SOLUTION

EE301 Signals and Systems — Spring 2016
Exam 2 — Thursday, Mar. 31, 2016

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chapter 4, Hmwks 6-7
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You can NOT do work on the back of a page unless permission is granted.
No work on the back of a page will be graded unless permission is granted.
You must show all work for each problem to receive full credit.

True False Questions. Circle the True (T) or False (F) for each part below.

(T) (F) Let $H(\omega)$ be the Fourier Transform of the impulse response of an LTI system $h(t)$; as you vary ω, a plot of $H(\omega)$ reveals how the system responds as a function of frequency.

(T) (F) Any real-valued signal has negative frequency content. If the Fourier Transform of a signal is zero for $\omega < 0$, then the signal must be complex-valued.

(T) (F) Multiple signals can be totally overlapping in time but, yet, we can extract one of them by itself if the signals lie in different frequency bands.

(T) (F) $X(j\omega)$ is good notation for the Fourier Transform, even if it makes it awkward to write the Duality Property of the Fourier Transform, or the Multiplication-in-Time = Convolution in Frequency Property, and even though the Laplace Transform of a sinewave turned-on forever is not defined, whereas it is defined for the Fourier Transform.

(T) (F) Intuition gleaned from the time-scaling/freqency-scaling property of the Fourier Transform is that the narrower you are in the time-domain (shorter duration), the wider you are in the frequency domain (take up more bandwidth.)

(T) (F) For any input signal, the energy distribution (in either the time or frequency domain) is the same at both the input and output of a square-law device system, $y(t) = x^2(t)$.

(T) (F) One of the most important practical implications of the convolution property of the Fourier Transform (convolution in time leads to multiplication in the frequency domain) is frequency selective linear filtering, that is, lowpass filtering, bandpass filtering, etc.

Problem 1. The signal $x(t) = \pi \left\{ \frac{\sin(t)}{\pi t} \right\}^2$ is input to an LTI system with impulse response

$$h(t) = \frac{\pi}{2} \frac{\sin(2t)}{\pi t} \frac{\sin(6t)}{\pi t}$$

The goal is to determine a time-domain expression for the output $y(t) = x(t) * h(t)$ according to the steps below. Clearly label and circle/box your final answer for each part.

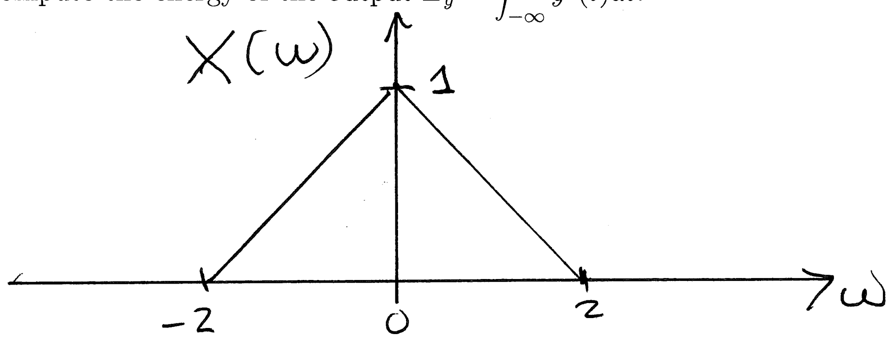
(a) Plot the Fourier Transform, $X(\omega)$, of the input signal $x(t) = \pi \left\{ \frac{\sin(t)}{\pi t} \right\}^2$

(b) Plot the Fourier Transform, $H(\omega)$, of the impulse response $h(t) = \frac{\pi}{2} \frac{\sin(2t)}{\pi t} \frac{\sin(6t)}{\pi t}$

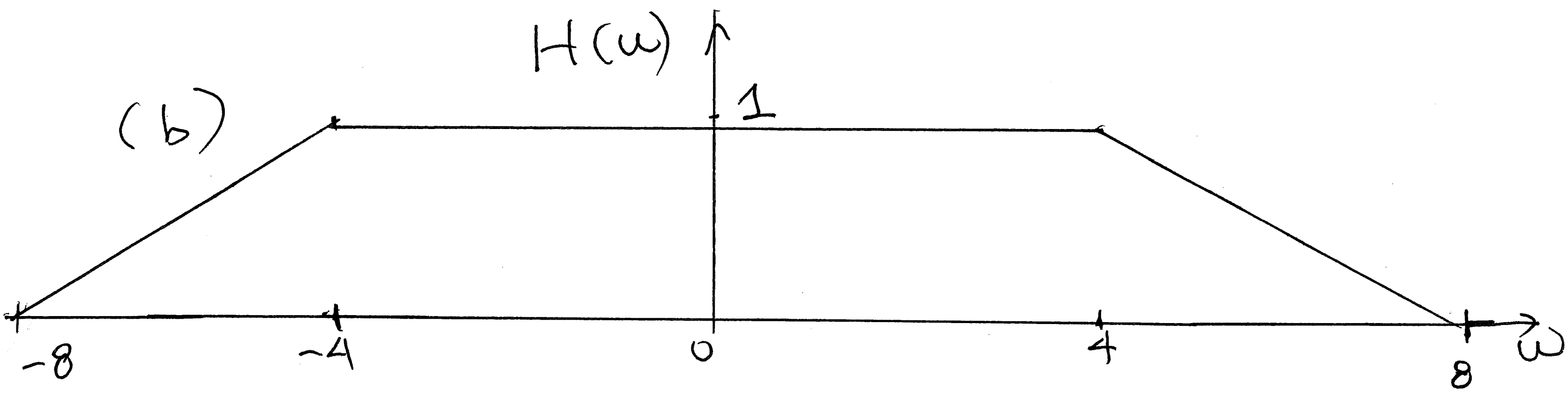
(c) Write a time-domain expression for the output $y(t)$.

(d) Compute the energy of the output $E_y = \int_{-\infty}^{\infty} y^2(t)dt$.

(a)



(b)



(c) $Y(\omega) = H(\omega) X(\omega) = X(\omega)$

$$y(t) = \pi \left\{ \frac{\sin(t)}{\pi t} \right\}^2 = x(t)$$

Additional Space for Problem 1 answer and work.

Since $y(t) = x(t)$, $E_y = E_x$

Parseval's Theorem $= \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-2}^{2} |X(\omega)|^2 d\omega$

$$= 2\frac{1}{2\pi}\int_0^2 \left[-\frac{1}{2}(\omega-2)\right]^2 d\omega$$

$$= \frac{1}{2\pi}2\left(\frac{1}{2}\right)^2 \int_0^2 (\omega-2)^2\,d\omega$$

$$= \frac{1}{4\pi}\left.\frac{(\omega-2)^3}{3}\right]_0^2$$

$$= \frac{1}{4\pi}\left(0 - \frac{(-2)^3}{3}\right)$$

$$= \frac{8}{12\pi} = \frac{2}{3\pi}$$

Problem 2. The signal

$$x(t) = 1 + 2\cos(t + \pi/3) + 3\cos(3t + \pi/4) + 6\cos(6t + \pi/4) + 8\cos(8t + \pi/3) + 9\cos(9t)$$

is input to an LTI system with impulse response $h(t) = \frac{\pi}{2}\frac{\sin(2t)}{\pi t}\frac{\sin(6t)}{\pi t}$; this is the same system as in Problem 1. Determine the output $y(t) = x(t) * h(t)$. Show all work and clearly indicate your final answer for $y(t)$ (in the time domain.)

$$A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t)} \longrightarrow A\,|H(\omega_0)| \cdot \cos(\omega_0 t + \theta + \angle H(\omega_0))$$

$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$$

LTI $\Rightarrow$ superposition applies!

$H(\omega)$ already plotted in Prob. 1 (b) $\}$ purely real-valued

$\angle H(\omega) = 0$ for all ω

$1 \to 1$

$2\cos\left(t + \frac{\pi}{3}\right) \longrightarrow 2\cos\left(t + \frac{\pi}{3}\right)$

$\omega_0 = 1$

$H(1) = 1$

$3\cos\left(3t + \frac{\pi}{4}\right) \longrightarrow 3\cos\left(3t + \frac{\pi}{4}\right)$

$\omega_0 = 3 \quad H(3) = 1$

$6\cos(6t + \pi/4) \longrightarrow 3\cos(6t + \pi/4)$

$\omega_0 = 6 \quad H(6) = 1/2$

since $H(8) = 0$ and $H(9) = 0$, final answer

$y(t) = 1 + 2\cos\left(t + \frac{\pi}{3}\right) + 3\cos\left(3t + \frac{\pi}{4}\right) + 3\cos\left(6t + \frac{\pi}{4}\right)$

Problem 3. The signal $x(t) = 2\pi \left\{ \frac{\sin(t)}{\pi t} \right\}^2 \cos(2t)$ is input to an LTI system with impulse response $h(t) = 2\frac{\sin(4t)}{\pi t}\sin(4t)$ to obtain the output $\hat{x}(t) = x(t) * h(t)$. This output is used to form the complex-valued signal

$$\tilde{x}(t) = x(t) + j\hat{x}(t)$$

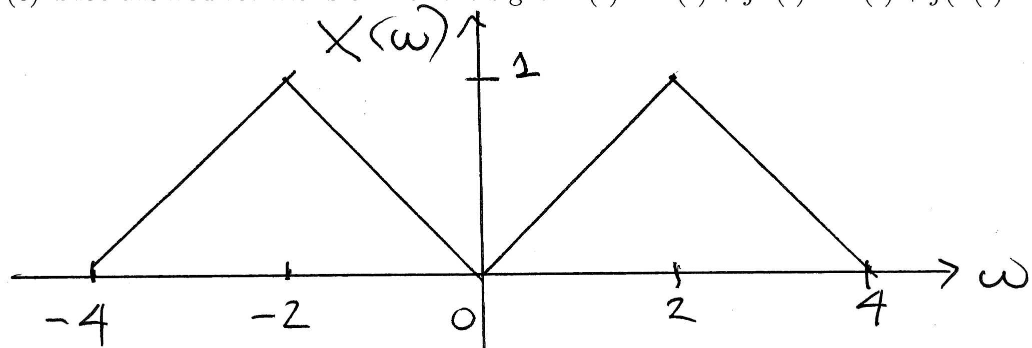
Determine and plot the Fourier Transform of $\tilde{x}(t)$. Show all work, which includes

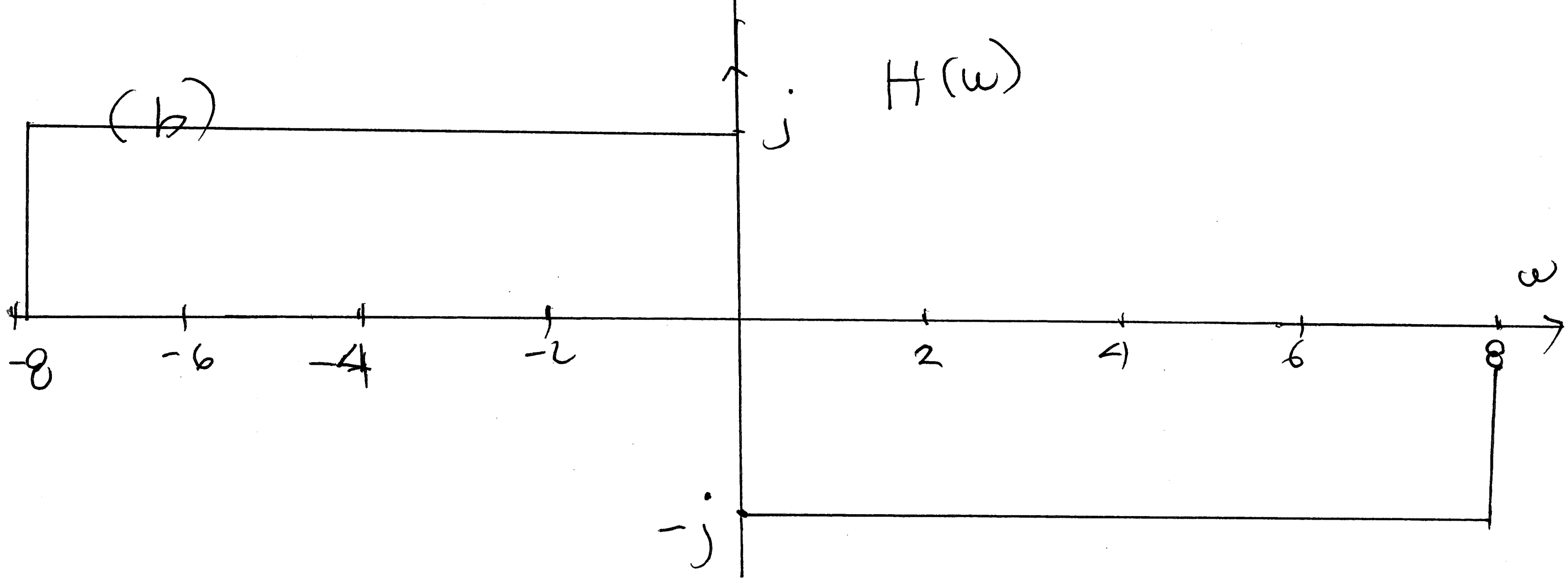
(a) Plot the Fourier Transform of the input signal $x(t) = 2\left\{ \frac{\sin(t)}{\pi t} \right\}^2 \cos(2t)$

(b) Plot the Fourier Transform of the impulse response $h(t) = 2\frac{\sin(4t)}{\pi t}\sin(4t)$

(c) Plot the Fourier Transform of the signal $\tilde{x}(t) = x(t) + j\hat{x}(t) = x(t) + j(x(t) * h(t))$

(a)

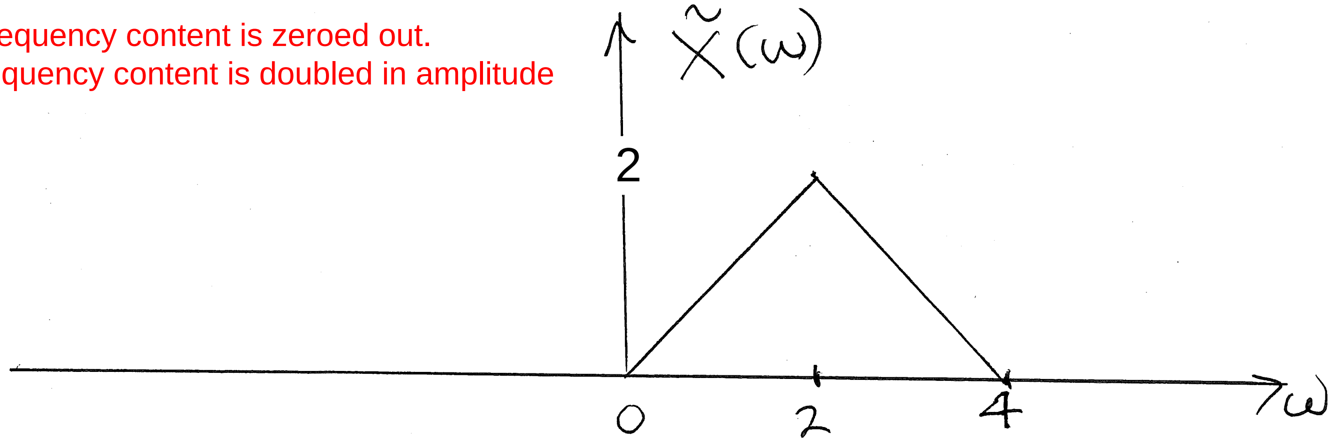




(c) Zeroes out the negative frequency content

Additional Space for Problem 3 answer and work.

negative frequency content is zeroed out.
positive frequency content is doubled in amplitude



Problem 4 This problem is about determining the Fourier Transform of the signal $x(t) = \tan^{-1}(t)$, working through a sequence of successive steps. Clearly delineate your work and circle your answer for each part below.

(a) Invoking the Duality Property and a standard Fourier Transform pair, determine the Fourier Transform of

$$x_1(t) = \frac{2}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) = ??$$

(b) Next, using the simple linearity property of the Fourier Transform, determine the Fourier Transform of

$$x_2(t) = \frac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega) = ??$$

(c) For $x(t) = \tan^{-1}(t)$, we have the known Calculus result: $\frac{d}{dt}\tan^{-1}(t) = \frac{1}{1+t^2}$. Take the Fourier Transform of both sides of the equation below, using the result in part (b), to solve for the Fourier Transform, $X(\omega)$, of $x(t) = \tan^{-1}(t)$:

$$\frac{d}{dt}x(t) = \frac{1}{1+t^2}$$

(d) Does your answer for part (c) have a real part? Explain how your answer is consistent with the symmetry properties of the Fourier Transform.

(e) Use one of the properties of the Fourier Transform to determine the Fourier Transform for the more general case below, where a is a real-valued positive constant.

$$x_a(t) = \tan^{-1}(at) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\omega) = ??$$

(a) $\left. e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{2a}{\omega^2 + a^2} \right\}$ For $a=1$: $e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{2}{\omega^2+1}$

Duality dictates:

$$\frac{2}{t^2+1} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$$

(b) Linearity dictates

$$\frac{1}{t^2+1} \overset{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}$$

Additional Space for Problem 4 answer and work.

(c) $$\frac{d}{dt} x(t) = \frac{1}{1+t^2}$$

$$j\omega X(\omega) = \pi e^{-|\omega|}$$

$$X(\omega) = \frac{\pi e^{-|\omega|}}{j\omega} = \mathcal{F}\{\tan^{-1}(t)\}$$

(d) $\tan^{-1}(t)$ is an odd function,

so $X(\omega)$ should be purely imaginary

which it is ✓ checks

(e) $$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$\tan^{-1}(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a} \frac{\pi e^{-\left|\frac{\omega}{a}\right|}}{j\frac{\omega}{a}}$$

Since stated that a is real and positive

$$= -j\pi \frac{e^{-\frac{|\omega|}{a}}}{\omega}$$