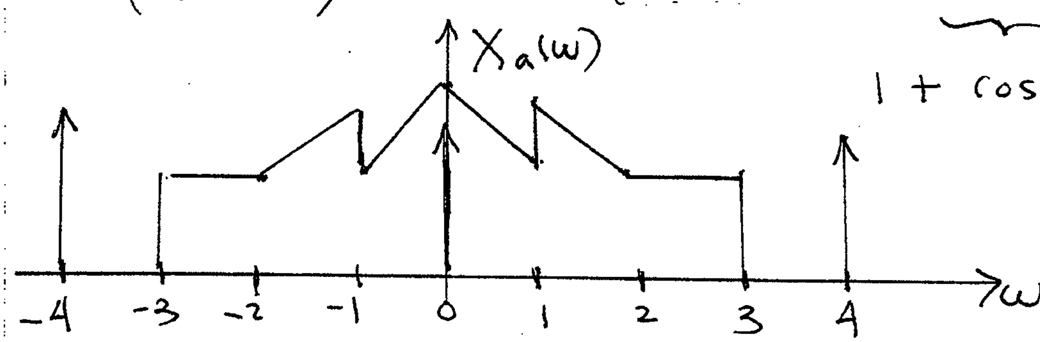
$= \frac{3\pi}{6} - \frac{\pi}{6}$

$= \frac{2\pi}{6}$

$= \frac{\pi}{3}$

Prob. 1 (f) $x_a(t) = 2T_s\left\{\frac{\sin(t)}{\pi t} + \cos(2t)\right\}^2$

$$= 2T_s\left(\frac{\sin(t)}{\pi t}\right)^2 + T_s\frac{\sin(t)}{\pi t}2\cos(2t) + \underbrace{T_s 2\cos^2(2t)}_{1+\cos(4t)}$$



Prob. 1 (f) Sol'n (cont.)

$\omega_s = 5$

$\omega_M = 4 \Rightarrow$ Nyquist rate $= 2(4) = 8$

Since $5 < 8 \Rightarrow$ aliasing

what about sinewave $\cos(4t)$:

$$\cos(4t)\Big|_{t = nT_s = n\frac{2\pi}{5}} = \cos\left(4n\frac{2\pi}{5}\right)$$

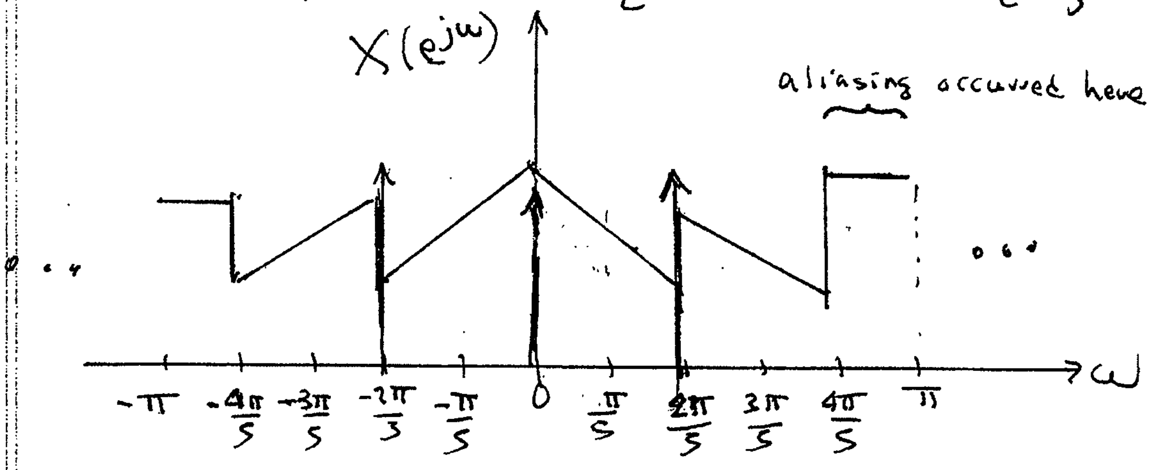
$$= \cos\left(n\frac{8}{5}\pi - n\frac{10}{5}\pi\right)$$

$$= \cos\left(-\frac{2\pi}{5}n\right) = \cos\left(+\frac{2\pi}{5}n\right)$$

In addition, aliasing also occurs in the range $2 \le \omega < 3 \Rightarrow$ overlap of rectangular regions

Again, analog frequency mapped to digital frequency by multiply by $T_s = \frac{2\pi}{5}$

For example, $\omega = 2.5 = \frac{5}{2}$ mapped to $\frac{5}{2} \cdot \frac{2\pi}{5} = \pi$



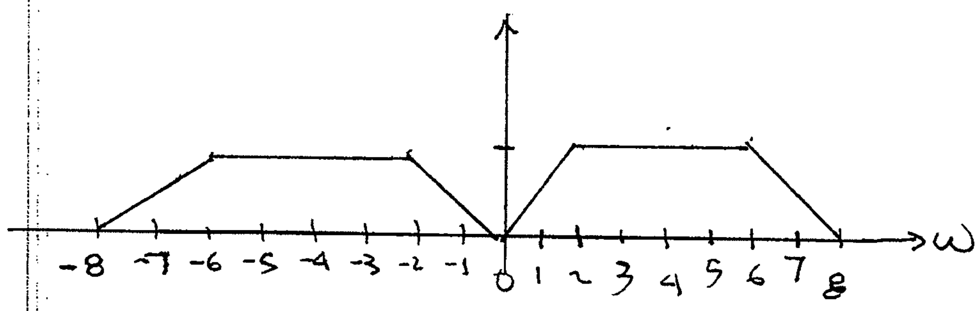
Prob. 1 (g)

$$T_s = \frac{2\pi}{32} \qquad x_a(t) = T_s \left\{ \frac{\sin(t)}{\pi t} \; \frac{\sin(3t)}{\pi t} \right\} 2\cos(4t)$$

$$\omega_M = 4 + (3+1) = 8$$

$$\omega_s = \frac{2\pi}{T_s} = 32 > 2(8) = 16 \qquad \text{no aliasing}$$

$X_a(\omega)$

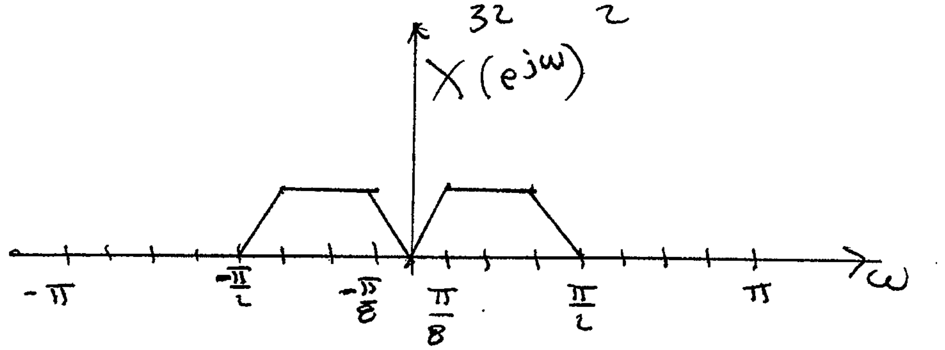


multiply each analog frequency by $T_s = \frac{2\pi}{32}$

2 mapped to $\frac{4\pi}{32} = \frac{\pi}{8}$

6 mapped to $\frac{12\pi}{32} = \frac{3\pi}{8}$

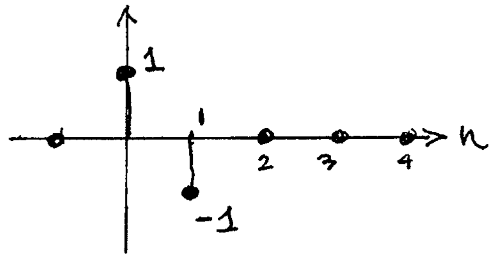
8 mapped to $\frac{16\pi}{32} = \frac{\pi}{2}$



Prob. 2 Sol'n. Exam 3

(a) $y[n] = x[n] - x[n-1]$

$h[n] = \delta[n] - \delta[n-1]$



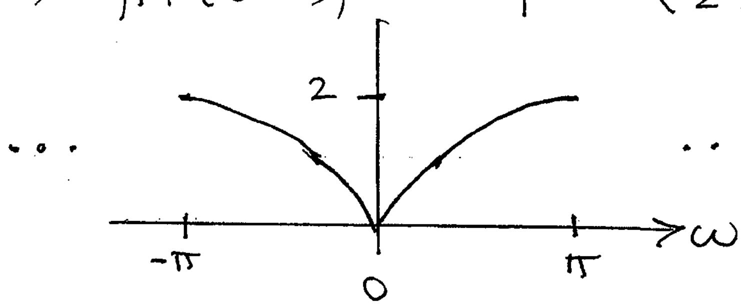
(b) $H(e^{j\omega}) = 1 - e^{-j\omega}$

$= e^{-j\frac{\omega}{2}} \left\{ e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}} \right\}$

$= 2j\, e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right)$

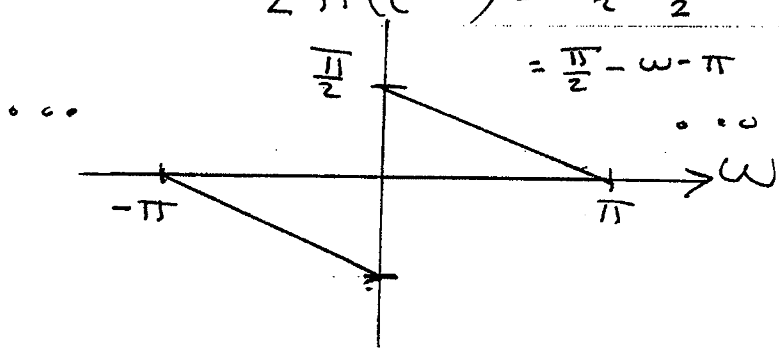
$= 2 \sin\left(\frac{\omega}{2}\right) e^{j\left(\frac{\pi}{2} - \omega\right)}$

(c) $|H(e^{j\omega})| = 2 \left|\sin\left(\frac{\omega}{2}\right)\right|$

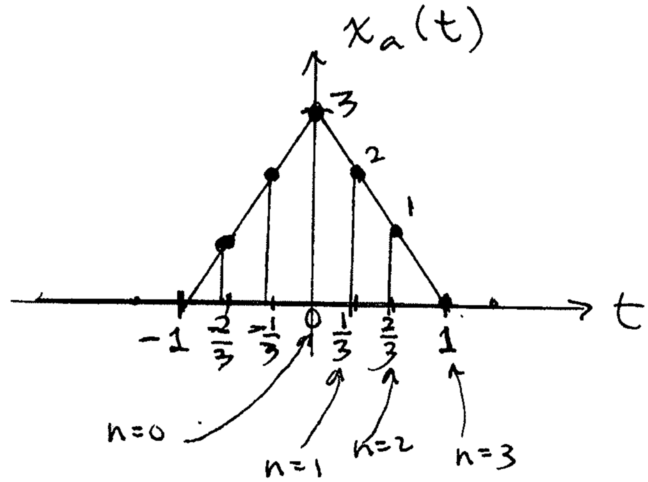


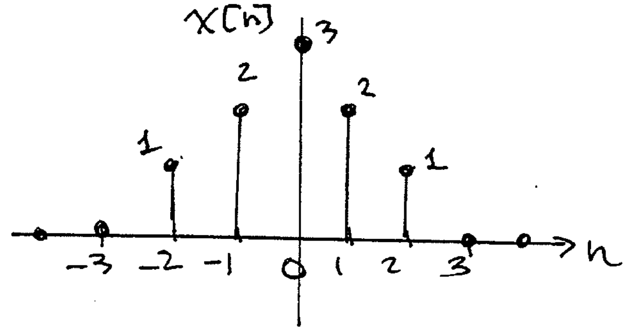
(d) $\angle H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2}$ $\quad 0 < \omega < \pi$

$= \frac{\pi}{2} - \omega - \pi$ $\quad -\pi < \omega < 0$

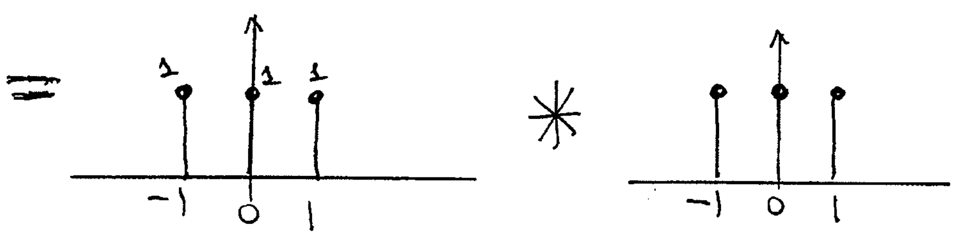


(e)





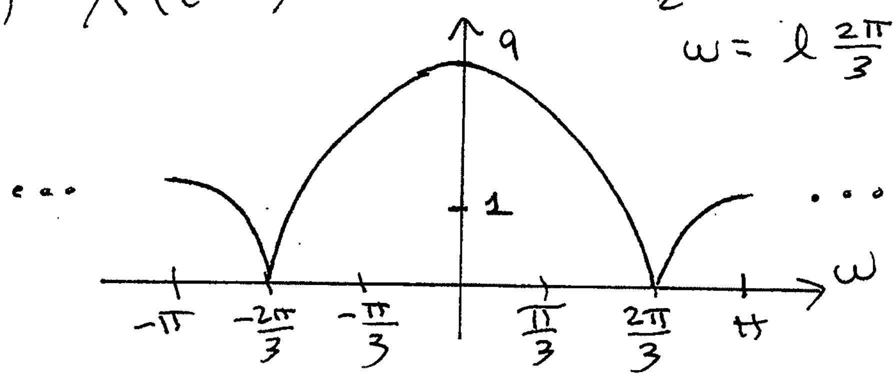
(f)



$$X(e^{j\omega}) = \frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} \cdot \frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} = \left(\frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right)^2$$

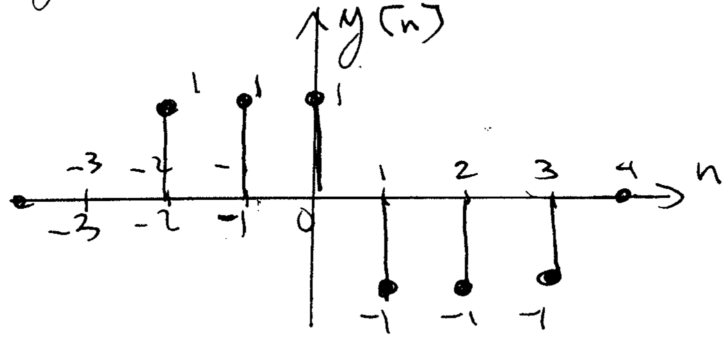
(g) $X(e^{j\omega}) = 0$ when $\frac{3}{2}\omega = l\pi$ l, integer

$$\omega = l\,\frac{2\pi}{3}$$



Prob. 2 Exam 3 (cont.)

(h) $y[n] = x[n] - x[n-1]$



(i) two methods: <u>Method 1:</u>

$$Y(e^{j\omega}) = \frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{j\omega} - \frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j2\omega}$$

$$= \frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{\omega}{2}} \left\{ e^{j\frac{3}{2}\omega} - e^{-j\frac{3}{2}\omega} \right\} \frac{2j}{2j}$$

$$= 2 \frac{\left(\sin\left(\frac{3}{2}\omega\right)\right)^2}{\sin\left(\frac{1}{2}\omega\right)} e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)}$$

$$= 2j \frac{\sin^2\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{\omega}{2}}$$

<u>Method 2:</u>

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$
$$= X(e^{j\omega}) H(e^{j\omega})$$

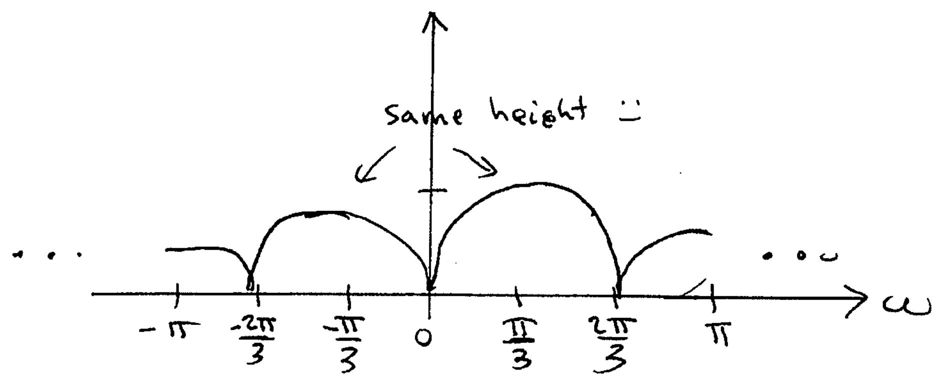
$$Y(e^{j\omega}) = \left(\frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right)^2 2j \sin\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}}$$

$$= \frac{\sin^2\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} 2j\, e^{-j\frac{\omega}{2}}$$

✓ checks

Prob. 2 Exam 3 (cont.)

(j) $|Y(e^{j\omega})| = |H(e^{j\omega})|\,|X(e^{j\omega})|$



(k) $\sum_{n=-\infty}^{\infty} y^2[n] = 3(-1)^2 + 3(1)^2 = 6$