SOLUTION

EE301 Signals and Systems — Spring 2013

Exam 2 — Wednesday, Mar. 27, 2013

Cover Sheet

Test Duration: 60 minutes.
Coverage: Chaps. 3,4 with emphasis on Chap. 4
Open Book but Closed Notes.
One 8.5 in. x 11 in. crib sheet
Calculators NOT allowed.
All work should be done on the sheets provided.
You must show all work for each problem to receive full credit.
Plot your answers on the graphs provided.

Yes or No Questions with Explanation. Circle Yes or No for each question below, and briefly **explain** your answer in the space provided. You must cite at least one Fourier Transform Property in your answer for each part.

Yes No Consider a signal $x(t)$ that is real-valued and non-negative for all time, i.e., $x(t) > 0$ for $-\infty < t < \infty$. Under these conditions, is it possible that the Fourier Transform $X(\omega)$ at $\omega = 0$ can be 0? That is, under these conditions, can $X(0) = 0$?

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

since $x(t) > 0 \ \forall t$ the area under $x(t)$ cannot be zero

Yes No Consider the signal $x(t) = \sqrt{2}\,\mathrm{rect}(2t) + \frac{sin(\pi t)}{\pi t} + \sqrt{\pi}\, e^{-t^2}$. Does the Fourier Transform of this signal, $X(\omega)$, have an imaginary part?

Since all three functions are real and even-symmetric $X(\omega)$ will be real and even-symmetric

Yes No Consider a signal $y(t) = t\, e^{-t^2}$ with Fourier Transform $Y(\omega)$. Is the area under $Y(\omega)$ integrated over all frequencies equal to zero?

$$y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)\, d\omega$$

Since $y(0) = 0$, the area under $Y(\omega)$ must be 0

$$y(0) = 0 \cdot e^{0} = 0$$

Yes No Consider $x(t)$ to be real-valued and having odd symmetry, $x(-t) = -x(t)$. Will the value of the Fourier Transform $X(\omega)$ at $\omega = 0$ always be zero, i.e., $X(0) = 0$ for a signal with odd symmetry?

Since $x(-t) = -x(t)$, the area under $x(t)$ will be 0

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt = 0 = \underbrace{\int_{-\infty}^{0} x(t)\, dt}_{\substack{\text{change of variables} \\ t' = -t}} + \int_{0}^{\infty} x(t)\, dt = 0$$

Yes No Is the energy distribution as a function of frequency for the signal at the output of an LTI filter the same as the energy distribution as a function of frequency for the corresponding input signal?

convolution-in-time property

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$\Rightarrow \underbrace{|Y(\omega)|^2}_{\text{energy distribution at output}} = \underbrace{|X(\omega)|^2}_{\text{energy distribution at input}} |H(\omega)|^2$$

can be totally different depending on $|H(\omega)|^2$

Yes No Supposed that $X(\omega) = 0$ for $|\omega| > 10$ rads/sec. Will the Fourier Transform of $y(t) = a\,x(t-t_1) + b\,x(t-t_2) + c\,x(t-t_3)$ also be equal to 0 for $|\omega| > 10$ rads/sec, regardless of the values of the time-delays t_1, t_2, and t_3 and the amplitude scalings a, b, and c?

$$Y(\omega) = X(\omega)\left\{a e^{-j\omega t_1} + b e^{-j\omega t_2} + c e^{-j\omega t_3}\right\}$$

Since $X(\omega) = 0$ for $|\omega| > 10$, guaranteed $Y(\omega) = 0$ for $|\omega| > 10$

convolution property in time

Yes No Let $y(t)$ denote the output when $x(t)$ is the signal input to an LTI system with impulse response $h(t)$. Supposed that $X(\omega) = 0$ for $|\omega| > 10$ rads/sec. Is it possible to design an LTI system such that $Y(\omega) = 0$ for $|\omega| > 5$ rads/sec? That is, is it possible to lower the bandwidth (the max frequency) of signal through LTI filtering?

Yes: $H(\omega) = \text{rect}\left(\frac{\omega}{10}\right)$ will lowpass filter $x(t)$ so that $Y(\omega) = 0$ for $|\omega| > 5$

convolution property in time

Yes No Consider the signal $y(t) = \frac{d}{dt}x(t)$ Is the Fourier Transform $Y(\omega)$ guaranteed to be 0 at at $\omega = 0$, i.e., $Y(0) = 0$, for any signal $x(t)$?

$$Y(\omega) = j\omega X(\omega) \Rightarrow Y(0) = j0\,X(0) = 0$$

Thus $Y(0) = 0$ $\Rightarrow$ Differentian-in-Time Property

Yes No Is ALL of the energy of a FINITE-duration sinewave totally concentrated at the frequency of the sinewave?

$y(t) = e^{j\omega_0 t}\,\text{rect}\left(\frac{t}{T}\right)$ T = duration of sinewave

$$Y(\omega) = \frac{\sin\left(T\frac{(\omega-\omega_0)}{2}\right)}{\frac{(\omega-\omega_0)}{2}}$$

only when $T \to \infty$ is all of the energy concentrated at $\omega = \omega_0$

Yes No Let $y(t)$ denote the output when a periodic signal $x(t)$ with period T is the signal input to an LTI system with impulse response $h(t)$. Will the output $y(t)$ be periodic with the same period T for any impulse response $h(t)$?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$y(t) = x(t) * h(t)$

$h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$

$\omega_0 = \frac{2\pi}{T}$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(k\omega_0)\, e^{jk\omega_0 t}$$

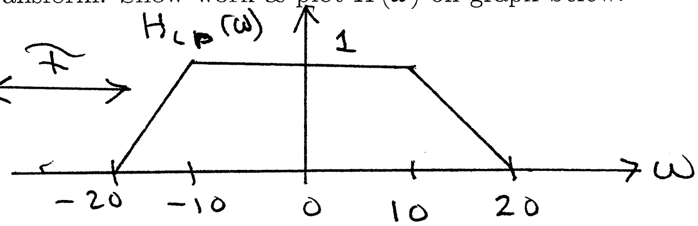
same period $\Rightarrow$ new FS coefficients

Problem 2 (a). Short Workout Problems Using Fourier Transform Properties. Consider an LTI system with the impulse response below. This filter is used in part (c) of Problem 2.

$$h(t) = \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t} \right\} 2j \sin(20t) \tag{1}$$

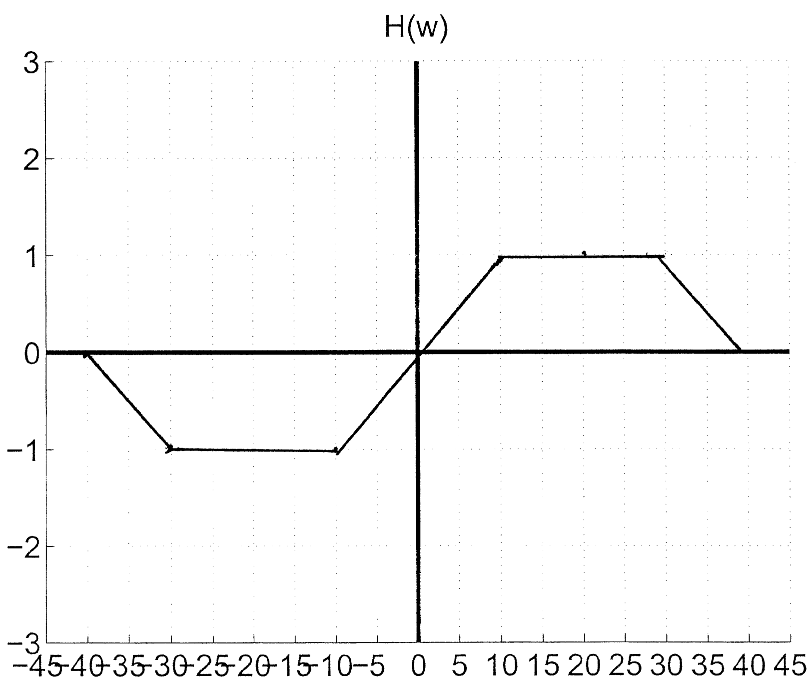
Determine the frequency response, $H(\omega)$, which is the Fourier Transform of $h(t)$. To this end, first determine the Fourier Transform of what's inside the brackets, and then apply the modulation property of the Fourier Transform. Show work & plot $H(\omega)$ on graph below.

$$h_{LP}(t) = \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t} \stackrel{\mathcal{F}}{\longleftrightarrow}$$

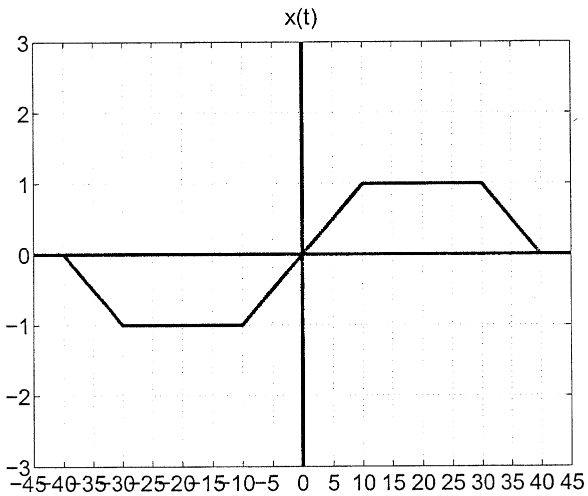


$$H(\omega) = \left\{ \frac{1}{2j} H_{LP}(\omega - 20) - \frac{1}{2j} H_{LP}(\omega + 20) \right\} 2j$$

$$= H_{LP}(\omega - 20) - H_{LP}(\omega + 20)$$



Problem 2 (b). Determine the Fourier Transform of the signal $x(t)$ below.



Show all work and write your expression for $X(\omega)$ in the space below. *HINT:* Duality.

This is identical to $H(\omega)$ for part (a)

Thus, Duality Property dictates:

$$X(\omega) = 2\pi \, \frac{\pi}{5} \, \frac{\sin(5(-\omega))}{\pi(-\omega)} \, \frac{\sin(15(-\omega))}{\pi(-\omega)} \, 2j \, \sin(20(-\omega))$$

$$X(\omega) = -\frac{4}{5} j \, \frac{\sin(5\omega)}{\omega} \, \frac{\sin(15\omega)}{\omega} \, \sin(20\omega)$$

Problem 2 (c). Consider an LTI system with impulse response

$$h(t) = \left\{ \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t} \right\} 2j\sin(20t) \tag{2}$$

You already determined the frequency response, $H(\omega)$, which is the Fourier Transform of $h(t)$, in part (a) of this problem. For the LTI system with impulse response above, determine the output $y(t)$ for the input $x(t)$ given below, which is the Fourier Series expansion for a periodic sawtooth waveform with fundamental frequency $\omega_0 = 5$ rads/sec.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk5t} \qquad a_k = \frac{j(-1)^k}{k\pi} \quad \text{for } k \neq 0, \ a_0 = 0$$

Show work and write your expression for $y(t)$ in the space directly below.

Any frequency $\omega_k = 5k$ for which $|\omega_k| \geq 40$ is rejected (not passed)

The freqs. $\pm 10, \pm 15, \pm 20, \pm 25, \pm 30$ are passed with $\pm$ unity gain $\Rightarrow$ no change in amplitude

The freqs. 5 and 35 are passed with a gain of 1/2, while the freqs. -5 and -35 are passed with a gain of $-1/2$

DC is rejected but $a_0 = 0$ any way

$$y(t) = -\frac{1}{2} a_{-7} e^{-j35t} + \sum_{\substack{k=-6 \\ k \neq \pm 1}}^{6} \text{sgn}(k)\, a_k e^{jk5t} + \frac{1}{2} a_7 e^{j35t}$$

$$-\frac{1}{2} a_{-1} e^{-5t} + \frac{1}{2} a_1 e^{j5t}$$

$$\text{sgn}(k) = \begin{cases} 1, & k > 0 \\ -1, & k < 0 \end{cases}$$

Workout Problem 3

(a) Determine and plot the magnitude of the Fourier Transform $X(\omega)$ of the signal $x(t)$ defined below. You must indicate which properties and pairs you are using as your arrive at your answer. Draw your plot on the graph provided on the following pages.

$$x(t) = \frac{d}{dt}\left\{\frac{\sin(10t)}{5\pi t}\right\} \quad (3)$$

$$= \frac{1}{5}\frac{d}{dt}\left\{\frac{\sin(10t)}{\pi t}\right\} \quad (4)$$

$$(5)$$

(b) Given $x(t)$ defined above, the signal $y(t)$ is created as shown below. Determine the Fourier Transform, $Y(\omega)$, of $y(t)$ and plot the magnitude $|Y(\omega)|$ as a function of frequency. Draw your plot on the graph provided on the following pages.

$$y(t) = x(t)\cos(30t) - \hat{x}(t)\sin(30t) \quad \text{where:} \quad \hat{x}(t) = x(t) * \frac{1}{\pi t}$$

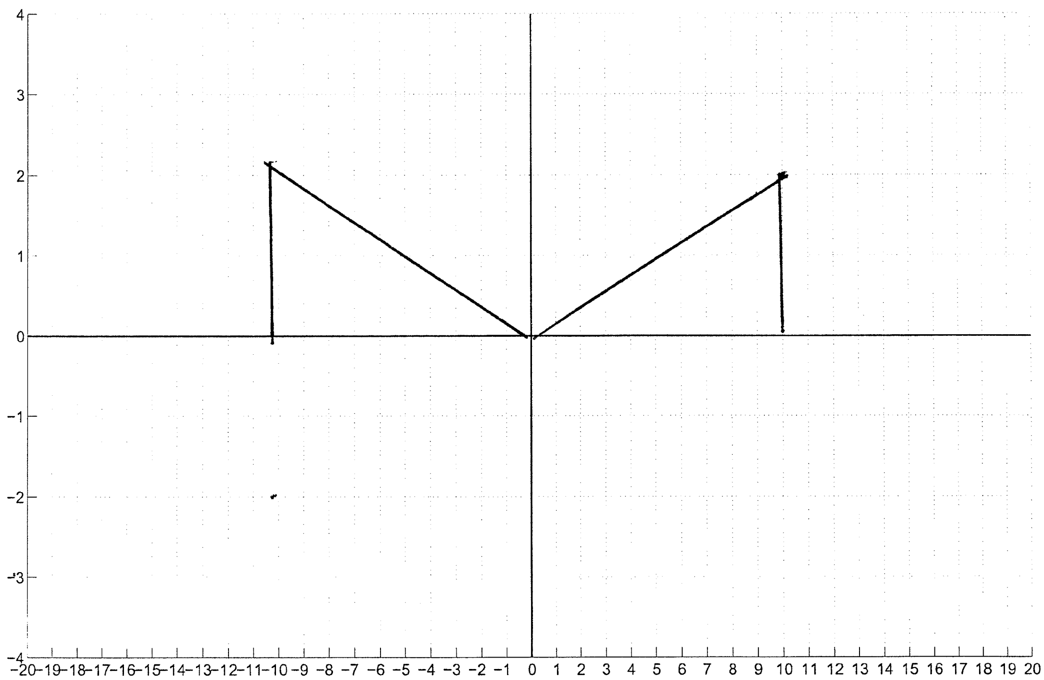
(c) Determine and plot the magnitude of the Fourier Transform $Z(\omega)$ of the signal $z(t)$ defined below. The trig identities $2\cos(\theta)\cos(\phi) = \cos(\theta+\phi)+\cos(\theta-\phi)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ should be useful. Draw your plot on the graph provided on the following pages.

$$z(t) = 2y(t)\cos(30t)$$

(d) The signal $w(t)$ is the output obtained with $z(t) = 2y(t)\cos(30t)$ from part (c) as the input to the lowpass filter with impulse response defined below. Plot the magnitude of the Fourier Transform $W(\omega)$ of $w(t)$. Draw your plot on the graph provided on the following pages. Does $w(t) = x(t)$?

$$w(t) = z(t) * h(t) \quad \text{where:} \quad h(t) = \frac{\pi}{5}\left\{\frac{\sin(5t)}{\pi t}\,\frac{\sin(15t)}{\pi t}\right\}$$

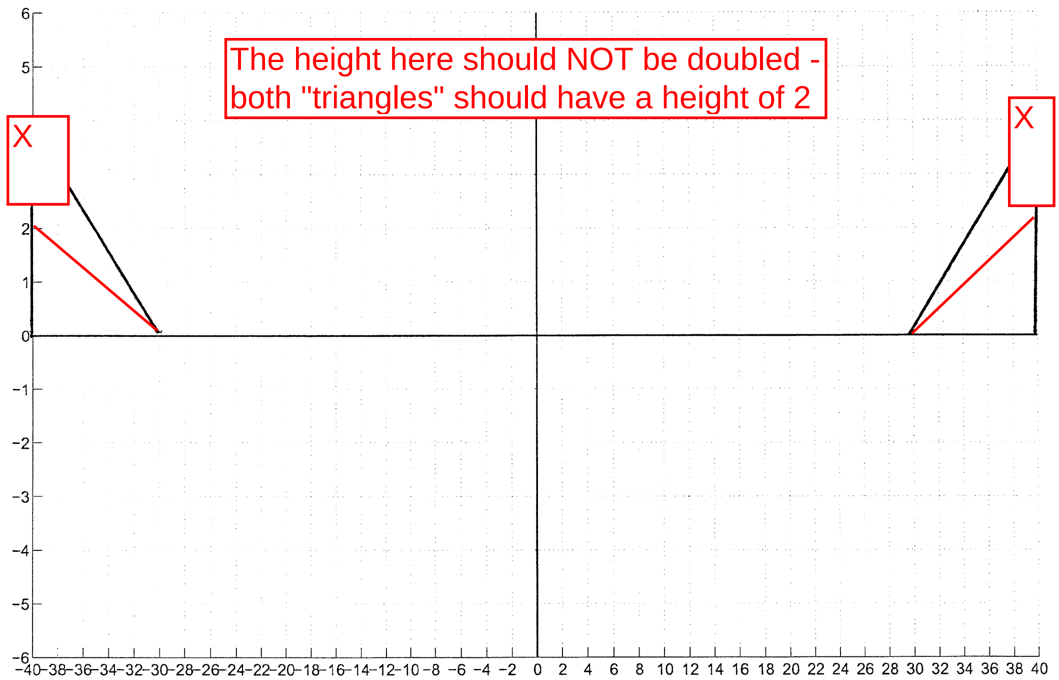
Plot your answer to Problem 3 (a) here. Show work below.



$$X(\omega) = \frac{1}{5} j\omega \, \mathrm{rect}\left(\frac{\omega}{20}\right)$$

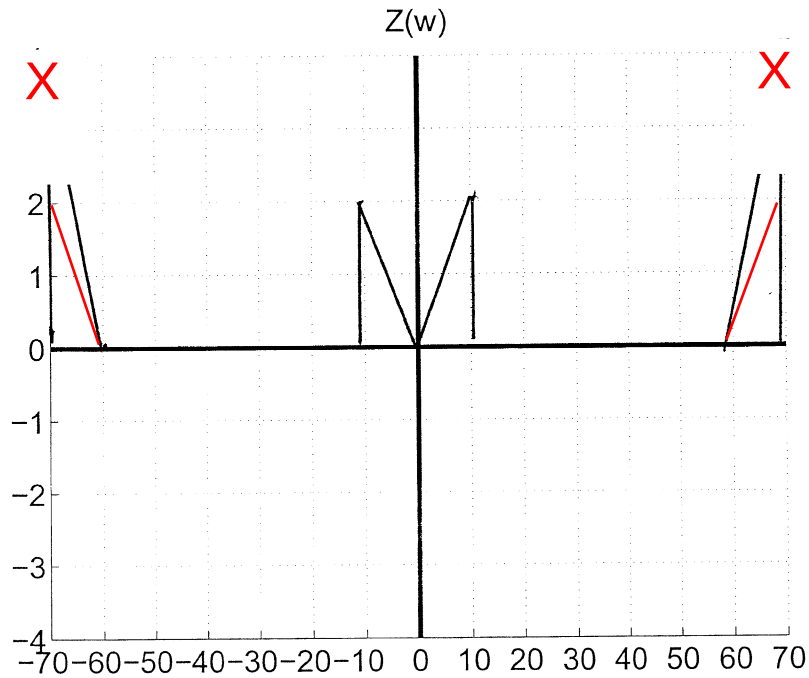
$$|X(\omega) = \frac{1}{5} |\omega| \quad \text{for} \quad |\omega| < 10$$

Plot your answer to Problem 3 (b) here. Show work below.



As derived in class and also for the Spring 2012 Exam 2 adding $-\hat{x}(t)\sin(30t)$ serves to cancel the negative frequency content comprising the lower sideband

Plot your answer to Problem 3 (c) here. Show work below.

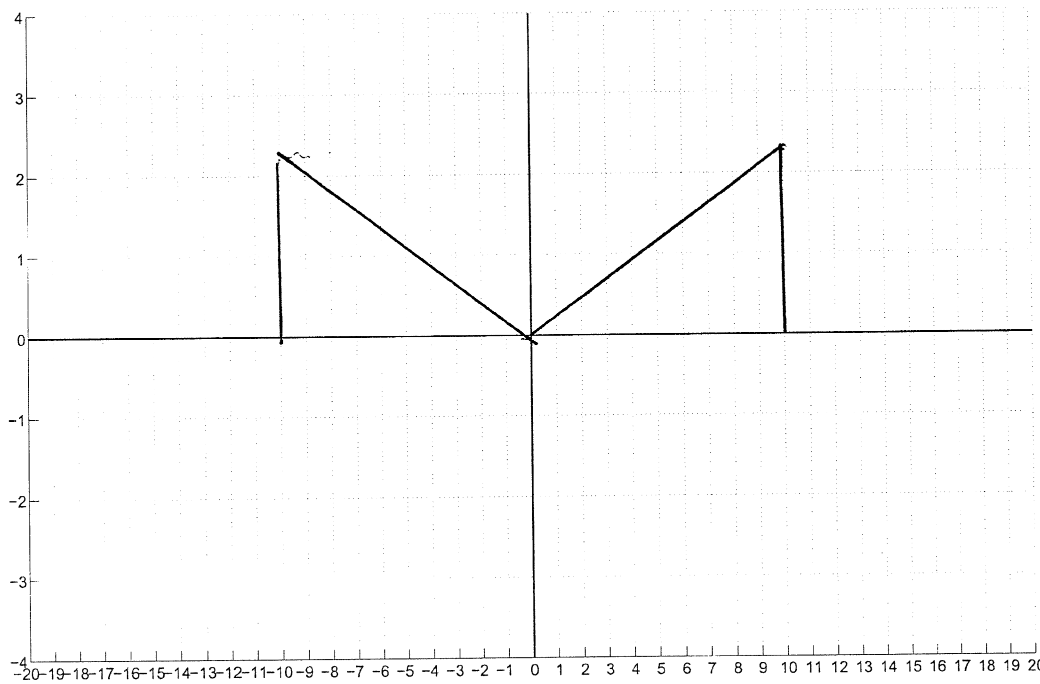


$$z(t) = 2\, y(t) \cos(30t)$$

$$= x(t)\, 2\cos^2(30t) - \hat{x}(t)\, 2\sin(30t)\cos(30t)$$

$$= x(t) + \underbrace{x(t)\cos(60t) - \hat{x}(t)\sin(60t)}_{\text{single sideband up at } \omega=60}$$

Plot your answer to Problem 3 (d) here. Show work below.



See Prob. 2(a) for plot of frequency response for $h_{LP}(t) = \frac{\pi}{5} \frac{\sin(5t)}{\pi t} \frac{\sin(15t)}{\pi t}$

It will only pass the original signal $x(t)$

$w(t) = x(t)$