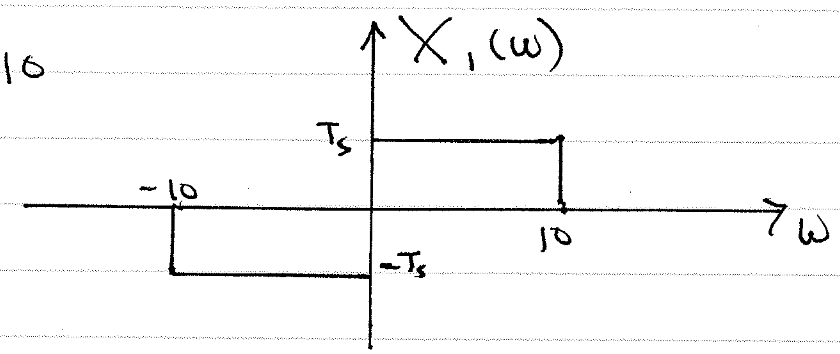
Exam 3 Solution Spring 2010

(a) $x_1(t) = T_s \, j \, 2\pi t \left\{ \dfrac{\sin(5t)}{\pi t} \right\}^2$ } from Exam 2

$$= T_s \left\{ \frac{\sin(5t)}{\pi t} \right\} 2j \sin(5t)$$

$$= T_s \frac{\sin(5t)}{\pi t} e^{j5t} - T_s \frac{\sin(5t)}{\pi t} e^{-j5t}$$

$\omega_{max} = 10$



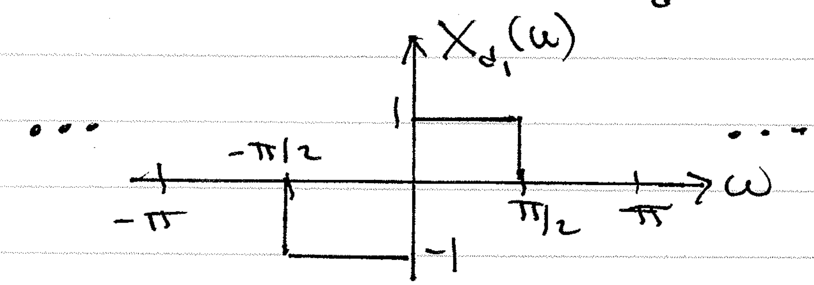
$\omega_s = 40$

Since $T_s = \dfrac{2\pi}{40}$ $\quad 40 > 2\omega_{max} = 20$

$\Rightarrow$ no aliasing

Thus, DTFT $X_{d_1}(\omega) = \dfrac{1}{T_s} X_1(F_s \omega) = \dfrac{1}{T_s} X_1\left(\dfrac{\omega}{T_s}\right)$

$\omega_a = 10$ gets mapped to $\omega_d = 10 T_s = 10 \cdot \dfrac{2\pi}{40} = \dfrac{\pi}{2}$

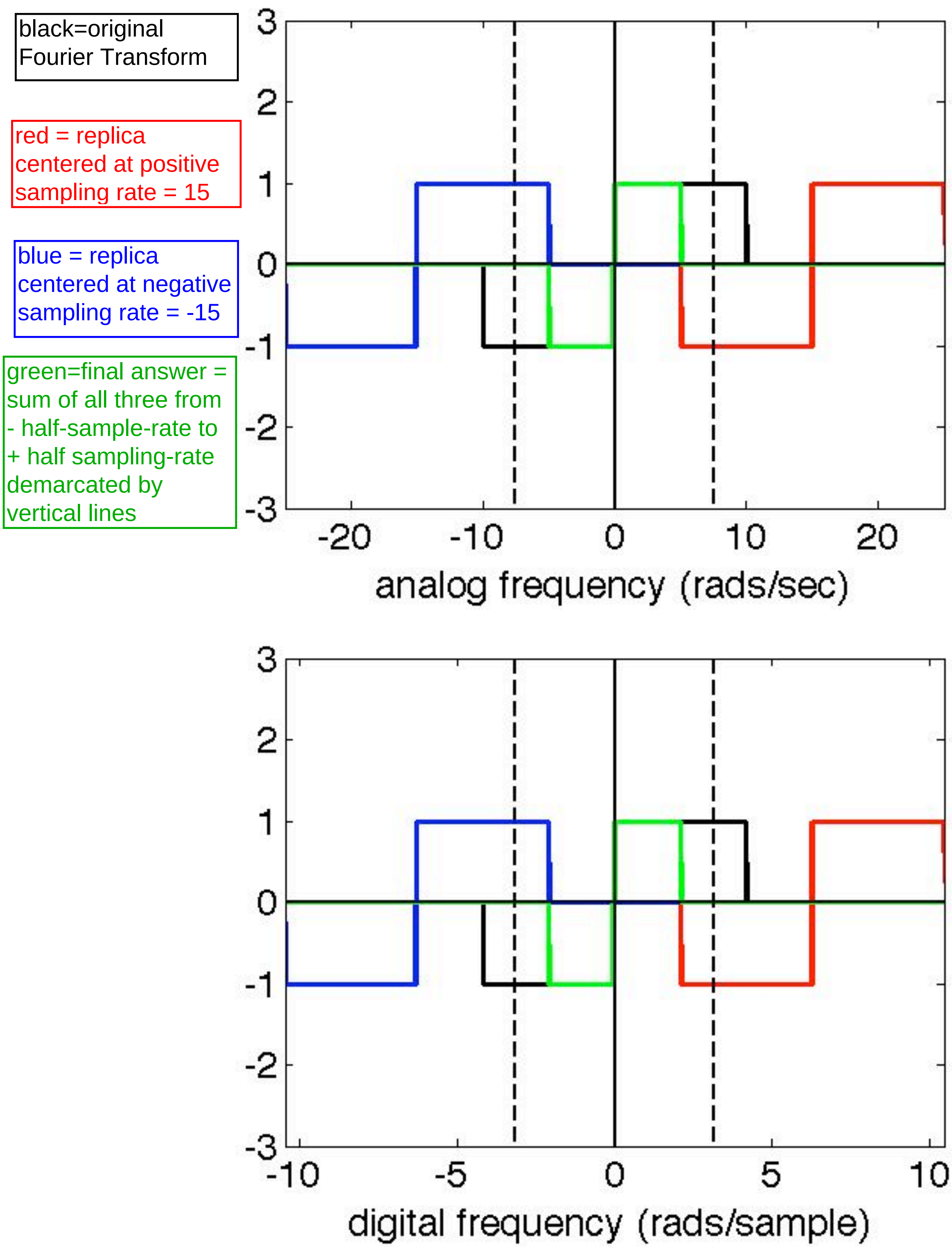


(b) $T_s = \dfrac{2\pi}{15}$ $\quad 15 < 2(10) = 20 \Rightarrow$ aliasing!

The aliasing starts at $(\omega_s - \omega_{max}) \cdot T_s$

$$= (15-10)\frac{2\pi}{15} = \frac{2\pi}{3}$$

See next page for plots


black=original Fourier Transform
red = replica centered at positive sampling rate = 15
blue = replica centered at negative sampling rate = -15
green=final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
3
2
1
0
-1
-2
-3
-20
-10
0
10
20
analog frequency (rads/sec)
3
2
1
0
-1
-2
-3
-10
-5
0
5
10
digital frequency (rads/sample)

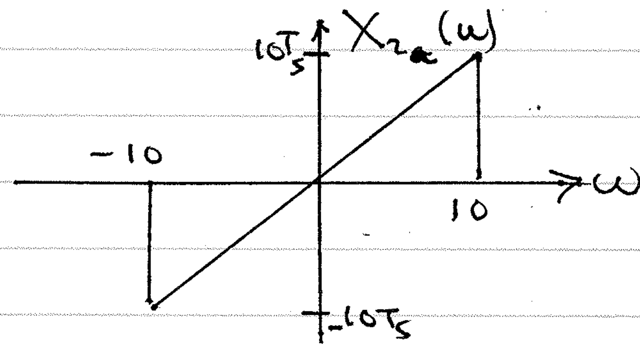
Part (c) $x_2(t) = -jT_s \frac{d}{dt}\left\{\frac{\sin(10t)}{\pi t}\right\}$

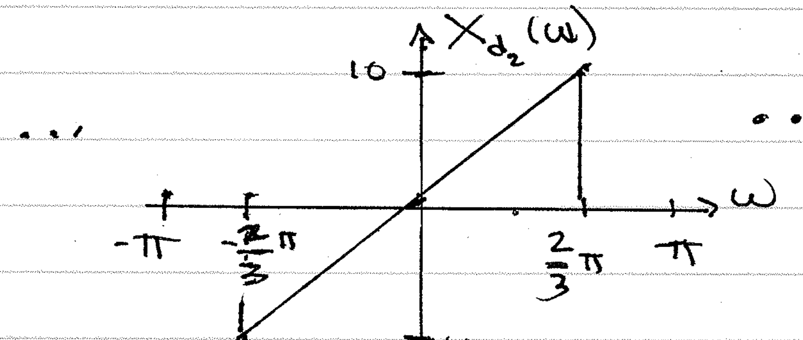
$$\xleftrightarrow{\mathcal{F}} T_s(-j)\, j\omega \,\mathrm{rect}\left(\frac{\omega}{20}\right)$$

$$= T_s\, \omega \,\mathrm{rect}\left(\frac{\omega}{20}\right) = X_{2_a}(\omega)$$

$\omega_{max} = 10$ $\quad \omega_s = 30 > 2(10) = 20 \Rightarrow$ no aliasing

$\omega_{max} = 10$ is mapped to $10\,T_s = 10\,\frac{2\pi}{30} = \frac{2}{3}\pi$



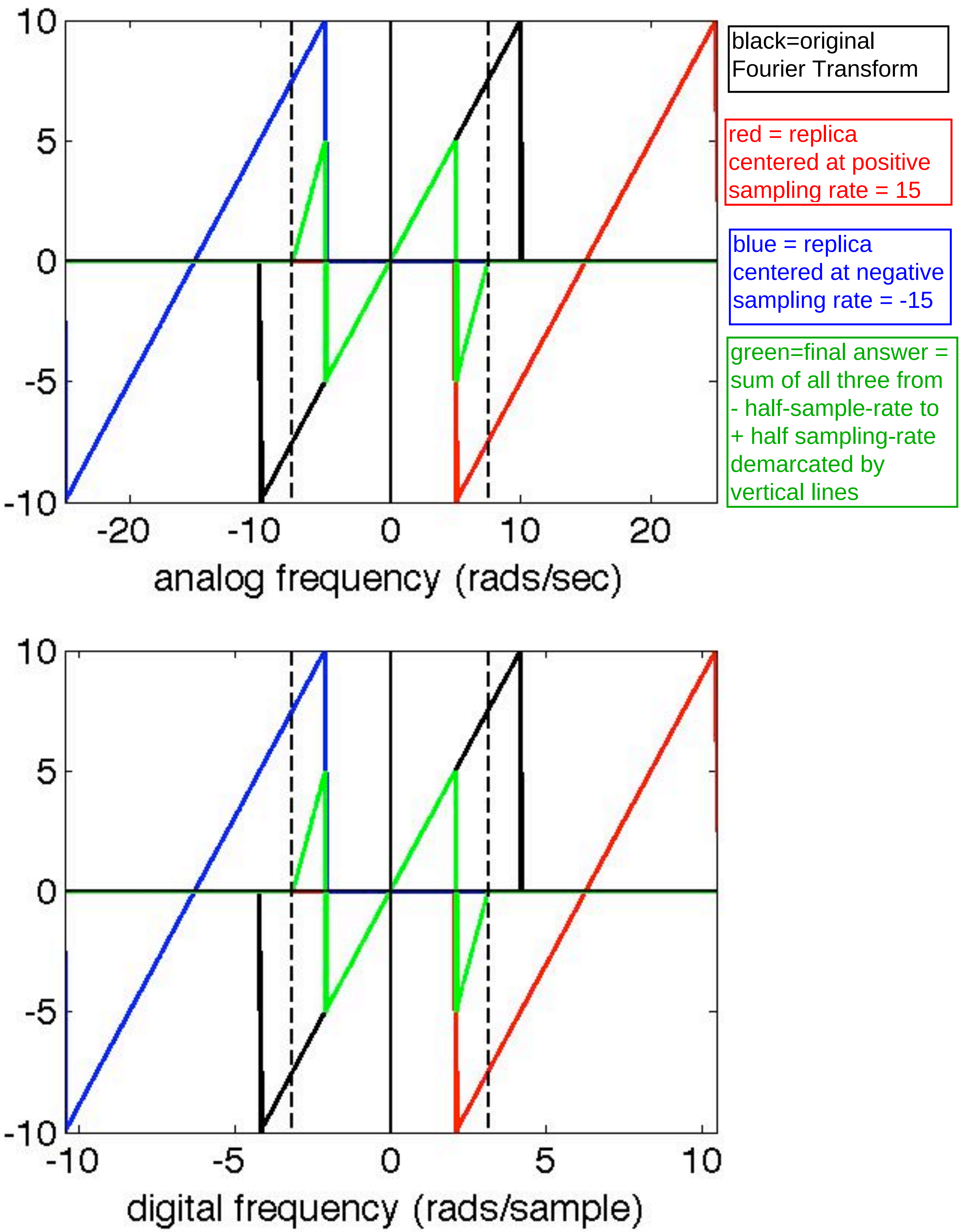


(d) $T_s = \frac{2\pi}{15}$ $\quad \omega_s = 15 < 2(10) \Rightarrow$ aliasing

The aliasing starts at :

$$(\omega_s - \omega_{max})\,T_s = (15-10)\,\frac{2\pi}{15} = \frac{2\pi}{3}$$

See plots on next page

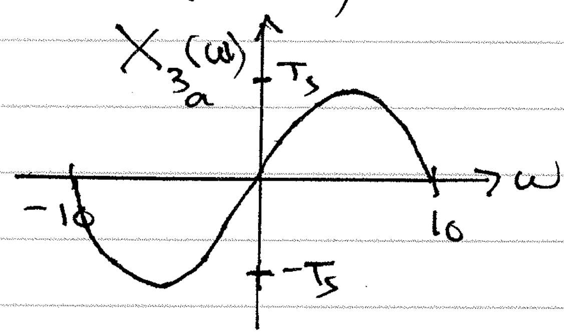

black=original Fourier Transform
red = replica centered at positive sampling rate = 15
blue = replica centered at negative sampling rate = -15
green=final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
10
5
0
-5
-10
-20
-10
0
10
20
analog frequency (rads/sec)
10
5
0
-5
-10
-10
-5
0
5
10
digital frequency (rads/sample)

(e) $x_3(t) = j T_s \frac{1}{2} \left\{ \frac{\sin(10(t-t_0))}{\pi(t-t_0)} - \frac{\sin(10(t+t_0))}{\pi(t+t_0)} \right\}$

$t_0 = \frac{\pi}{10}$

$$X_{3_a}(\omega) = T_s \, \text{rect}\left(\frac{\omega}{20}\right) \frac{1}{-2j} \left\{ e^{-j\omega\frac{\pi}{10}} - e^{j\omega\frac{\pi}{10}} \right\}$$

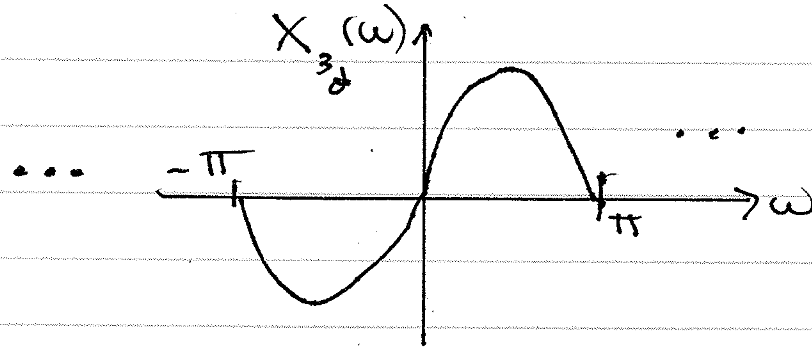
$$= T_s \sin\left(\omega \frac{\pi}{10}\right) \text{rect}\left(\frac{\omega}{20}\right)$$



$\omega_{max} = 10$

$T_s = \frac{2\pi}{20}$ $\quad \omega_s = 20 = 2\omega_{max} \Rightarrow$ Nyquist rate

$\omega_{max} = 0$ mapped to $10 T_s = 10 \frac{2\pi}{20} = \pi$

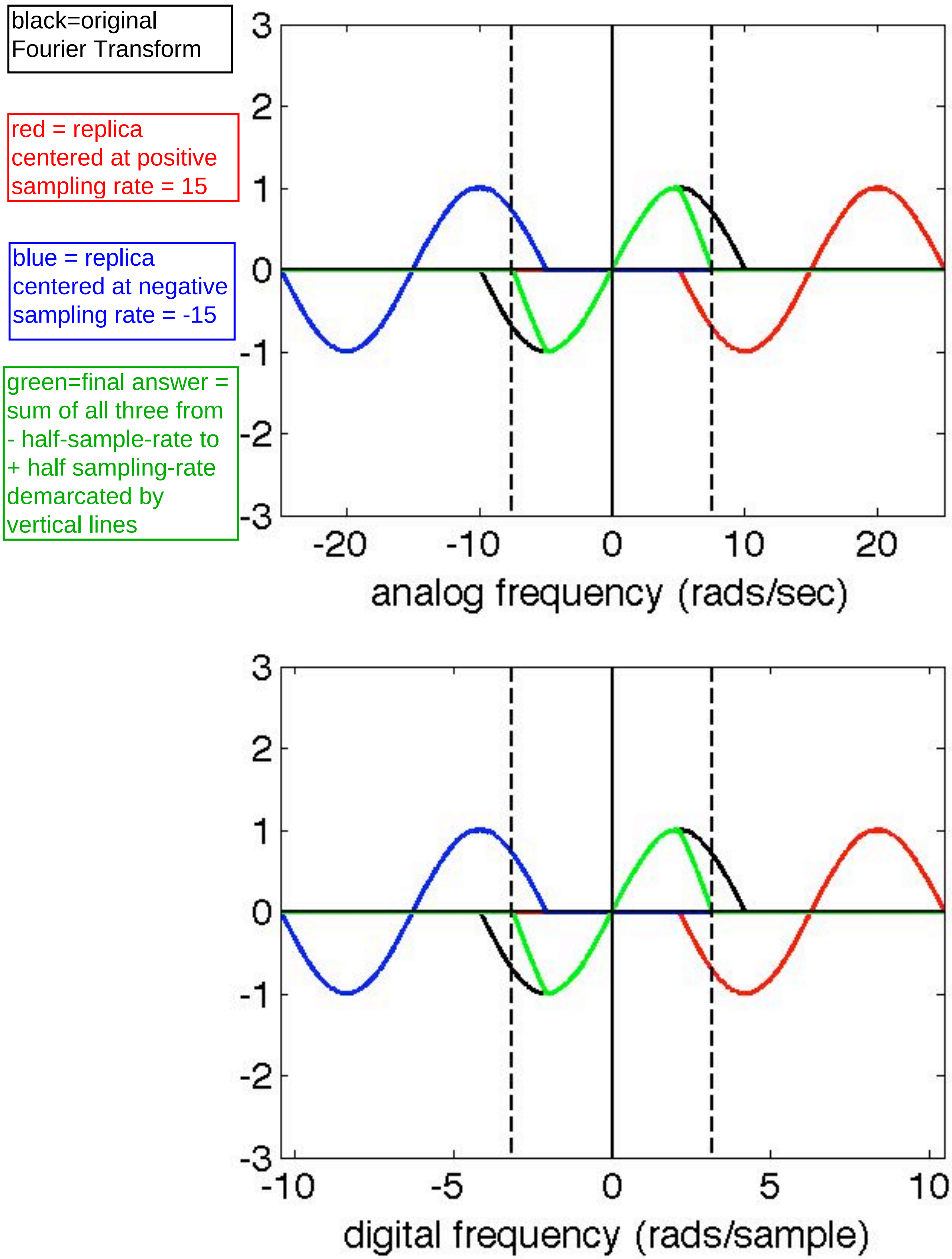


(f) $T_s = \frac{2\pi}{15}$ $\quad \omega_s = 15 < 2\omega_{max} = 20$

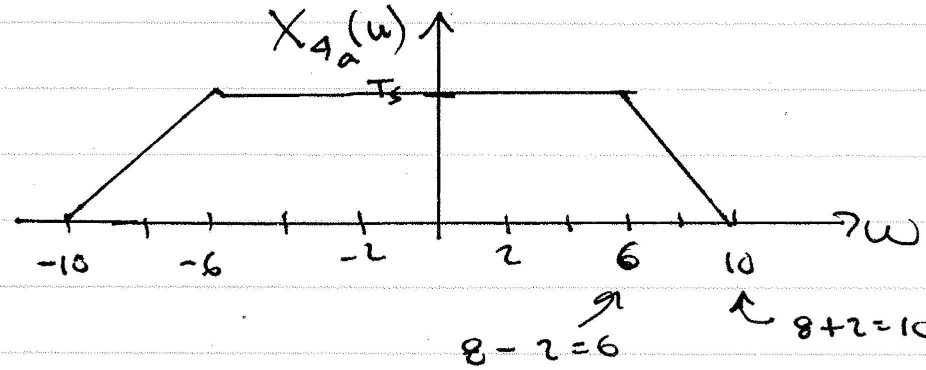
$\Rightarrow$ aliasing

aliasing starts at $(\omega_s - \omega_{max}) T_s = \frac{2\pi}{3}$

See plots on next page


black=original Fourier Transform
red = replica centered at positive sampling rate = 15
blue = replica centered at negative sampling rate = -15
green=final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
3
2
1
0
-1
-2
-3
-20
-10
0
10
20
analog frequency (rads/sec)
3
2
1
0
-1
-2
-3
-10
-5
0
5
10
digital frequency (rads/sample)

(g) $x_4(t) = T_s \frac{\pi}{2} \left\{ \frac{\sin(2t)}{\pi t} \frac{\sin(8t)}{\pi t} \right\}$

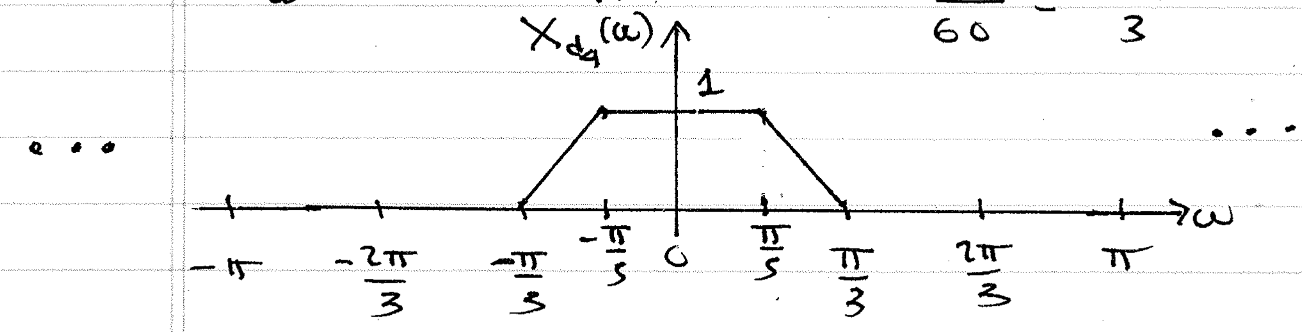


$\omega_{max} = 10$

$T_s = \frac{2\pi}{60}$ $\quad 60 > 2(\omega_{max}) = 20 \Rightarrow$ no aliasing

$\omega_a = 6$ is mapped to $6 \frac{2\pi}{60} = \frac{\pi}{5}$

$\omega_a = 10$ is mapped to $10 \frac{2\pi}{60} = \frac{\pi}{3}$



(h) $T_s = \frac{2\pi}{18}$ $\quad 18 < 20 \Rightarrow$ aliasing starts at

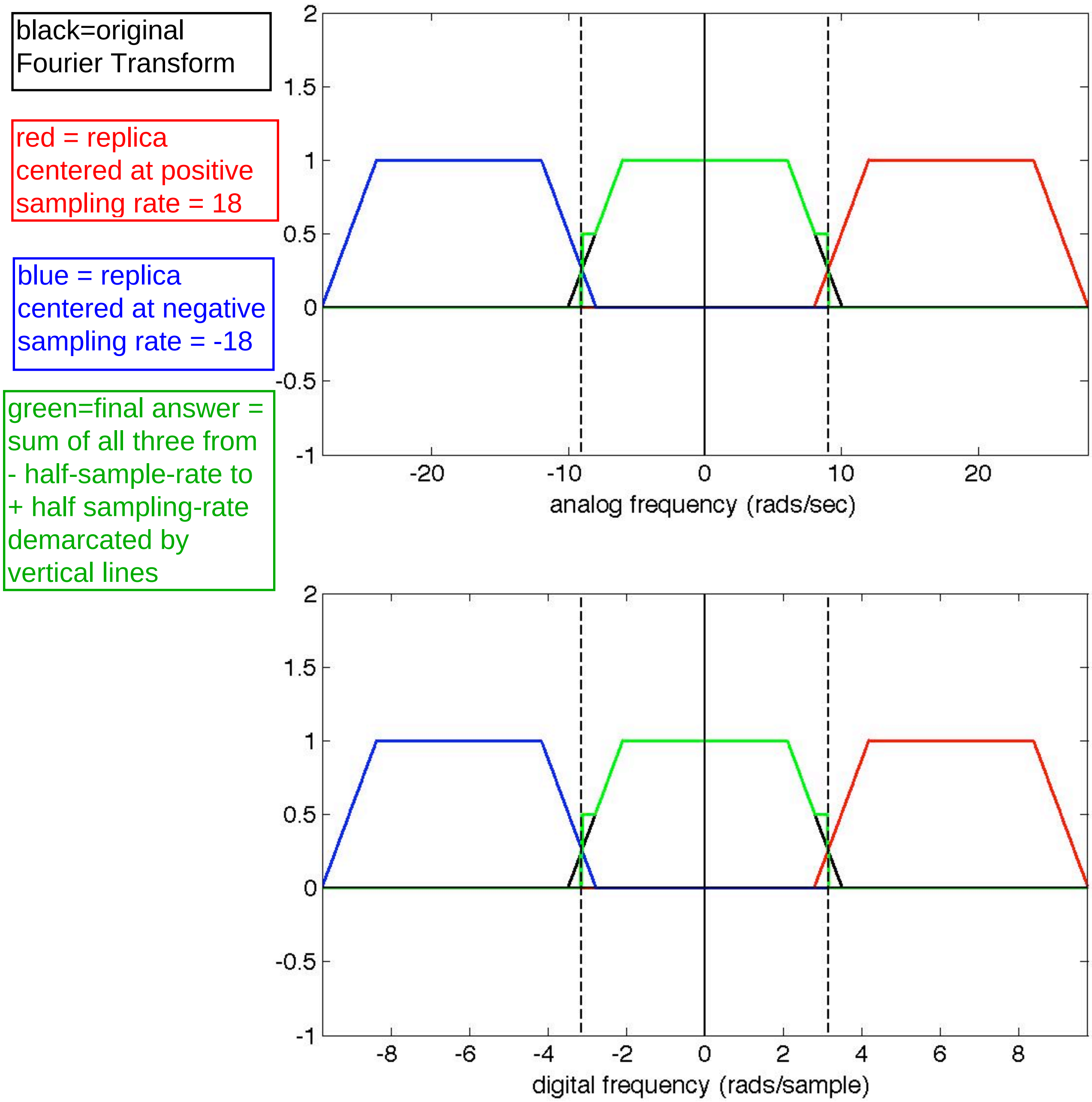
$$(\omega_s - \omega_{max})T_s = (18-10)\frac{2\pi}{18} = \frac{8\pi}{9}$$

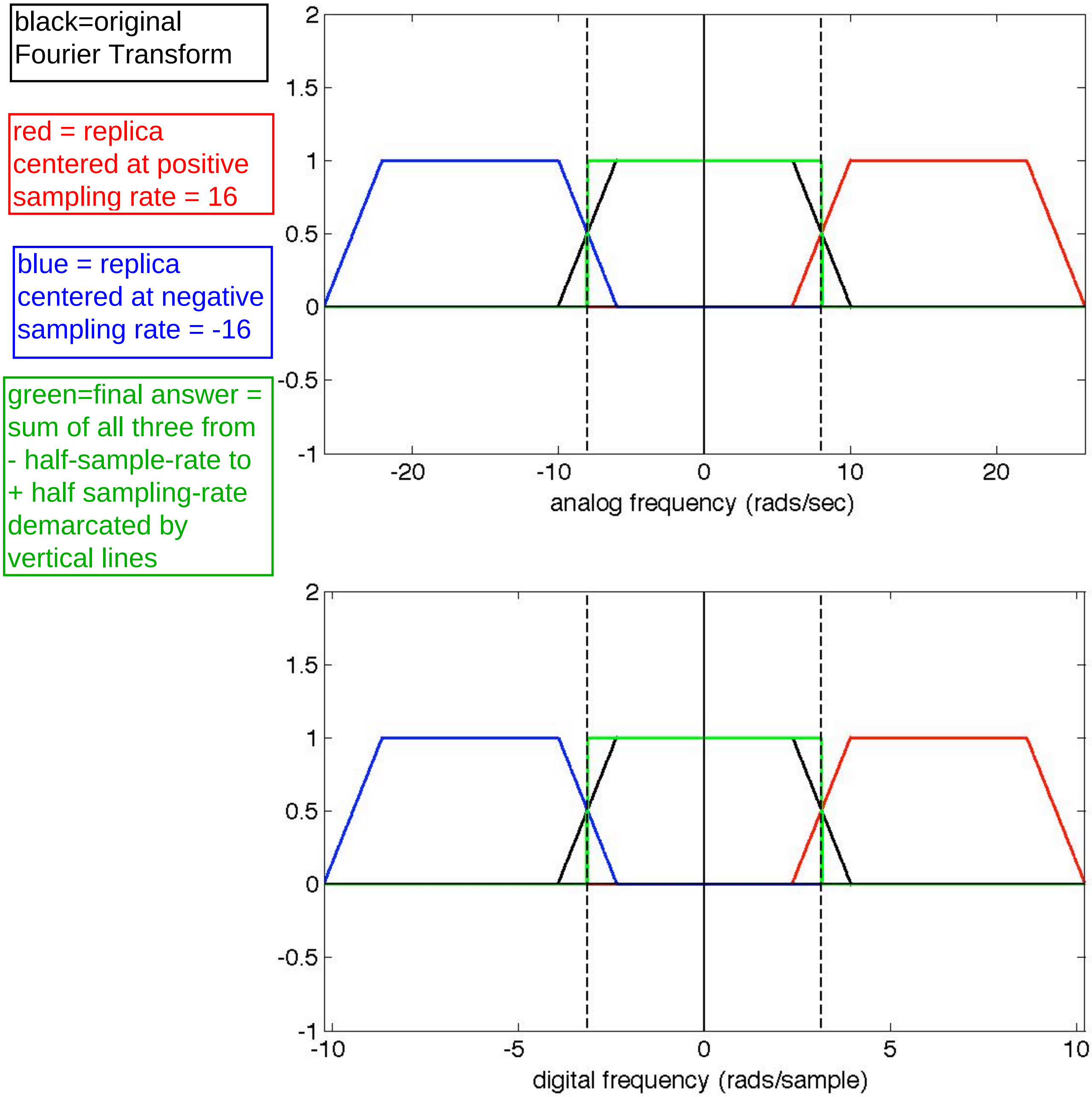
See plots on next page

(i) $T_s = \frac{2\pi}{16}$ $\quad 16 < 20 \Rightarrow$ aliasing starts at

$$(\omega_s - \omega_{max})T_s = (16-10)\frac{2\pi}{16} = \frac{3\pi}{4}$$

See plots on next page after next page


black=original Fourier Transform
red = replica centered at positive sampling rate = 18
blue = replica centered at negative sampling rate = -18
green=final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
analog frequency (rads/sec)
digital frequency (rads/sample)


black=original Fourier Transform
red = replica centered at positive sampling rate = 16
blue = replica centered at negative sampling rate = -16
green=final answer = sum of all three from - half-sample-rate to + half sampling-rate demarcated by vertical lines
2
1.5
1
0.5
0
-0.5
-1
-20
-10
0
10
20
analog frequency (rads/sec)
2
1.5
1
0.5
0
-0.5
-1
-10
-5
0
5
10
digital frequency (rads/sample)

(j) $T_s = \frac{2\pi}{48}$

Six analog frequencies are mapped to:

$$6\,\frac{2\pi}{48} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$12\,\frac{2\pi}{48} = \frac{\pi}{2}$$

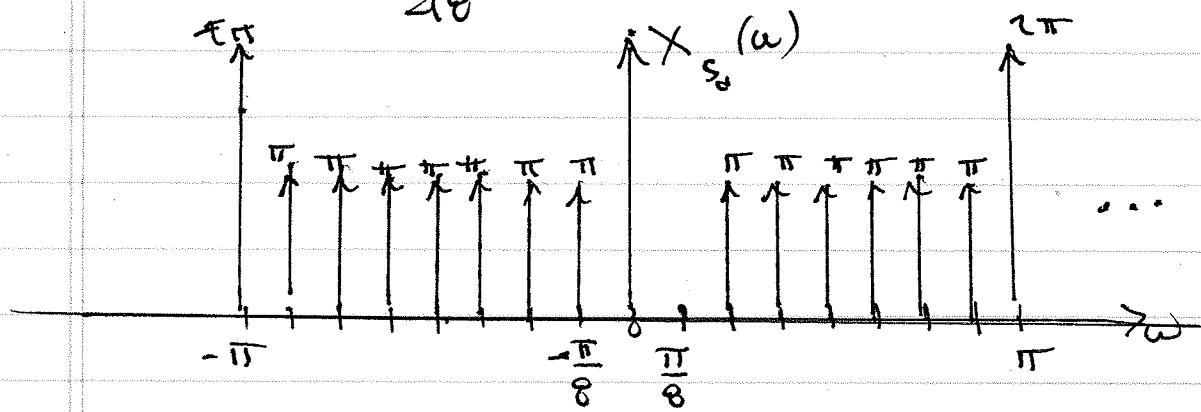
$$15\,\frac{2\pi}{48} = \frac{5}{8}\pi = \frac{5\pi}{8}$$

$$18\,\frac{2\pi}{48} = \frac{3(2\pi)}{8} = \frac{3\pi}{4}$$

$$21\left(\frac{2\pi}{48}\right) = \frac{7(2\pi)}{16} = \frac{7\pi}{8}$$

all less than π, so no aliasing

$$24\left(\frac{2\pi}{48}\right) = \pi$$



(k) $T_s = \frac{2\pi}{24}$ $\quad \omega_s = 24$

$$6 \cdot \frac{2\pi}{24} = \frac{\pi}{2}$$

$$12 \cdot \frac{2\pi}{24} = \pi$$

$$15 \cdot \frac{2\pi}{24} = \frac{5\pi}{4} \Rightarrow \frac{5\pi}{4} - \frac{8\pi}{4} = \frac{-3\pi}{4}$$

note: $\cos\left(-\frac{3\pi}{4}n\right) = \cos\left(\frac{3\pi}{4}n\right)$

$$18 \cdot \frac{2\pi}{24} = \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} - \frac{4\pi}{2} = \frac{-\pi}{2}$$

note: $\cos\left(-\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right)$

$$21 \cdot \frac{2\pi}{24} = \frac{7\pi}{4} \Rightarrow \frac{7\pi}{4} - \frac{8\pi}{4} = \frac{-\pi}{4}$$

note: $\cos\left(-\frac{\pi}{4}n\right) = \cos\left(\frac{\pi}{4}n\right)$

$24 \cdot \frac{2\pi}{24} = 2\pi \Rightarrow$ aliased to zero