SOLUTION

EE301 Signals and Systems — Spring 2015

Exam 2 — Tuesday, Mar. 31, 2015

Cover Sheet

Test Duration: 75 minutes.
Coverage: Chaps. 3,4 with emphasis on Chap. 4
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You can NOT do work on the back of a page unless permission is granted.
No work on the back of a page will be graded unless permission is granted.
You must show all work for each problem to receive full credit.

Problem 1. The sum of (infinite-duration) sinewaves below

$$x(t) = 1 + \pi \cos(\pi t) + \pi \cos(2\pi t)$$

is input to an LTI system with impulse response given by

$$h(t) = \cos(\pi t)\mathrm{rect}(t)$$

Determine and write a closed-form expression for the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$x(t) = \sum_{k=-2}^{2} a_k e^{jk\pi t}$$

$$a_{-2} = a_2 = \frac{\pi}{2} \qquad a_{-1} = a_1 = \frac{\pi}{2} \qquad a_0 = 1$$

$$y(t) = \sum_{k=-2}^{2} a_k H(k\pi) e^{jk\pi t}$$

$$H(\omega) = \mathcal{F}\{\cos(\pi t)\,\mathrm{rect}(t)\}$$

similar to H mwk. Prob. 4.21 (c) done in class

$$\mathcal{F}\left\{(1 + \cos(\pi t))\,\mathrm{rect}\left(\frac{t}{2}\right)\right\}$$

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} = 2\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}$$

$$\mathrm{rect}(t)\cos(\pi t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin\left(\frac{\omega-\pi}{2}\right)}{\omega-\pi} + \frac{\sin\left(\frac{\omega+\pi}{2}\right)}{\omega+\pi}$$

real-valued and Symmetric!

$$H(0) = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\pi} + \frac{\sin\left(\frac{\pi}{2}\right)}{\pi} = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

Additional Space for Problem 1 answer and work.

$$H(\pi) = \underbrace{\frac{\sin\left(\frac{\pi-\pi}{2}\right)}{\pi-\pi}}_{\text{use L'Hospital's Rule or what we learned in-class about FT of finite-length sinewaves}} + \frac{\cancel{\sin\left(\frac{2\pi}{2}\right)}\to 0}{2\pi} \quad \text{since } \sin(\pi)=0$$

$$H(\pi) = \frac{1}{2} = H(-\pi)$$

$$H(2\pi) = \frac{\sin\left(\frac{2\pi-\pi}{2}\right)}{2\pi-\pi} + \frac{\sin\left(\frac{2\pi+\pi}{2}\right)}{2\pi+\pi}$$

$$= \frac{1}{\pi} + \frac{-1}{3\pi} \qquad \sin\left(\frac{3\pi}{2}\right) = -1$$

$$= \frac{3-1}{3\pi} = \frac{2}{3\pi} = H(-2\pi)$$

$$y(t) = \frac{2}{\pi} + \frac{\pi}{2}\cos(\pi t) + \frac{2}{3}\cos(2\pi t)$$

Problem 2 (a). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is multiplied by a complex-valued sinewave to form $y(t) = e^{j3t}x(t)$. Find the numerical value of $E = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}|Y(\omega)|^2 d\omega$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

basic result: $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(t-t_0)^2}{2\sigma^2}} dt = \sqrt{2\pi}\,\sigma$

consider $t_0=0$ and square to get energy.

$$\left(e^{-\frac{t^2}{2\sigma^2}}\right)^2 = e^{-\frac{2t^2}{2\sigma^2}} = e^{-\frac{t^2}{2(\sigma/\sqrt{2})^2}}$$

squaring reduces std. dev, by $\sqrt{2}$

Thus,

$$\int_{-\infty}^{\infty}\left(e^{-\frac{t^2}{2\sigma^2}}\right)^2 dt = \sqrt{2\pi}\,\frac{\sigma}{\sqrt{2}} = \sqrt{\pi}\,\sigma$$

and: $$\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^2\left(e^{-\frac{t^2}{2\sigma^2}}\right)^2 dt = \frac{1}{2\pi\sigma^2}\sqrt{\pi}\,\sigma$$
$$= \frac{1}{2\sqrt{\pi}\,\sigma}$$

Thus;
energy of $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}}$ is $\frac{1}{2\sigma\sqrt{\pi}}$

Now: $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2 d\omega = \int_{-\infty}^{\infty}|y(t)|^2 dt \Rightarrow$

$$|y(t)|^2 = |e^{j3t}x(t)|^2 = |e^{j3t}|^2\,|x(t)|^2 = |x(t)|^2$$

Answer $= \dfrac{1}{2\cdot 6\sqrt{\pi}} = \dfrac{1}{12\sqrt{\pi}}$

Problem 2 (b). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is time-shifted to form $y(t) = x(t-3)$. Find the numerical value of $E = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

A time-shift does not change the total energy, just how the energy is distributed as a function of time.

$$\int_{-\infty}^{\infty} x^2(t)\,dt \equiv \int_{-\infty}^{\infty} x^2(t-3)\,dt = E$$

Answer is same as in (a) $= \dfrac{1}{12\sqrt{\pi}}$

Problem 2 (c). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is differentiated to form $y(t) = \frac{d}{dt}x(t)$. Find the numerical value of $A = \int_{-\infty}^{\infty} y(t)dt$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$y(t) = \frac{d}{dt}x(t) \xleftrightarrow{\mathcal{F}} j\omega X(\omega) = Y(\omega)$$

$$A = \int_{-\infty}^{\infty} y(t)\,dt = Y(0)$$

$$Y(0) = j \cdot 0 \cdot X(0) = 0$$

$$A = 0$$

Problem 2 (d). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is multiplied by t to form $y(t) = tx(t)$. Find the numerical value of $A = \int_{-\infty}^{\infty} y(t)dt$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega)$$

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{t^2}{2\sigma^2}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\frac{\omega^2\sigma^2}{2}}$$

$$\underbrace{\frac{1}{\sigma\sqrt{2\pi}}\,t\,e^{-\frac{t^2}{2\sigma^2}}}_{y(t)} \longleftrightarrow j\frac{d}{d\omega}\left\{e^{-\frac{\omega^2\sigma^2}{2}}\right\}$$

$$= j\left(-\frac{1}{2}\sigma^2\right)2\omega e^{-\frac{\omega^2\sigma^2}{2}}$$

$$= \underbrace{-j\sigma^2\omega e^{-\frac{\omega^2\sigma^2}{2}}}_{Y(\omega)}$$

$$A = \int_{-\infty}^{\infty} y(t)\,dt = Y(0) = -j\sigma^2(0)e^0 = 0$$

$$A = 0$$

Problem 2 (e). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is compressed in time by a factor of 2 to form $y(t) = x(2t)$. Find the numerical value of the energy $E = \int_{-\infty}^{\infty} y^2(t)dt$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$y(t) = x(2t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2\cdot 36}(2t)^2}$$

$$= \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\cdot 9}}$$

$$= \frac{1}{2} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}}_{}$$ where $\sigma = 3$

from 2(a), we know energy is $\frac{1}{2\sigma\sqrt{\pi}} = \frac{1}{2\cdot 3\sqrt{\pi}}$

Final answer: $\left(\frac{1}{2}\right)^2 \frac{1}{6\sqrt{\pi}} = \frac{1}{24\sqrt{\pi}}$

Problem 2 (f). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{8\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 8^2$. Determine a simple, closed-form expression for the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$\frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\cdot 6^2}} * \frac{1}{8\sqrt{2\pi}} e^{-\frac{t^2}{2\cdot 8^2}}$$

$$= \frac{1}{10\sqrt{2\pi}} e^{-\frac{t^2}{2\cdot 10^2}}$$

since variances add

$10^2 = 8^2 + 6^2$

as proved in class using:

FT pair: $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\frac{\omega^2\sigma^2}{2}}$

FT property:

$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = X(\omega)H(\omega)$

Problem 2 (g). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{8\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 8^2$. Determine the numerical value of the area, $A = \int_{-\infty}^{\infty} y(t)dt$, under the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$A = \int_{-\infty}^{\infty} y(t)\,dt = Y(0) = e^{-\frac{\omega^2 10^2}{2}} \Big|_{\omega = 0}$$

$$= 1$$

$$A = 1$$

Problem 2 (h). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{8\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 8^2$. Determine the numerical value of the energy $E = \int_{-\infty}^{\infty} y^2(t)dt$ of the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

From 2(a), energy of

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{t^2}{2\sigma^2}} \text{ is } \frac{1}{2\sigma\sqrt{\pi}}$$

Since $\sigma = 10$ for $y(t)$,

$$E = \frac{1}{20\sqrt{\pi}}$$

Problem 2 (i). The Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}}e^{-\frac{(t-2)^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, is input to an LTI system with a Gaussian shaped impulse response, $h(t) = \frac{1}{8\sqrt{2\pi}}e^{-\frac{(t-3)^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 8^2$. Determine the numerical value of the energy $E = \int_{-\infty}^{\infty} y^2(t)dt$ of the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

Time-Invariance dictates that

answer is $\tilde{y}(t-5)$

$t_1 + t_2$

where $\tilde{y}(t)$ is answer to 2(f)

time-shift doesn't change total energy

so E is same as answer for 2(h)

$$\frac{1}{20\sqrt{\pi}}$$

Problem 2 (j).

The signal $z(t) = x(t)y(t)$ is the PRODUCT of the Gaussian pulse $x(t) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}}$, with $\sigma_1^2 = 6^2$, and the Gaussian pulse $y(t) = \frac{1}{8\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$, with $\sigma_2^2 = 8^2$. Determine a simple expression for the Fourier Transform, $Z(\omega)$, of $z(t) = x(t)y(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

$$z(t) = x(t)\, y(t)$$

$$= \frac{1}{\sigma_1 \sqrt{2\pi}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_3^2}}$$

$$\text{where: } \sigma_3^2 = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{36 \cdot 64}{36 + 64}$$

$$= \frac{36 \cdot 64}{100} = \frac{6^2 \cdot 8^2}{2^2 \cdot 5^2} \approx 23$$

(23.04)

$$\frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi}} \frac{\sqrt{23}}{\sqrt{23}\sqrt{2\pi}} e^{-\frac{t^2}{2 \cdot 23}} \xleftrightarrow{\mathcal{F}}$$

$$\frac{\sqrt{23}}{48\sqrt{2\pi}} e^{-\frac{\omega^2 23}{2}}$$

Problem 3. The signal

$$x(t) = \frac{1}{t^2+1}\cos(20t) - \left\{\frac{1}{t^2+1} * \frac{1}{\pi t}\right\}\sin(20t)$$

is input to an LTI system with impulse response

$$h(t) = \left\{\frac{\pi}{5}\,\frac{\sin(5t)}{\pi t}\,\frac{\sin(15t)}{\pi t}\right\}$$

Determine the output $y(t) = x(t) * h(t)$. Show all work, state which Fourier Transform pairs and/or properties you are using, and clearly indicate your final answer.

Hint during exam: from Table

$$e^{-a|t|} \overset{F}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$$

for $a = 1$

$$e^{-|t|} \overset{F}{\longleftrightarrow} \frac{2}{\omega^2+1}$$

duality:

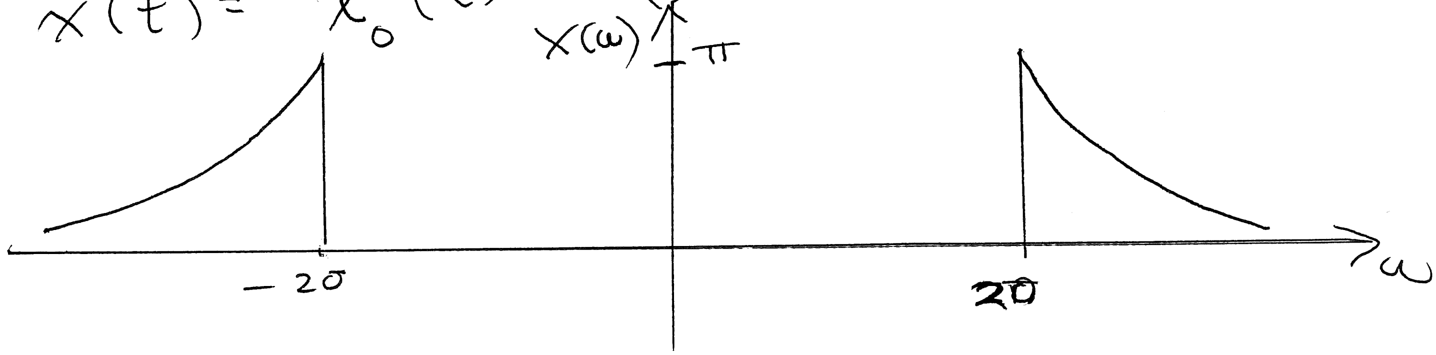
$$\frac{2}{t^2+1} \overset{F}{\longleftrightarrow} 2\pi e^{-|-\omega|}$$

dividing by 2 on both sides

$$\frac{1}{t^2+1} \overset{F}{\longleftrightarrow} \pi e^{-|\omega|}$$

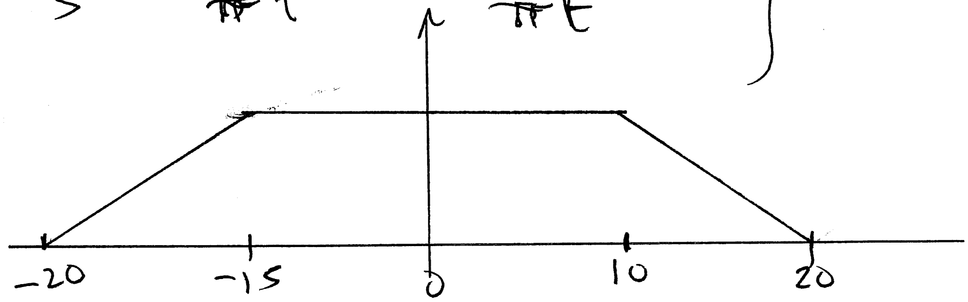
learned in class

$$x(t) = x_0(t)\cos(20t) - \hat{x}_0(t)\sin(20t)$$



Additional Space for Problem 3 answer and work.

$$h(t) = \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t}$$ } used on many old exams.



Thus. $\therefore$ $Y(\omega) = X(\omega) H(\omega) = 0$ no overlap

$$y(t) = 0$$

Note: let $x_0(t) = \frac{1}{1+t^2}$ $\hat{x}_0(t) = x_0(t) * \frac{1}{\pi t}$

$$x_0(t) \cos(\omega_0 t) \xleftrightarrow{f} \frac{1}{2} X_0(\omega - \omega_0) + \frac{1}{2} X_0(\omega + \omega_0)$$

$$\hat{x}_0(t) \sin(\omega_0 t) \xleftrightarrow{f} \frac{1}{2j} \hat{X}_0(\omega - \omega_0) - \frac{1}{2j} \hat{X}_0(\omega + \omega_0)$$

---

substitute: $\hat{X}_0(\omega) = -j \operatorname{sgn}(\omega) X_0(\omega)$

$$\hat{x}_0(t) \sin(\omega_0 t) \xleftrightarrow{f} -\frac{1}{2} \operatorname{sgn}(\omega - \omega_0) X_0(\omega - \omega_0)$$
$$+ \frac{1}{2} \operatorname{sgn}(\omega + \omega_0) X(\omega + \omega_0)$$

Thus:

$$x_0(t) \cos(\omega_0 t) - \hat{x}_0(t) \sin(\omega_0 t) \xleftrightarrow{f} \frac{1}{2}\left(1 + \operatorname{sgn}(\omega - \omega_0)\right) X(\omega - \omega_0) + \frac{1}{2}\left(1 - \operatorname{sgn}(\omega + \omega_0)\right) X(\omega + \omega_0)$$

$$\frac{1}{2}\left(1 + \operatorname{sgn}(\omega - \omega_0)\right) = \begin{cases} 1, & \omega > \omega_0 \\ 0, & \omega < \omega_0 \end{cases}$$

$$\frac{1}{2}\left(1 - \operatorname{sgn}(\omega + \omega_0)\right) = \begin{cases} 1, & \omega < -\omega_0 \\ 0, & \omega > -\omega_0 \end{cases}$$