EE301 Signals and Systems | **Exam 2**
Exam 2 | **Thursday, Mar. 29, 2018**

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 1,2,3, and 4, Emphasis on Chap. 4
Open Book but Closed Notes. One double-sided handwritten crib sheet.
Calculators NOT allowed.
This test contains **three** problems, each with multiple parts.
You must show all work for each problem to receive full credit.
Always state which Fourier Transform Property or Pair is being used.

Problem 1. You are given the Fourier Transform pair below

$$x(t) = \cos\left(\frac{\pi t}{2}\right)\text{rect}\left(\frac{t}{2}\right) \longleftrightarrow X(\omega) = \frac{4\pi\cos(\omega)}{\pi^2 - 4\omega^2}$$

(a) Determine the numerical value of $A_1 = \int_{-\infty}^{\infty} X(\omega)d\omega$?

Initial Value Theorem : $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$

$A_1 = 2\pi\{\cos(0)\} = 2\pi$

(b) Determine the numerical value of $A_2 = \int_{-\infty}^{\infty} x(t)dt$?

$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$

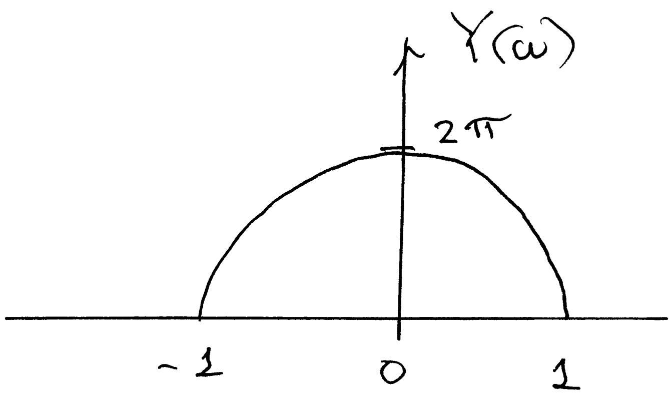
$= \frac{4\pi\cos(0)}{\pi^2 - 4(0)^2} = \frac{4}{\pi}$

(c) Determine and plot the Fourier Transform of

$$y(t) = \frac{4\pi\cos(t)}{\pi^2 - 4t^2}$$

Duality : $Y(\omega) = 2\pi\cos\left(\frac{\pi(-\omega)}{2}\right)\text{rect}\left(\frac{-\omega}{2}\right)$

$= 2\pi\cos\left(\frac{\pi}{2}\omega\right)\text{rect}\left(\frac{\omega}{2}\right)$



(d) Determine the numerical value of the energy of $y(t)$ defined in part (c), $E_y = \int_{-\infty}^{\infty} y^2(t)\,dt$. The following results may be helpful: $2\cos^2(x) = 1 + \cos(2x)$ and $\int \cos(x)\,dx = \sin(x)$.

Parseval's Theorem

$$E_y = \frac{2}{2\pi}\int_0^1 |Y(\omega)|^2\, d\omega$$

$$= \frac{1}{\pi}\int_0^1 (2\pi)^2 \cos^2\left(\frac{\pi}{2}\omega\right) d\omega$$

$$= \frac{1}{\pi}\, 4\pi^2 \int_0^1 \left(\frac{1}{2} + \frac{1}{2}\cos(\pi\omega)\right) d\omega$$

$$= \pi 2 \left(\omega \Big]_0^1 + \frac{1}{\pi} \sin(\pi\omega) \Big]_0^1 \right)$$

$$= 2\pi\,(1 + (0 - 0))$$

$$= 2\pi$$

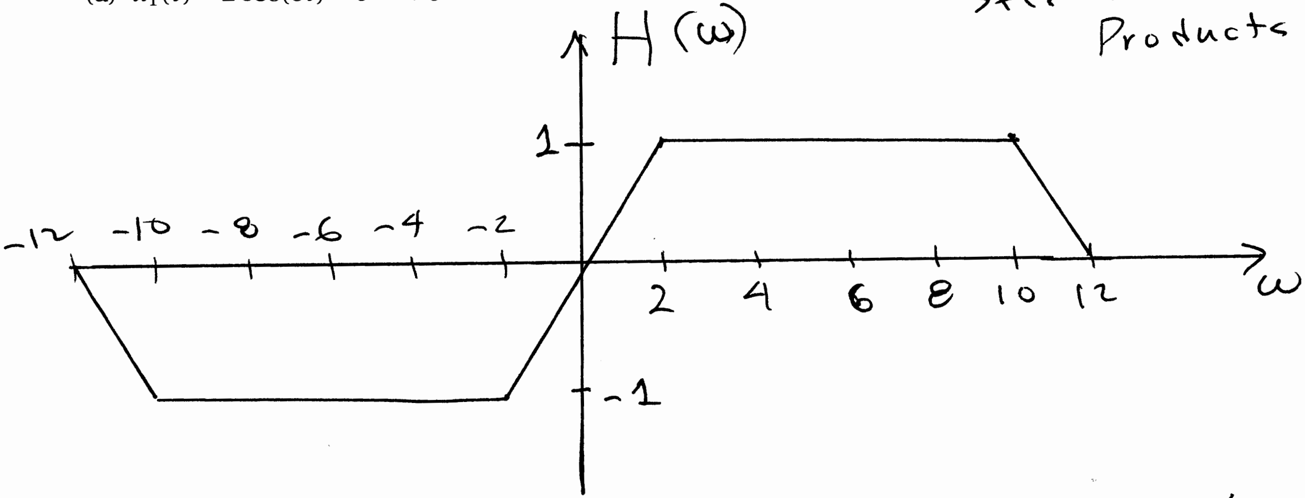
Problem 2. Consider an LTI system with impulse response

$$h(t) = \pi \frac{\sin(t)}{\pi t}\frac{\sin(5t)}{\pi t} 2j\sin(6t)$$

Plot the frequency response, $H(\omega)$**, for this system** in part (a) (you need this for each part) and determine the respective output for each input below (four parts = four different inputs). **Write a closed-form expression for the output in the time domain for each part.**

(a) $x_1(t) = 2\cos(6t) = e^{j6t} + e^{-j6t}$

See: Sinc Functions Products Handout



$$e^{j6t} \longrightarrow \boxed{} \longrightarrow H(6)e^{j6t} = (1)e^{j6t}$$

$$e^{-j6t} \longrightarrow \boxed{} \longrightarrow H(-6)e^{-j6t} = (-1)e^{-j6t}$$

$$y_1(t) = e^{j6t} - e^{-j6t} = 2j\sin(6t)$$

(b) $x_2(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} e^{j2kt}$

Frequencies ≥ 12 are rejected
≤ -12 are rejected

rejected $=$ not passed $=$ multiplied by zero

$\omega = 0$ is rejected as well.

$$y_2(t) = \sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k} e^{j2kt}$$

$$- \sum_{k=-5}^{-1} \left(\frac{1}{2}\right)^{-k} e^{jkt2}$$

$$= 2j \sum_{k=1}^{5} \left(\frac{1}{2}\right)^{k} \sin(2kt)$$

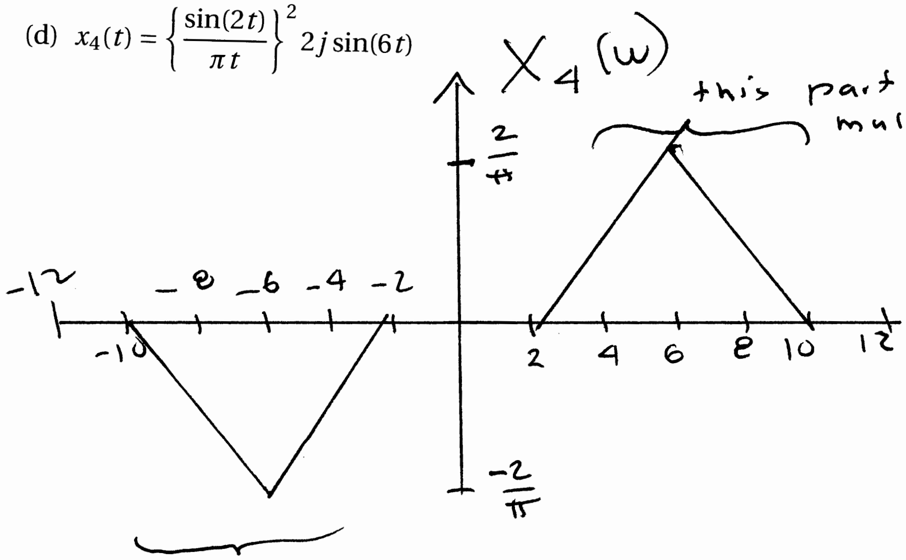
(c) this part is tricky: $x_3(t) = \dfrac{\sin(2t)}{\pi t}$

$$Y_3(\omega) = \frac{1}{2}\,\omega\, X_3(\omega)\left(\frac{j}{j}\right)$$

Differentiation in Time Property: $\quad y_3(t) = \frac{1}{2}\,\frac{1}{j}\,\frac{d}{dt}\,x_3(t)$

$$y_3(t) = \frac{-j}{2}\,\frac{d}{dt}\left\{\frac{\sin(2t)}{\pi t}\right\}$$

(d) $x_4(t) = \left\{\dfrac{\sin(2t)}{\pi t}\right\}^2 2j\sin(6t)$



this part gets multiplied by +1

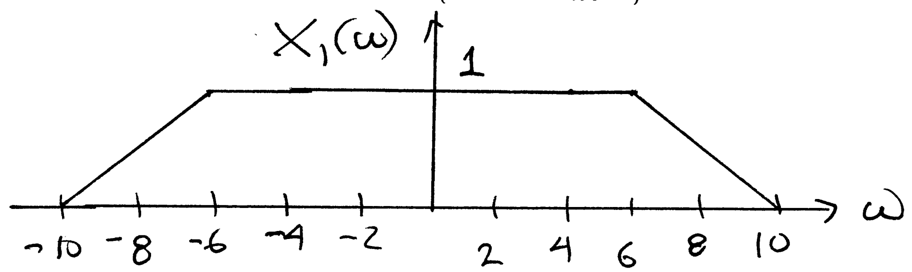
filter multiplies this by -1

$$y_4(t) = \left\{\frac{\sin(2t)}{\pi t}\right\}^2 2\cos(6t)$$

Problem 3.

(a) Plot the Fourier Transform $X_1(\omega)$ of $x_1(t)$ below. ALSO, determine the energy of $x_1(t)$, $E_{x_1} = \int_{-\infty}^{\infty} x_1^2(t)\,dt$.

$$x_1(t) = \frac{\pi}{2}\left\{\frac{\sin(2t)}{\pi t}\frac{\sin(8t)}{\pi t}\right\}$$



$$E_{x_1} = \frac{2}{2\pi}\int_0^{10} X_1^2(\omega)\,d\omega = \frac{1}{\pi}(6)(1)^2$$

$$+ \underbrace{\frac{1}{\pi}\int_6^{10}\left(-\frac{1}{4}(\omega-10)\right)^2 d\omega}$$

$$= \frac{1}{\pi}\frac{1}{16}\left.\frac{(\omega-10)^3}{3}\right]_6^{10}$$

$$= \frac{1}{\pi}\frac{1}{4^2}\left(0-(-1)\frac{4^3}{3}\right) = \frac{4}{3\pi}$$

$$\boxed{\text{Ans: } \frac{1}{\pi}\left(6+\frac{4}{3}\right) = \frac{1}{\pi}\frac{22}{3}}$$

(b) Plot the Fourier Transform $X_2(\omega)$ of $x_2(t)$ below. What is the energy of $x_2(t)$, $E_{x_2} = \int_{-\infty}^{\infty} |x_2(t)|^2 dt$?

$$x_2(t) = \left\{\frac{\pi}{2}\left\{\frac{\sin(2t)}{\pi t}\frac{\sin(8t)}{\pi t}\right\}\right\} e^{j10t}$$

$$E_{x_2} = \int_{-\infty}^{\infty} |x_1(t)e^{j10t}|^2\,dt$$

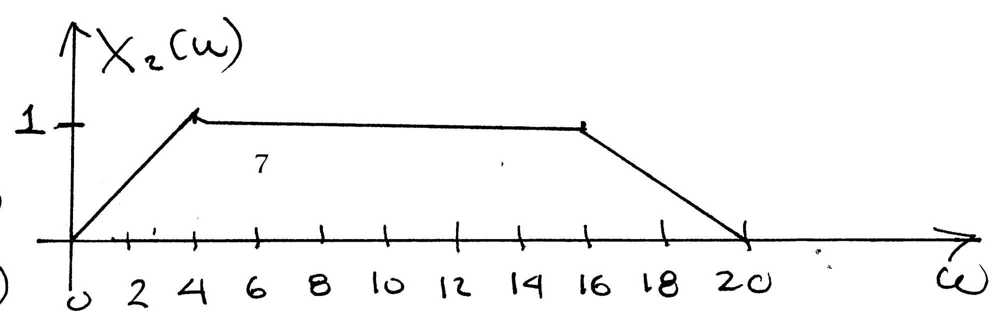
$$= \int_{-\infty}^{\infty} |x_1(t)|^2 \underbrace{|e^{j10t}|^2}_{=1}\,dt$$

$$= \int_{-\infty}^{\infty} x_1^2(t)\,dt = \frac{22}{3\pi}$$

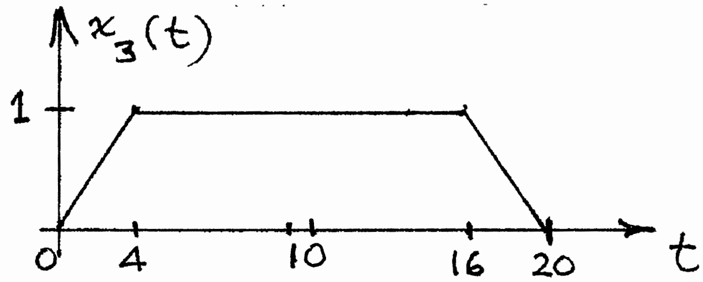
Modulation Property

$x_2(t) = e^{j10t}x_1(t)$

$X_2(\omega) = X_1(\omega-10)$



(c) Determine a closed-form expression for the Fourier Transform $X_3(\omega)$ of $x_3(t)$ plotted below. (You do NOT need to plot $X_3(\omega)$.) *Hint: Duality.* What is the energy of $x_3(t)$, $E_{x3} = \int_{-\infty}^{\infty} x_3^2(t)dt$?



This is, in fact, the Fourier Transform of $x_2(t)$.

Thus, duality states:

$$X_3(\omega) = 2\pi\, x_2(-\omega)$$

$$= 2\pi \frac{\pi}{2} \frac{\sin(-2\omega)}{-\pi\omega} \frac{\sin(-8\omega)}{-\pi\omega} e^{-j10\omega}$$

$$X_3(\omega) = \pi^2 \frac{\sin(2\omega)}{\pi\omega} \frac{\sin(8\omega)}{\pi\omega} e^{-j10\omega}$$

Energy in frequency domain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X_3(\omega)|^2 d\omega = \frac{\pi^2}{2\pi} \int_{-\infty}^{\infty} \left|\frac{\sin(2\omega)}{\pi\omega}\right|^2 \left|\frac{\sin(8\omega)}{\pi\omega}\right|^2 \left|e^{-j10\omega}\right|^2 d\omega$$

$$E_{x_3} = \frac{\pi}{2}\left(\frac{2}{\pi}\right)^2\left(\frac{\pi}{2}\right)^2 \int_{-\infty}^{\infty} \left\{\frac{\sin(2\omega)}{\pi\omega}\right\}^2 \left\{\frac{\sin(8\omega)}{\pi\omega}\right\}^2 d\omega$$

$$\boxed{E_{x_3} = 2\pi E_{x_1}}$$

In time-domain:

$$E_{x_3} = 2\left\{6(1)^2 + \int_{16}^{20} \left(-\tfrac{1}{4}(t-20)\right)^2 dt\right\}$$

$$= 12 + \frac{2}{4^2}\left\{0 - (-1)\frac{4^3}{3}\right\} = 12 + 2\,\frac{4}{3} = \frac{36+8}{3} = \frac{44}{3}$$

(d) Determine a closed-form expression for the Fourier Transform $X_4(\omega)$ of $x_4(t)$ below. Plot $x_4(t)$. (You do NOT need to plot $X_4(\omega)$.) What is the value of $E_{x_4} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_4(\omega)|^2 d\omega$?

$$x_4(t) = \frac{1}{4} rect\left\{\frac{t-2}{4}\right\} * rect\left\{\frac{t-8}{16}\right\}$$

$$X_4(\omega) = \frac{1}{4} \frac{\sin\left(4\frac{\omega}{2}\right)}{\frac{\omega}{2}} e^{-j2\omega} \frac{\sin\left(16\frac{\omega}{2}\right)}{\frac{\omega}{2}} e^{-j8\omega}$$

$$= \frac{\sin(2\omega)}{\omega} \frac{\sin(8\omega)}{\omega} e^{-j10\omega}$$

$$= \pi^2 \frac{\sin(2\omega)}{\pi\omega} \frac{\sin(8\omega)}{\pi\omega} e^{-j10\omega}$$

$$x_4(t) = x_3(t)$$

$$E_{x_4} = E_{\cancel{x}3} = \frac{44}{3}$$

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(\omega)$
		$y(t)$	$Y(\omega)$
4.3.0	Duality	$X(t)$	$2\pi\, x(-\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
4.3.3	Conjugation	$x^*(t)$	$X^*(-\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(\omega)Y(\omega)$
4.5	Multiplication	$x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}X(\omega)*Y(\omega) =$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\theta)Y(\omega - \theta)\,d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(\omega) = X^*(-\omega) \\ \mathcal{Re}\{X(\omega)\} = \mathcal{Re}\{X(-\omega)\} \\ \mathcal{Im}\{X(\omega)\} = -\mathcal{Im}\{X(-\omega)\} \\ \lvert X(\omega)\rvert = \lvert X(-\omega)\rvert \\ \sphericalangle X(\omega) = -\sphericalangle X(-\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(\omega)\}$ $j\mathcal{Im}\{X(\omega)\}$

Initial Value Theorems: $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$ $\quad X(0) = \int_{-\infty}^{\infty} x(t)\,dt$

4.3.7 Parseval's Relation for Aperiodic Signals

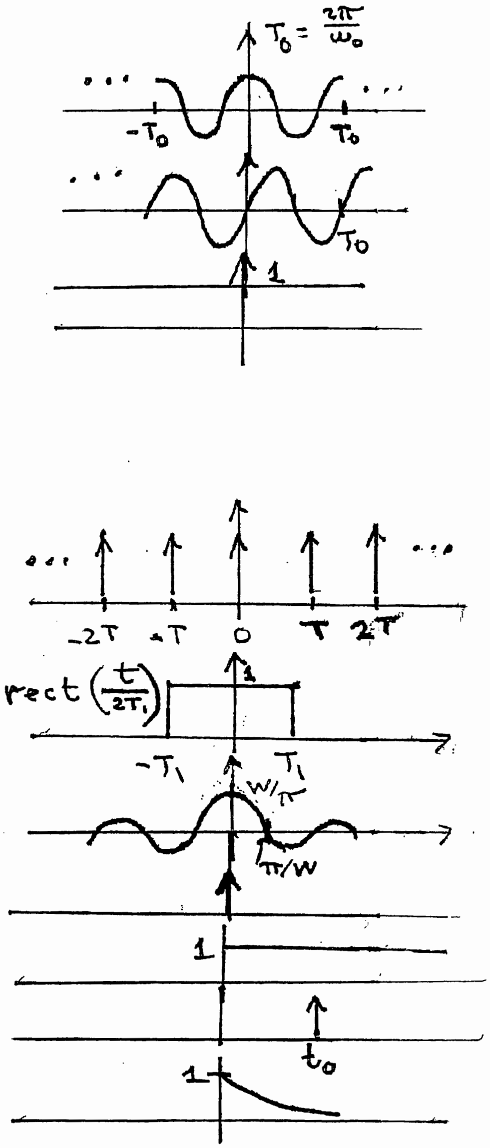
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

4.3.8 Frequency Shift Variants

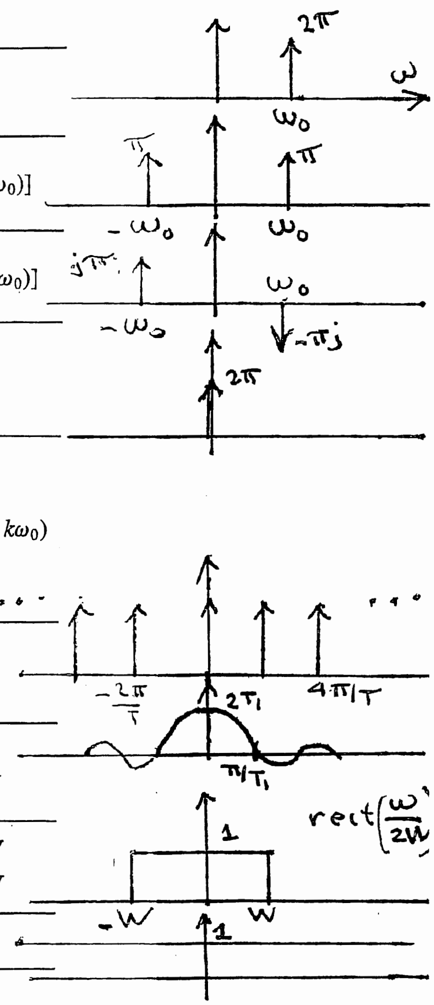
$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$

$$x(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j}X(\omega-\omega_0)-\frac{1}{2j}X(\omega+\omega_0)$$

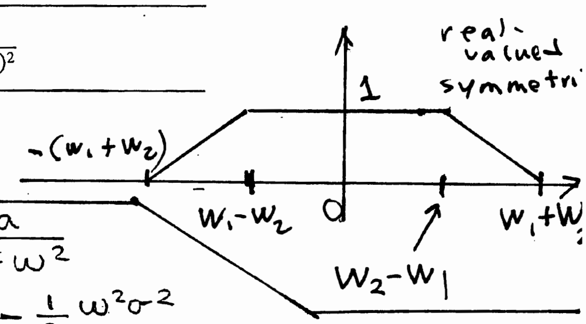
TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS



Signal	Fourier transform
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$x(t) = 1$	$2\pi\,\delta(\omega)$
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$



$$\frac{\pi}{W_1}\;\frac{\sin(W_1 t)}{\pi t}\cdot\frac{\sin(W_2 t)}{\pi t}$$



$$e^{-a|t|} \qquad \frac{2a}{a^2+\omega^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}} \qquad e^{-\frac{1}{2}\omega^2\sigma^2}$$

1

πt

-j sgn(ω) = j for ω<0
-j for ω>0